Michel van den Bergh:

Three-dimensional flops and non-commutative rings

$k=\bar{k}$ char 0

$$\begin{array}{ccl} & \exists\, Y & \text{resolution of singularities} \\ \text{Smooth} & \pi \downarrow & \rightarrow \text{projective birational map} \\ & X & \text{~~normal~~ variety} \end{array}$$

everything is Gorenstein and normal

π is *crepant* if $\pi^*\omega_X=\omega_Y$.

By adjointness $\pi_*\omega_Y=\omega_X$

So if $\exists$ crepant resolution, X has rational singularities

- Does not always exist

e.g. a non-rational singularity

e.g. $k[X,Y,Z,T]/(X^2+Y^2+Z^2+T^3)$ has rational singularities but no crepant resolution.

- If a crepant resolution exists, it is usually not unique.

$X=\mathrm{Spec}\; k[X,Y,Z,T]/(XY-ZT)$

- Blow up (X,Z) or (Y,T).

General principle (?)

Different crepant resolutions have similar properties

$$\begin{array}{ccc} Y_1 & & Y_2 \\ & \searrow \;\; \swarrow & \text{crepant} \\ & X & \end{array}$$

Theorem (Kontsevich, Batyrev)

Hodge #s: $h_{pq}(Y_1)=h_{pq}(Y_2)\;\left(=h_{pq}(X)_{\text{stringy}}\right)$

Conjecture (Bondal–Orlov)

$D^b \operatorname{Coh} Y_1 \cong D^b \operatorname{coh} Y_2$

Proved in dim 3 by Bridgeland (2000)

Modified Proof (—) using "non-commutative crepant resolutions".

Bold Question: $\exists\, D^b(\operatorname{Coh}(X))_{\text{stringy}}$?

(— thinks not)

McKay correspondence:

$G \subseteq SL(V)$ finite subgroup, $\dim V = 2, 3$.

$X = \operatorname{Spec}(\overset{\text{symmetric algebra of } V}{SV})^G$

Thm (Bridgeland–King–Reid, Kapranov–Vasserot ($\dim 2$))

- $\exists$ crepant resolution Y of X.
- $D^b(\operatorname{coh}(Y)) \cong D^b(\operatorname{mod}(\underbrace{SV * G}_{\text{non-comm. ring}}))$

$SV * G = \operatorname{End}_{SV^G}(\underbrace{SV}_{\text{reflexive } (SV)^G\text{-module}})$

Def $X = \operatorname{Spec} R$ (normal Gorenstein)

A n.c. crepant resolution of R (or X) is an R-algebra

A of the form $\operatorname{End}_R(M)$, such that:

(1) M is reflexive and finitely generated
(2) $\operatorname{gl\,dim} A < \infty$, and
(3) A is a mCM R-module.

Conjecture (—): All crepant resolutions of X (comm. and n.c.) are derived equivalent

Thm: This conjecture is true in dimension 3.

Relation between comm. and n.c. crepant resolutions

resolutions

$\boxed{\text{comm.} \to \text{nc.}}$

Thm (no Gorenstein hypothesis)

Assume (1) $f : Y \to X$ projective, $X = \operatorname{Spec} R$
(2) $Rf_* \mathcal{O}_Y = \mathcal{O}_X$
(3) dim. fibers $f \leq 1$.

Then $\exists$ vector bundle M on Y such that if

$A = \operatorname{End}_{\mathcal{O}_Y}(M)$, then

$D^b(\operatorname{Coh}(Y)) \cong D^b(\operatorname{mod}(A))$ (see e.g. tilting theory)

X normal Gorenstein, rational singularities

$$\begin{tikzcd} Y \arrow[d, "\pi"'] & \text{resolution} \\ X \end{tikzcd}$$

$\pi_* \omega_Y = \omega_X$, $\quad \pi^* \omega_X \hookrightarrow \omega_Y$

We say X has <u>terminal singularities</u> if $\mathrm{Supp}(\omega_Y / \pi^* \omega_X)$ contains all exceptional divisors.

