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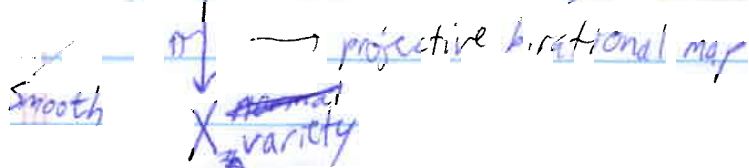
(1)

# Three-dimensional flops and non-commutative rings

$k = \bar{k}$  char 0

$\exists Y$  resolution of singularities

↓ everything is Gorenstein and normal



$\pi$  is crepant if  $\pi^* \omega_X = \omega_Y$ .

By adjointness  $\pi_* \omega_Y = \omega_X$

So if  $\exists$  crepant resolution,  $X$  has rational singularities

- Does not always exist

e.g. a non-rational singularity

e.g.  $k[X, Y, Z, T] / (X^2 + Y^2 + Z^2 + T^3)$  is <sup>has</sup> rational singularities but no crepant resolution.

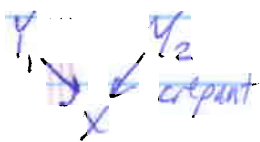
- If a crepant resolution exists, it is usually not unique.

$$X = \text{Spec } k[X, Y, Z, T] / (XY - ZT)$$

- Blow up  $(X, Z)$   
or  $(Y, T)$ .

## General principle (?)

Different crepant resolutions have similar properties



Theorem (Kontsevich, Borcea)

Hodge #s:  $h_{pq}(Y_1) = h_{pq}(Y_2) (= h_{pq}(\text{string}))$

Conjecture (Bondal-Orlov)

$D^b \text{Coh } Y_1 \cong D^b \text{Coh } Y_2$

Proved in dim 3 by Bridgeland (2000)

Modified Proof (-) using "non-commutative crepant resolutions"

Bold Question:  $\exists D^b(\text{Coh}(X))_{\text{stringy}}?$

(- thinks not.)

McKay correspondence:

$G \subseteq SL(V)$  finite subgroup,  $\dim V = 2, 3$ , symmetric algebra of  $V$   
 $X = \text{Spec}(SV)^G$

Thm (Bridgeland-King-Reid, Kpsnov-Vasserot (dim 2))

$\exists$  crepant resolution  $Y$  of  $X$ .

$D^b(\text{Coh}(Y)) \cong D^b(\text{mod}(SV * G))$   
non-comm. ring

$SV * G = \text{End}_{SV^G}(SV)$

reflexive  $(SV^G)$ -module

Def  $X = \text{Spec } R$  (normal Gorenstein)

A n.c. crepant resolution of  $R$  (or  $X$ ) is an  $R$ -algebra

A defn from  $\text{End}_R(M)$ , such that:

- (1)  $M$  is reflexive and finitely generated
- (2)  $\text{gl dim } A < \infty$ , and
- (3)  $A$  is an  $\text{CM } R$ -module

Conjecture (-): All crepant resolutions of  $X$  (comm. and n.c.) are derived equivalent

Thm: This conjecture is true in dimension 3.

Relation between comm. and n.c. crepant resolutions

resolutions

Comm.  $\rightarrow$  n.c.

Thm (no Gorenstein hypothesis)

Assume (1)  $f: Y \rightarrow X$  projective,  $X = \text{Spec } R$

(2)  $Rf_* \mathcal{O}_Y = \mathcal{O}_X$

(3)  $\dim$  fibers  $f \leq 1$ .

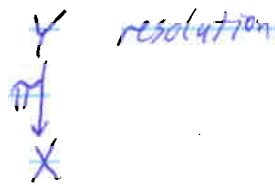
Then  $\exists$  vector bundle  $M$  on  $Y$  such that if

$A = \text{End}_Y(M)$ , then

$D^b(\text{coh}(Y)) = D^b(\text{mod}(A))$

(see e.g. tilting theory)

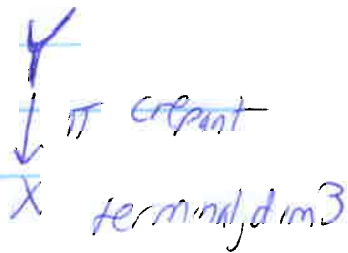
X normal Gorenstein, rational singularities



$$\pi_* \omega_Y = \omega_X, \quad \pi^* \omega_X \hookrightarrow \omega_Y$$

We say X has terminal singularities if Supp( $\omega_Y / \pi^* \omega_X$ )  
contains all exceptional divisors.

