MSRI- Workshop. Feb. 10^{th} 2003

1 Frobenius endomorphism:

- (R, m, k) a local (commutative, noetherian) ring.

- Examples: $\frac{k[[x]]}{(I)}$, $\frac{k[x]_{(x)}}{(I)}$. $\frac{k[x]}{(x^4)}$, $\frac{k[x,y]}{(x^2, xy)}$, ...

. Suppose that $\mathrm{char}(R) = p$ (a positive prime number).

$\varphi: R \to R$ $\varphi(r) = r^p$. is a ring homomorphism.
- Frobenius endomorphism of R
- $\varphi^n :=$ Iterates of the Frobenius.

Main pt: The homological behaviour of φ^n is similar to that of k.

Example: Suppose $m^d = 0$ for some d. (∴ R is artinian).
Then:

$$\begin{tikzcd} R \arrow[rr, "\varphi^n"] \arrow[dr, "\pi"'] & & R \\ & k \arrow[ur, "\text{flat}"'] & \end{tikzcd} \qquad \text{for } n \geq d.$$

∴ As far as this R is concerned, that R is a k- module.
i.e. a k vector space. Thus. Observe: I have not said anything about what happens for smaller values of n.

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Joke: This and that....

${}^{\varphi^n}R :=$ R Viewed as a (left R- module) via φ^n.

Exhibit C.

(A) $fd_R({}^{\varphi^n}R) < \infty \iff fd_R\, k < \infty$

Kunz + Rodicio. $\Updownarrow$ R regular. $\Updownarrow$ A-B-S. (True for any local ring R).

- Talk about B.

Want to go further in this direction:

: Suppose that M is an R-module: It has a flat resolution:

$$F: \quad \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0$$

$fd_R M :=$ Length of the shortest possible resolution.

Even when $fd_R M = \infty$, one can associated other invariants that measure how fast the resolution grows.

- Want to analyze ${}^{\varphi^n}R$ from this point of view.

One sticky pt.: ${}^{\varphi^n}R$ may not be f.g. as an R-module.

There are two ways to ~~overcome~~ deal with this:

- $\mathrm{Tor}^R(k, {}^{\varphi}R)$ - View it as an R-module.

or:

Cohen factorizations:

$\varphi : (R, m, k) \longmapsto (S, n, \ell)$ a local homomorphism $\varphi(m) \subseteq n$.

Fact: (Avramov - B. Herzog - Foxby)

$$\begin{tikzcd}
R \arrow[r, dashed, tail] \arrow[d, "\varphi"'] & R' \arrow[d, "\varphi'"] \\
S \arrow[r] & \hat{S}
\end{tikzcd}$$

$R \to R'$: flat + (R'/mR') regular.

φ': surjective. (Say fibration + cofibration).

$\hat{S}$: completion at n

- φ and φ' behave the same (Explain).

Example: $\operatorname{fd}_R S < \infty \iff \operatorname{fd}_{R'} \hat{S} < \infty$.

Examples: (a) $k[[x, y]] \longrightarrow k[[t]]$

$x \longmapsto t^2$ and $y \longmapsto t^3$.

$$\begin{tikzcd}
k[[x,y]] \arrow[r, hook] \arrow[d] & k[[x,y,t]] \arrow[d] \\
k[[t]] \arrow[r, equal] & k[[t]]
\end{tikzcd}$$

$z \to t$, $x \to t^2$, $y \to t^3$

(b) Frobenius

$$\begin{tikzcd}
& k[[x]] \arrow[dr, bend left] & \\
k[[x]] \arrow[ur, bend left, "\varphi|_k"] \arrow[rr] \arrow[d, "x \to x^p"', "\varphi(a) = a^p"] & & k[[x, y]] \arrow[d] \\
k[[x]] \arrow[rr, equal] & & k[[x]]
\end{tikzcd}$$

$x \longmapsto x^p$, $y \longmapsto x$.

(c) $R = \dfrac{k[[X]]}{f(X)}$, $\quad f(X) = \sum a_I\, x^I$

$$\begin{tikzcd}
 & \dfrac{k[[X]]}{(\sum a_I^{\,p} X^I)} \arrow[dr, hook] & \\
\dfrac{k[[X]]}{(f(X))} \arrow[ur] \arrow[rr] \arrow[d] & & \dfrac{k[[X,Y]]}{(\sum a_I^{\,p} X^I)} \arrow[d, two heads] \\
\dfrac{k[[X]]}{f(X)} \arrow[rr, equal] & & \dfrac{k[[X]]}{f(X)}
\end{tikzcd}
\qquad
\begin{aligned} X &\to X^p \\ Y &\to X^p \end{aligned}$$

(III) Invariants of local homomorphisms:

$\varphi : R \to S \qquad N$ is a finite S module.