X terminal singularities $\Rightarrow$ codim Sing $X \geq 3$

$$\begin{tikzcd} Y \arrow[d, "\pi\ \text{crepant}"] \\ X \ \text{terminal, dim } 3 \end{tikzcd}$$

$\pi^* \omega_X = \omega_Y$

$\Rightarrow \pi$ contracts only a finite number of curves.

<u>Assume now</u>: X 3-dim'l terminal

$f : Y \to X$ crepant. $\mathcal{M}$ as in theorem, $X = \mathrm{Spec}\, R$.

$M = \Gamma(Y, \mathcal{M}) \Rightarrow A = \mathrm{End}_{\mathcal{O}_Y}(\mathcal{M}) = \mathrm{End}_R(M)$ is a n.c. crepant resolution of R.

Non comm. ⟹ comm. crepant res.

R 3 dim'l normal Gorenstein, rational singularities, which is not regular.

$A = \mathrm{End}_R(M)$, M n.c. crepant resolution

Using ring theory, $\Rightarrow$ $\mathrm{gldim}\, A = 3$

M is reflexive as an A module

$\Rightarrow \mathrm{pd}_A M \leq 3-2 = 1.$

$\Rightarrow \mathrm{pd}_A M = 1.$

$$0 \to P \to A^{\oplus d} \to M \to 0 \quad (**)$$

projective res.

$$P \oplus Q = A^{\oplus c} \quad (*)$$

Reflexive Morita theory

$P = \mathrm{Hom}_R(M, M_1)$

$Q = \mathrm{Hom}_R(M, M_2)$

$M_1 \oplus M_2 = M^{\oplus c}$, by $(*)$.

Using $(**)$ we can assume $(\mathrm{rk}_R M_1, \mathrm{rk}_R M_2) = 1$.

Replace: $M \rightsquigarrow M^{\oplus c}$

$A \rightsquigarrow M_c(A)$.

$e_1, e_2 \in A$ idempotents corresponding to

projections $\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ M_1 & & M_2 \end{tikzcd}$, e_1, e_2 orthogonal, $e_1 + e_2 = 1$.

For V a f.d. A-representation,

$\underline{\dim V} := (\dim e_1 V, \dim e_2 V)$

$\delta = (rk_R M_1, rk_R M_2)$

Choose $\theta \in \mathbb{Z}^2$ such that $\theta \cdot \delta = 0$,

but $\theta \cdot \alpha \neq 0$ for all $0 < \alpha < \delta$

We say V is $\underline{\theta\text{-stable}}$ if for all $0 \subsetneq W \subsetneq V$,

$\theta \cdot \dim(V) = 0$.

$$Z = \left\{ V \in \text{f.d.}(A) \mid \underline{\dim} V = \delta, \; V \text{ is } \theta\text{-stable} \right\} / \cong$$

Y: conn. component dominating X.

$$\begin{tikzcd} \supset Y \arrow[dr] & \\ Z \arrow[r, "\text{proper}"'] & X \end{tikzcd} \qquad (X = \operatorname{Spec} R)$$

$V \longmapsto \operatorname{Ann}_R V$

Thm: Y is a crepant resolution of X, and

$D^b(\operatorname{coh} Y) = D^b(\operatorname{mod} A)$.

proof: Similar to the proof of the McKay correspondence due to Bridgeland-King-Reid.

Conjecture 1: (dim 3, Gorenstein, rational sings)

(a) X has a comm. crepant resolution $\Leftrightarrow$ X has a n.c. crepant res.

$\Rightarrow$ OK for terminal sings

$\Leftarrow$ OK by Bridgeland-King-Reid generalization

(b) Every crepant resolution of X may be obtained (locally) from a single non-comm. crepant resolution through the choice of some θ.

~ (b) is true for terminal sings (vdB)

and for $SV * G$, G abelian (Craw, Ishii)