Karen Smith: Jumping coefficients and the uniform Artin-Rees theorem

w/ L. Ein, R. Lazarsfeld (and Varolin)

Jumping Numbers of an ideal

- X (smooth) variety ($/\mathbb{C}$) (Think $X = \mathbb{A}^N_{\mathbb{C}}$)
- $\mathcal{O}_X$ regular functions (Think $\mathbb{C}[x_1, \dots, x_n]$)

$\mathfrak{a} \subseteq \mathcal{O}_X$

Associate a family of ideals $\{\mathcal{J}(\mathfrak{a}^c)\}_{c \in \mathbb{Q}_{\geq 0}}$ of $\mathcal{O}_X$

(the multiplier ideals of $\mathfrak{a}$ with coefficient c).

Idea: – capture the singularities of closed subscheme of X defined by $\mathfrak{a}$
– good for finding uniform results.

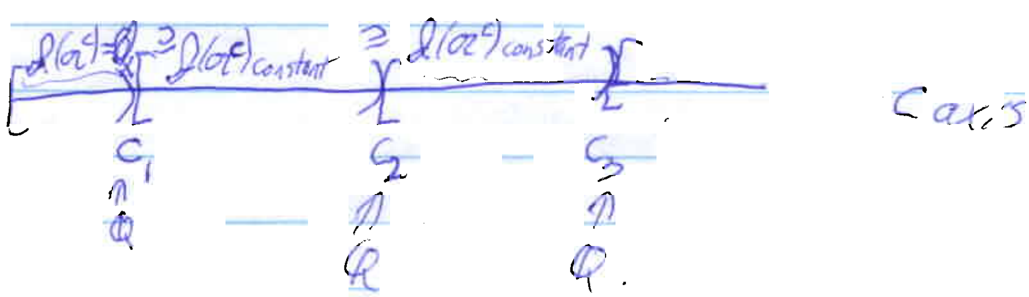
Theorem (Siu) (Invariance of Plurigenera)
Let $X \to T$ smooth projective family of irred. smooth varieties of general type, $\{X_t\}_{t \in T}$. (i.e. canonical bundle is "big".)
Then the plurigenera of the X_t are independent of t.

$(P_m = \dim H^0(X_t, \omega_{X_t}^m))$

Theorem (Ein, Lazarsfeld, —): If $P \subseteq \mathcal{O}_X$ is prime in the regular ring $\mathcal{O}_X$, dim d, then $P^{(Nd)} \subseteq P^N \quad \forall N \geq 0$

[Swanson: Given P $\exists k$ (depending on P) such that
$$P^{(Nk)} \subseteq P^N \quad \forall N \geq 0.]$$

Behavior of $\{\mathcal{J}(\mathfrak{a}^c)\}_{c \in \mathbb{Q}_{\geq 0}}$. All multiplier ideals are integrally closed.



Def: The numbers c_i are the jumping numbers (or "jumping coefficients") of $\mathfrak{a} \subseteq \mathcal{O}_X$.

Fact:
- The c_i are discrete
- The c_i are eventually periodic beyond dim X (or the analytic spread of A), of period 1.

Example: $(f = s^3 + t^4) \subseteq \mathbb{C}[s,t]$

Relation to Bernstein-Sato polynomial

$\mathfrak{a} = (f) \subseteq \mathbb{C}[x_1, \ldots, x_d]$.

B-S poly (also called b-polynomial) is defined as follows:

$\exists$ diff. op. P with coefficients in $\mathbb{C}[s, x_1, \ldots, x_d]$ (s a new variable) such that

$$P * f^{s+1} = b(s) f^s$$ for some polynomial $b(s) \in \mathbb{C}[s]$.

The set of all such $b(s) \in \mathbb{C}[s]$ forms an ideal, and the B-S poly. is its unique monic generator.

Example $f = x$

$$\frac{\partial}{\partial x} * x^{s+1} = (s+1)x^s.$$

So B-S poly of (x) is $(s+1)$.

(We can similarly find B-S poly. by hand for monic f...)

Fact: The first jumping number c_1 is the log-canonical threshold of the ideal. (Analysts have a name for $\frac{1}{c_1}$.)

Theorem: If c is a jumping number of $(f) \subseteq \mathbb{C}[x_1, \dots, x_d]$ in the range $(0,1]$, then $-c$ is a root of the BS poly. for f.

(Yang Lichen, Kollar for log-canon. thr. of an isolated sing.)

