

① Frobenius endomorphism:

$(R, m, k)$  a local (commutative, noetherian) ring.

Examples:  $\frac{k[x]}{(I)}$ ,  $\frac{k[x]_{(x)}}{(I)}$ ,  $\frac{k[x]}{(x^2)}$ ,  $\frac{k[x, y]}{(x^2, xy)}$ , ...

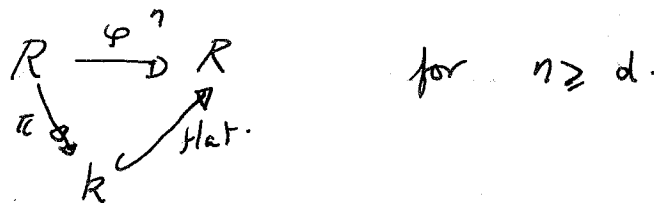
Suppose that  $\text{char}(R) = p$  (a positive prime number).

- $\varphi: R \rightarrow R$   $\varphi(r) = r^p$  is a ring homomorphism.
- Frobenius endomorphism of  $R$
- $\varphi^n =$  Iterates of the Frobenius.

Main pt: The homological behaviour of  $\varphi^n$  is similar to that of  $k$ .

Example: Suppose  $r^d = 0$  for some  $d$ . ( $\therefore R$  is artinian).

Then:



$\therefore$  As far as this  $R$  is concerned, that  $R$  is a  $k$ -module.

i.e. a  $k$ -vector spaces. Thus, observe: I have not said anything

- x - about what happens for smaller values of  $n$ .

Take: This and that....

$\varphi^n R = R$  viewed as a (left  $R$ -module) via  $\varphi^n$ .

Exhib: 5.

(A)

$$\text{fd}_R({}^{\varphi_n}R) < \infty \iff \text{fd}_R k < \infty$$

Kunz +  
Rud.lio.



R regular.



A-B-S. (True for any local ring R).

- Talk about B.

Want to go further in this direction:

: Suppose that M is an R-module: It has a flat resolution:

$$F: \dots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow 0$$

$\text{fd}_R M :=$  length of the shortest possible resolution.

Even when  $\text{fd}_R M = \infty$ , one can associate other invariants that measure how fast the resolution grows.

- Want to analyze  ${}^{\varphi_n}R$  from this point of view.

One sticky pt.:  ${}^{\varphi_n}R$  may not be f.g. as an R-module.

There are two ways to overcome deal with this:

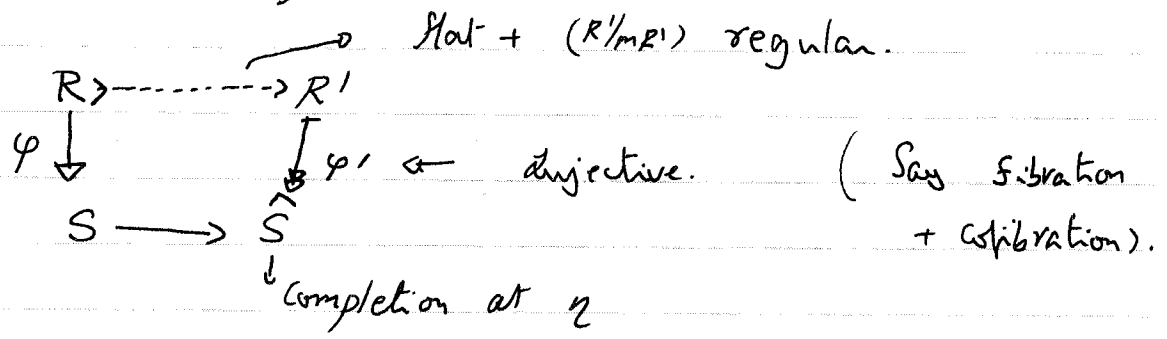
- $\text{Tor}_R^R(k, {}^{\varphi}R)$  - View it as an R-module.

or:

Other factorizations:

$\varphi: (R, m, k) \mapsto (S, n, l)$  a local homomorphism  
 $\varphi(m) \subseteq n$ .