X terminal singularities  $\Rightarrow$  codim Sing X  $\geq 3$



$$\pi^* \omega_X = \omega_Y$$

$\Rightarrow \pi$  contracts only a finite number of curves.

Assume now: X ~~is~~ 3-dim terminal

$f: Y \rightarrow X$  crepant.  $\mathcal{M}$  as in theorem,  $X = \text{Spec } R$ .

$M = F(Y, \mathcal{M}) \Rightarrow A = \text{End}_Y(M) = \text{End}_R(M)$  is a n.c.  
crepant resolution of  $R$ .



Non comm.  $\Rightarrow$  comm. crepant res.

$R$  3 dim'l normal Gorenstein, rational singularities,  
which is not regular.

$A = \text{End}_R(M)$ ,  $M$  n.g. crepant resolution

Using ring theory,  $\Rightarrow \text{gl.dim } A = 3$

$M$  is reflexive as an  $A$  module

$$\Rightarrow \text{pd}_A M \leq 3 - 2 = 1.$$

$$\Rightarrow \text{pd}_A M = 1.$$

~~$$0 \rightarrow P \rightarrow A^{\oplus c} \rightarrow M \rightarrow 0$$~~

$$0 \rightarrow P \rightarrow A^{\oplus c} \rightarrow M \rightarrow 0 \quad (**)$$

projective  
res.

$$P \oplus Q = A^{\oplus c} \quad (*)$$

Reflexive Morita Theory

$$P = \text{Hom}_R(M, M_1)$$

$$Q = \text{Hom}_R(M, M_2)$$

$$M_1 \oplus M_2 = M^{\oplus c}, \text{ by } (*).$$

Using  $(**)$  we can assume  $(\text{rk}_R M_1, \text{rk}_R M_2) = 1$ .

Replace:

$$M \rightsquigarrow M^{\oplus c}$$

$$A \rightsquigarrow M_2(A).$$

$e_1, e_2 \in A$  idempotents corresponding to

projections  $M \begin{matrix} \swarrow M_1 \\ \searrow M_2 \end{matrix}$ ,  $e_1, e_2$  orthogonal,  $e_1 + e_2 = 1$

For  $V$  a f.d.  $A$ -representation,

$$\underline{\dim} V := (\dim e_1 V, \dim e_2 V)$$

$$\delta = (\text{rk}_R M_1, \text{rk}_R M_2)$$

Choose  $\Theta \in \mathbb{Z}^2$  such that  $\Theta \cdot \delta = 0$ ,

but  $\Theta \cdot \alpha \neq 0$  for all  $0 < \alpha < \delta$

We say  $V$  is  $\Theta$ -stable if for all  $0 \subsetneq W \subsetneq V$ ,

$$\Theta \cdot \dim(V) = 0$$

$$\mathbb{Z} = \left\{ V \in \text{f.d.}(A) \mid \underline{\dim} V = \delta, \right. \\ \left. V \text{ is } \Theta\text{-stable} \right\} / \cong$$

f. cons. component dominating  $X$ .

$$\mathbb{Z} \xrightarrow{\text{proper}} X \quad (X = \text{spec } R)$$

$$V \mapsto \text{Ann}_R V$$

Thm:  $Y$  is a crepant resolution of  $X$ , and

$$D^b(\text{coh } Y) = D^b(\text{mod } A)$$

Proof: similar to the proof of the McKay correspondence due to Bridgeland-King-Reid.

Conjecture 1: (dim 3, Gorenstein, rational sing.)

(a)  $X$  has a comm. crepant resolution  $\Leftrightarrow X$  has a  
 $\mathbb{Q}$ -c. crepant res.

$\begin{cases} \Rightarrow \text{OK for terminal sing.} \\ \Leftarrow \text{OK by Bridgeland-King-Reid generalization} \end{cases}$

(b) Every crepant resolution of  $X$  may be obtained (locally) from a single non-comm. crepant resolution through the choice of some  $\theta$ .

~ (b) is true for terminal sing. (v.d.B.)  
 and  $\sum_{\mathbb{Q}}^{\text{for}} \mathbb{Q} \times G, G$  abelian (Crow, Ishii)