~~Also:~~

Also: If f has an iso. sing., Steenbrink defined the Spectrum of f. c is a Jumping number in range $(0,1]$ $\Leftrightarrow$ $(c-1)$ is in the spectrum of (f).
(Varchenko, Vaquié)

Uniform Artin-Rees

Fix a pair of modules $N \subseteq M$ f.g. over a Noetherian ring R

The (weak form of the) Artin-Rees Lemma states that:

For every ideal $I \subseteq R$, $\exists$ n (depending on I) such that $\forall k \geq 0$, $I^{n+k} M \cap N \subseteq I^k N$.

If $M \supseteq IM \supseteq I^2M \supseteq \dots$

$N \supseteq IM \cap N \supseteq I^2M \cap N \supseteq \dots$
$\cup\!\!\mid \qquad \cup\!\!\mid$
$N \supseteq IN \supseteq I^2N \supseteq \dots$

are two sub filtrations induced by N.

(the AR lemma says that if you shift, you get the containment the other way.)

Smith (5)

Q1: Can you choose n that works for all I?
(Eisenbud-Hochster)

Huneke: Yes (if excellent local) (Uniform Artin-Rees Theorem)

Q2: What is n?

Our contribution is that when R is regular, (ess. of finite type over k char $k=0$), can find n explicitly, in the case where $N \subseteq M$ is $J \subseteq R$, where J is a multiplier ideal.

Def: Say $J \subseteq R$ is a multiplier ideal (i.e. $J = \mathcal{J}(\mathfrak{a}^c)$ for some $\mathfrak{a} \subseteq R$ and some $c \in \mathbb{Q}_{\geq 0}$). Then the jumping length is the length of the shortest saturated chain of the form

$$\mathcal{O}_X \supseteq \mathcal{J}(\mathfrak{a}^{c_1}) \supseteq \mathcal{J}(\mathfrak{a}^{c_2}) \supseteq \cdots \supseteq \mathcal{J}(\mathfrak{a}^{c}) = J.$$

(i.e. the # of distinct jumping numbers $\leq c$).
(i.e. # of mult. ideals in family $\{\mathcal{J}(\mathfrak{a}^d)\}_{d \leq c}$)

Fact: $\mathcal{J}((f)^1) = (f)$.

In the example on page (2), the jumping length of the principal ideal (f) is (4).

Theorem Let $J \subseteq R$ be a multiplier ideal in a regular ring (ess. of fin type over k, char $k=0$). Say $\dim R = d$ and jumping length of J is ℓ.
Then $\forall I \subseteq R$, $I^{\ell d + k} \cap J \subseteq I^k J \quad \forall k \geq 0$

Effective uniform Artin-Rees

Two Questions

Q1: How big is the class of multiplier ideals?
Is every integrally closed ideal a multiplier ideal?

Q2: How to compute (or bound) the jumping length?

Theorem (Lipman-Watanabe, Favre-Jonsson (analytic setting)): For R if $\dim = 2$, every integrally closed ideal is a multiplier ideal.

Fact: Every principal ideal is a multiplier ideal.

Theorem: Say f defines an isolated singularity.

i.e. $\tau = \lambda\left(\frac{R}{(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N})}\right) < \infty$.

"Tjurina number"

The jumping length of (f) is bounded by $\tau + 1$.

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Analytic def. of $\mathcal{J}(\mathfrak{a}^c)$: Fix $\mathfrak{a} = (g_1, \ldots, g_r)$

$$\mathcal{J}(\mathfrak{a}^c) = \left\{ f \;\middle|\; \int \frac{|f|^2}{\sum |g_i|^{2c}} < \infty \ \text{(as a germ)} \right\}.$$

Algebro-geometric def. of $\mathcal{J}(\mathfrak{a}^c)$:

Fix a log. resolution of $\mathfrak{a}$

i.e. $Y \xrightarrow[\text{proper, birat.}]{\pi} X$, Y smooth,

$\mathfrak{a}\mathcal{O}_Y$ principal, and

$\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$,

$F + \mathrm{exc}(\pi)$ have normal crossing support

(i.e. a monomial in local coords.)

Then $$\mathcal{J}(\mathfrak{a}^c) = \pi_* \mathcal{O}_Y(K_{Y/X} - \lfloor c \cdot F \rfloor)$$

$$\subseteq \pi_* \mathcal{O}_Y(K_{Y/X}) \subseteq \mathcal{O}_X.$$

relative canonical module

$K_{Y/X} = \sum a_i E_i$

$F = \sum b_i E_i$

$$K_{Y/X} - \lfloor cF \rfloor = \sum (a_i - \lfloor c b_i \rfloor) E_i.$$

possible jumping #'s are a subset of the set of c's such that $a_i - c b_i \in \mathbb{Z}$.