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R' \arrow[d, two heads] \\
S \arrow[r] & \hat{S}
\end{tikzcd}
\quad \hat{N} = \hat{S} \otimes_S N.$$

May assume: $\operatorname{edim}(R'/m_{R'}) = \operatorname{edim}(S/m_S)$.

Min. res. of $\hat{N}$ over R':

$$F: \quad \cdots \to (R')^{b_2} \to (R')^{b_1} \to (R')^{b_0} \to 0$$

$$\partial(F) \subseteq m F.$$

$\cancel{R(t) =}$ $\quad P_N^{\varphi}(t) := \sum_{n \geq 0} b_n t^n$: Poincaré series of N over φ.

- Independent of Cohen factorizations.

Invariants:

(1) $\mathrm{pd}_\varphi(N) :=$ degree of $P^\varphi_N(t)$ − edim $(S/\mathfrak{m}S)$
$= [\mathrm{pd}_{R'} \hat{N} - \mathrm{edim}(S/\mathfrak{m}S)]$

(2) $\mathrm{cx}_\varphi(N) := \inf\{ d \in \mathbb{N} \mid b_n \leq a n^{d-1}$ for $n \gg 0 \}$.
$\hat{R}$

Eg: $\mathrm{cx}_\varphi(N) = 0 \iff b_n = 0$ for $n \gg 0$.
– borrowed from g.c.
– Recall rep. theory of finite groups.

(3) $\mathrm{curv}_\varphi(N)$
$= \cancel{\sup} \limsup_{n\to\infty} \sqrt[n]{b_n}$

(Because $\frac{1}{\text{r.o.c.}}$ of $P^\varphi_N(t)$).

Remarks:
- $\mathrm{pd}_\varphi N < \infty \iff \mathrm{cx}_\varphi N = 0 \iff \mathrm{curv}_\varphi N = 0$.
- $\mathrm{pd}_\varphi N = \infty \iff \mathrm{cx}_\varphi N \geq 1 \iff \mathrm{curv}_\varphi N \geq 1$.
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$\mathrm{cx}_\varphi N < \infty \implies \mathrm{curv}_\varphi N \leq 1$.

- $\mathrm{curv}_\varphi N \leq \mathrm{curv}_R k < \infty$.
describe c.f.

Remark: • When $\varphi = \mathrm{id}$: $R = R$. Recover invariants
N

for f.g. modules over local rings. These had been studied before.

- If N happens to be finite $/R$. Then: all equal.

Examples: Absolute case: $R = \dfrac{k[x]}{(x^2)}$ $\quad N = k.$

(a) $\quad \cdots \to R \xrightarrow{x} R \xrightarrow{x} R \to 0 \qquad P_k^R(t) = \dfrac{1}{(1-t)}$

$\operatorname{Cx}_R k = 1 = \operatorname{Curv}_R k$

(b) $\quad \dfrac{k[x,y]}{(x^2, y^2)} \qquad P_k^R(t) = \dfrac{1}{(1-t)^2} \quad \therefore \quad \operatorname{Cx}_R k = 2$

$\operatorname{Curv}_R k = 1.$

(c) $\quad \dfrac{k[x,y]}{(x^2, xy, y^2)} \qquad 0 \to k^2 \to R \to k \to 0$

$$P_k^R(t) = 1 + 2t\, P_k^R(t)$$

$$P_k^R(t) = \frac{1}{1-2t} \qquad b_n = 2^n.$$

$\therefore \operatorname{Cx}_R k = \infty$. However, $\operatorname{Curv}_R k = 2$.

Fact:
- $\operatorname{Cx}_R k < \infty \overset{\text{Gulliksen}}{\iff} R$ is c.i. (define c.i.)
- $\operatorname{Curv}_R k < \infty$ always.

  $\operatorname{Curv}_R k \leq 1 \underset{\text{Avramov.}}{\iff} R$ is c.i.

...ben ...

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<u>Theorem</u>: B (Avramov, I., Miller). (R, m, k) local ring of char,

$$\mathrm{cx}_{\varphi^n}(R) = \mathrm{cx}_R(k) \quad \text{and} \quad \mathrm{curv}_{\varphi^n}(R) = \mathrm{curv}_R(k)$$

for each integer $n \geq 1$.

- Contains Kunz's theorem:

$$\mathrm{cx}_{\varphi^n}(R) = 0$$
$$\Updownarrow$$
$$\mathrm{pd}_{\varphi^n} R < \infty \iff \mathrm{fd}_{\varphi^n} R < \infty \iff \mathrm{fd}_{\varphi^n} R = 0$$

<u>Theorem</u>: T.F.A.E.