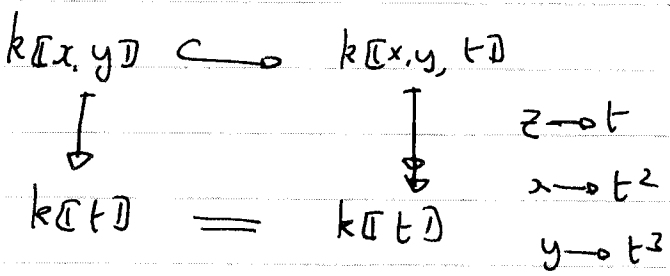
Fact: (Avramov-B. Herzog - Foxby)



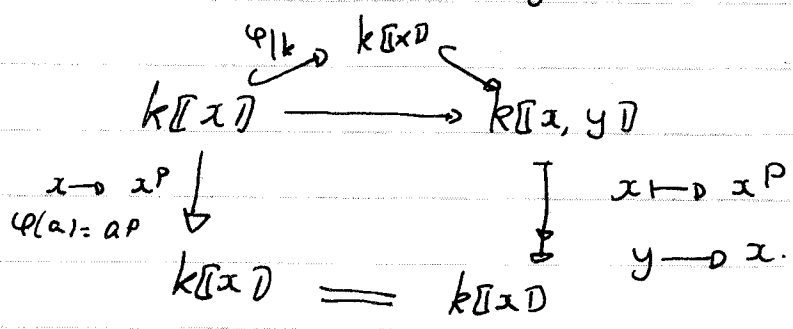
-  $\varphi$  and  $\varphi'$  behave the same (Explain).

Example:  $\text{fd}_R S < \infty \iff \text{fd}_{R'} \hat{S} < \infty$ .

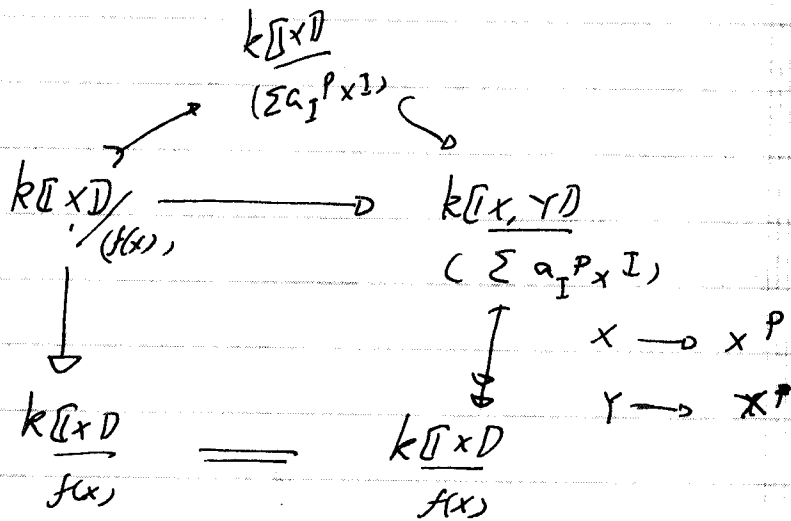
Examples: (a)  $k[x, y] \mapsto k[t]$   
 $x \mapsto t^2$  and  $y \mapsto t^3$ .



(b) Frobenius



©  $R = \frac{k[x]}{f(x)}$   
 $\sum a_i x^i$



III Invariants of local homomorphisms:

$\varphi: R \rightarrow S$   $N$  is a finite  $S$  module.

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ S & \longrightarrow & \hat{S} \end{array} \quad \hat{N} = \hat{S} \otimes N.$$

May assume:  $\text{pdim}(R'/m_{R'}) = \text{edim}(S/m_S)$ .

Min. res. of  $\hat{N}$  are  $R'$ :

$$F: \dots \rightarrow (R')^{b_2} \rightarrow (R')^{b_1} \rightarrow (R')^{b_0} \rightarrow 0$$

$\partial(F) \subseteq m F.$

$P_N^\varphi(t) := \sum_{n \geq 0} b_n t^n$  : Poincaré series of  $N$  over  $\varphi$ .

- Independent of Cohen factorizations.

variants:

$$\textcircled{1} \quad \text{pd}_\varphi(N) := \text{degree of } P_N^\varphi(t) - \text{edim}(S/\mathfrak{m}_S) \\ = \left[ \text{pd}_{\hat{R}} \hat{N} - \text{edim}(S/\mathfrak{m}_S) \right]$$

$$\textcircled{2} \quad \alpha_\varphi(N) := \inf \{ d \in \mathbb{N} \mid b_n \leq \binom{d-1}{n} \text{ for } n \gg 0 \}$$

Eg:  $\alpha_\varphi(N) = 0 \iff b_n = 0 \text{ for } n \gg 0$ .  
 - borrowed from g.c.  
 - Recall rep. theory of finite groups.

$$\textcircled{3} \quad \text{Curv}_\varphi(N) = \limsup_{n \rightarrow \infty} \sqrt[n]{b_n}$$

(Because  $\frac{1}{r.o.c.}$  of  $P_N^\varphi(t)$ ).

Remarks:

- $\text{pd}_\varphi N < \infty \iff \alpha_\varphi N = 0 \iff \text{Curv}_\varphi N = 0$ .
- $\text{pd}_\varphi N = \infty \iff \alpha_\varphi N \geq 1 \iff \text{Curv}_\varphi N \geq 1$ .

$$\alpha_\varphi N < \infty \implies \text{Curv}_\varphi N \leq 1$$

$\text{Curv}_\varphi N \leq \infty$ .  $\text{Curv}_R k \leq \infty$ .  
 describe c.f.

Remark: When  $\varphi = \text{id}$ :  $R = R$ . Recover invariants

for f.g. modules over local rings. These had been studied before.

If  $N$  happens to be finite  $k$ . Then:  
 all equal.

Examples: Absolute case:  $R = \frac{k[x]}{(x^2)}$   $N = k$ .

(a)  $\dots \rightarrow R \xrightarrow{x} R \xrightarrow{x} R \rightarrow 0$   $P_k^R(t) = \frac{1}{(1-t)}$   
 $C_{R,k} = 1 = \text{Cov}_R k$

(b)  $\frac{k[x, y]}{(x^2, y^2)}$   $P_k^R(t) = \frac{1}{(1-t)^2}$   $\therefore C_{R,k} = 2$   
 $\text{Cov}_R k = 1$

(c)  $\frac{k[x, y]}{(x^2, xy, y^2)}$   $0 \rightarrow k^2 \rightarrow R \rightarrow k \rightarrow 0$

$$P_k^R(t) = 1 + 2t P_k^R(t)$$

$$P_k^R(t) = \frac{1}{1-2t} \quad b_n = 2^n$$

$\therefore C_{R,k} = \infty$ . However,  $\text{Cov}_R k = 2$ .

Fact:  $C_{R,k} < \infty \stackrel{\text{Gulliksen}}{\iff} R$  is c.i. (define c.i.)  
 $\text{Cov}_R k < \infty$  always.  
 $\text{Cov}_R k \leq 1 \stackrel{\text{Avramov}}{\iff} R$  is c.i.

Theorem B (Avramov, I., M. Herz).  $(R, \mathfrak{m}, k)$  local ring of char,

$$cx_{\varphi^n}(R) = cx_R(k) \quad \text{and} \quad \text{Cov}_{\varphi^n}(R) = \text{Cov}_{\varphi^n}(k)$$

for each integer  $n \geq 1$ .

- Contains Kunz's Theorem:

$$cx_{\varphi^n}(R) = 0$$



$$\text{pd}_{\varphi^n} R < \infty \Leftrightarrow \text{fd}_{\varphi^n} R < \infty \Leftrightarrow \text{fd}_{\varphi^n} R = 0$$

Theorem: T.F.A.E.

- ①  $R$  is C.I.
- ②  $cx_{\varphi^n}(R) < \infty$  for some  $n$ .
- ③  $cx_{\varphi^n}(R) = \text{e-dim } R - \dim R$  for ~~some~~<sup>all</sup>  $n \geq 1$ .
- ④  $\text{Cov}_{\varphi^n}(R) \leq 1$  for some  $n \geq 1$ .
- ⑤  $\text{Cov}_{\varphi^n}(R) = 1 \quad \forall \quad n \geq 1$ .