1. R is C.I.
2. $\mathrm{cx}_{\varphi^n}(R) < \infty$ for some n.
3. $\mathrm{cx}_{\varphi^n}(R) = \mathrm{edim}\, R - \dim R$ for ~~some~~ all $n \geq 1$
4. $\mathrm{curv}_{\varphi^n}(R) \leq 1$ for some $n \geq 1$.
5. $\mathrm{curv}_{\varphi^n}(R) = 1 \quad \forall \; n \geq 1$

<u>Steps in the proof</u>:

Think about the artinian case.

- <u>Step 1</u>: Establish desired result for some power of φ.

<u>Step 2</u>: Use the following result to obtain the result for

every φ^n:

$$R \xrightarrow{\varphi} S \xrightarrow{\psi} T$$

$$\operatorname{curv}(\psi \circ \varphi) \le \operatorname{Max}\{\operatorname{curv}_\psi(T), \operatorname{curv}_\varphi(S)\}$$

About Step 1: Three possible ways. Talk about $\operatorname{Tor}^R_*(k, {}^{\varphi^n}K^R)$.

(a) Koh. Lee: to estimate $\operatorname{Tor}^R_*(k, {}^{\varphi^n}R)$. $\operatorname{Tor}^{R}(k, {}^{\varphi}R)$

(b)

$$\begin{array}{l} R \xrightarrow{\varphi^n} R \to K^R \\ \qquad\quad\ \downarrow \end{array}$$

$$0 \to K_e \to K_{e-1} \to \cdots \to K_0 \to 0$$

$$\cup\!\!\mid \qquad \cup\!\!\mid$$

$$E^i \quad 0 \to m^{i-e} K_e \to \cdots \to m^{i-1} K_1 \to m^i K_0 \to 0$$

$$H(C^i) = 0 \quad \text{for } i \ge s.$$

$$\begin{array}{ccccc} R & \xrightarrow{\varphi^n} & R & \to & K^R \\ \downarrow & & & & \downarrow \cong \\ k & & \longrightarrow & & \frac{K^R}{E} \end{array}$$

$$\operatorname{Tor}^R_*(k, {}^{\varphi^n}K^R) \cong \operatorname{Tor}^R(k,k) \otimes_k H(K^R).$$

$$P^{\varphi^n}_R(t) := P^R_k(t) \cdot \operatorname{Hilb}_{H(K^R)}(t).$$

(c) Third method: Same formula, $n \ge \left\lceil \frac{\operatorname{edim} R - \operatorname{depth} R}{2} \right\rceil$
from alg. top.

V Frobenius & G-dimensions:

$R \xrightarrow{\varphi} S$ M a finite S-module.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ S \arrow[r] & \hat{S} \end{tikzcd} \qquad \mathrm{Gdim}_\varphi M = \mathrm{Gdim}_{R'} \hat{M} - \mathrm{edim}(R'/\mathfrak{m}R').$$

(Remark: If M finite $/R$, then $\mathrm{Gdim}_\varphi M = \mathrm{Gdim}_R M$).

- Fact: $G\text{-}\dim_R k < \infty \iff R$ is Gorenstein (Auslander-Bridger).

Theorem: (–, Sather-Wagstaff). T.F.A.E.

1. R Gorenstein.
2. $\mathrm{Gdim}_{\varphi^n} R < \infty$ for all $n \geq 1$
3. $\mathrm{Gdim}_{\varphi^n} R < \infty$ for some $n \geq 1$.

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Mention: Takahashi & Yoshino.

VI

$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & R \\ k & \end{tikzcd}$. ${}^{\varphi}R$ and k should be the same as R-algebras.

i.e: R-modules viewed through φ $\equiv$ R-modules viewed through k.

– okay for φ^n for $n \geq \left\lceil \dfrac{\mathrm{edim} R - \mathrm{depth} R}{2} \right\rceil$

Expect: $\operatorname{curv}_\varphi M = \operatorname{curv}_R k$. $\forall$ M f.g.

- Do know if $\operatorname{pd}_R M < \infty$.

This follows from:

Theorem: $\varphi: R \to S$ M a finite S-module with $\operatorname{pd}_S M < \infty$.
Then

(i) $\operatorname{pd}_\varphi M = \operatorname{pd}_\varphi S + \operatorname{pd}_S M$

(ii) $\operatorname{Gdim}_\varphi M = \operatorname{Gdim}_\varphi S + \operatorname{pd}_S M$.

(iii) $\operatorname{cx}_\varphi(S) \leq \operatorname{cx}_R(M)$

$$\operatorname{curv}_\varphi(S) \leq \operatorname{curv}_\varphi(M)$$

- This is the start of a different story, so I should end here.