Steps in the proof:

Think about the artinian case.

• Step 1: Establish desired result for some power of  $\varphi$ .

Step 2: Use the following result to obtain the result for

every  $\varphi^n$ :

$$R \xrightarrow{\varphi} S \xrightarrow{\psi} T$$

$$\text{Corv}(\varphi \circ \psi) \leq \text{Max} \{ \text{Corv}_{\psi}(T), \text{Corv}_{\varphi}(S) \} \leq \text{Corv}_{\varphi}$$

About step 1: Three possible ways: Talk about  $\text{Tor}_*^R(k, \varphi^n R)$ .

(a) Koh-Lee: to estimate  $\text{Tor}_*^R(k, \varphi^n R)$ .  $\text{Tor}_*^R(k, \varphi^n R)$

$$(b) \begin{array}{ccc} R \xrightarrow{\varphi^n} R \rightarrow X^R & & 0 \rightarrow X_e \rightarrow X_{e-1} \rightarrow \dots \rightarrow X_0 \rightarrow 0 \\ \downarrow & & \cup \\ E^* \rightarrow M^{i-1} X_e \rightarrow \dots \rightarrow M^{i-1} X_1 \rightarrow M^i X_0 \rightarrow 0 \end{array}$$

$$H(C^i) = 0 \quad \text{for } i \geq 3.$$

$$\begin{array}{ccc} R \xrightarrow{\varphi^n} R \rightarrow X^R \\ \downarrow \quad \downarrow \cong \\ k \rightarrow \frac{X^R}{E} \end{array}$$

$$\text{Tor}_*^R(k, X^R) \cong \text{Tor}_*^R(k, k) \oplus H(k^R)$$

as left

$$P_R^{\varphi^n}(t) := P_k^R(t) \cdot \text{char Hilb}_{H(k^R)}(t)$$

(c) Third method: same formula,  $n \geq \lceil \frac{\text{edim } R - \text{dep } R}{2} \rceil$   
from alg. top.



⑤ Frobenius & G-dimensions:

$R \xrightarrow{\varphi} S$   $M$  a finite  $S$ -module.

$$\begin{array}{ccc}
 R & \longrightarrow & R' \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & \hat{S}
 \end{array}
 \quad \text{gdim}_{\varphi} M = \text{gdim}_{R'} \hat{M} - \text{edim}(R'/MR').$$

(Remark: If  $M$  finite  $R$ , then  $\text{gdim}_{\varphi} M = \text{gdim}_R M$ ).

Fact:  $\text{g-dim}_R k < \infty \iff R$  is Gorenstein (Auslander-Bridger).

Theorem: (-, Saka-Wagstaff). T.F.A.E.

- ①  $R$  Gorenstein.
- ②  $\text{gdim}_{\varphi^n} R < \infty$  for all  $n \geq 1$
- ③  $\text{gdim}_{\varphi^n} R < \infty$  for some  $n \geq 1$ .

-x-

Mention: Takahashi & Yoshino.

⑥  $R \xrightarrow{\varphi} R$  and  $k$  should be the same as  $R$ -algebras.

i.e.  $R$ -modules viewed through  $\varphi \equiv R$ -modules viewed through  $k$ .

- okay for  $\varphi^n$  for  $n \geq \lceil \frac{\text{edim} R - \text{dep}^k R}{2} \rceil$ .

Expt:  $\text{Cov}_\varphi M = \text{Cov}_R k \quad \forall \quad M \text{ f.g.}$

- Do know if  $\text{pd}_R M < \infty$ .

This follows from:

Theorem:  $\varphi: R \rightarrow S$   $M$  a finite  $S$ -module with  $\text{pd}_S M < \infty$ .

Then

(i)  $\text{pd}_\varphi M = \text{pd}_R M + \text{pd}_S M$

(ii)  $\text{gdim}_\varphi M = \text{gdim}_R M + \text{pd}_S M$

(iii)  $\text{cx}_\varphi(S) \leq \text{cx}_R(M)$

$$\text{Cov}_\varphi(S) \leq \text{Cov}_\varphi(M)$$

- This is the start of a different story, so I should end here.