Koszul duality in representation theory,
Koszul dual operads?
Why not ...

Koszul algebras in combinatorics

Vic Reiner
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
reiner@math.umn.edu

Outline

I. Review of Koszul algebras

II. Combinatorial examples/questions

A. Is a certain algebra Koszul?

B. Consequences for Hilbert function?

III. Koszul $\overset{interaction}{\longleftrightarrow}$ real roots/PF-sequences

I. Review of Koszul algebras

Definition (Priddy 1970).
A a finitely generated, associative, standard graded k-algebra,

$$A = k\langle x_1, \ldots, x_n \rangle / J$$

for some homogeneous (two-sided) ideal J.

A is Koszul if $k = A/A_+$ has a linear A-free resolution

$$\cdots \to A(-2)^{\beta_2} \to A(-1)^{\beta_1} \to A \to k \to 0$$

that is, all maps have only k-linear entries in the x_i's.

NB: In this case, this is a minimal free resolution, and

$$\beta_i = \dim_k \operatorname{Tor}_i^A(k,k) = \dim_k \operatorname{Ext}_i^A(k,k).$$

Two consequences.

Firstly, define theHilbert and Poincaré series

$$\text{Hilb}(A,t) := \sum_{i\geq 0} \dim_k A_i t^i$$
$$\text{Poin}(A,t) := \sum_{i\geq 0} \beta_i t^i.$$

Then Euler characteristic in each degree of the exact sequence

$$\cdots \to A(-2)^{\beta_2} \to A(-1)^{\beta_1} \to A \to k \to 0$$

yields

$$\text{Poin}(A,-t)\text{Hilb}(A,t) = 1.$$

In particular,

$$\frac{1}{\text{Hilb}(A,-t)} = \text{Poin}(A,t) \in \mathbb{N}[[t]].$$

Secondly, assuming W.L.O.G. that

$$A = k\langle x_1, \ldots, x_n \rangle / J$$

has no redundant generators x_i,

A Koszul $\Rightarrow$ J quadratically generated

as a minimal resolution starts

$$\begin{array}{ccccccccc} \cdots & \to A^{\beta_2} & \to & A(-1)^n & \to & A & \to k & \to & 0 \\ & & & y_i & \mapsto & x_i & & & \end{array}$$

and minimal generators of J in degree d lead to elements in the kernel of $A(-1)^n \to A$ with degree d.

Converse (quadratic $\Rightarrow$ Koszul) is false,
but true for monomial ideals (Fröberg 1975)

- in the purely non-commutative setting

$$J = \langle x_i x_j, \ldots \rangle$$

- or in the commutative setting

$$J = \langle x_i x_j - x_j x_i : i < j \rangle + \langle x_i x_j, \ldots \rangle$$

- or in the anticommutative setting

$$J = \langle x_i x_j + x_j x_i, x_i^2 : i < j \rangle + \langle x_i x_j, \ldots \rangle$$

So by deformation argument, one can prove Koszul-ness via Gröbner bases by exhibiting a quadratic initial ideal $\mathrm{init}_{\prec}(J)$.

Duality

When A is Koszul,

$$\text{Poin}(A,t) = \text{Hilb}(A^!,t)$$

where $A^!$ is another Koszul algebra called the Koszul dual $A^!$. Thus

$$\text{Hilb}(A^!,-t)\text{Hilb}(A,t) = 1.$$

In fact, the linear minimal free resolution can be constructed explicitly by a natural differential on $A \otimes_k A^!$.

And it really is a duality: $(A^!)^! = A$.

(Recipe for $A^!$?
Think of $x_1,\ldots,x_n$ as a basis for a k-space V.
Thus $A = T^{\cdot}(V)/\langle J_2\rangle$ with $J_2 \subset V \otimes V$.
Let $A^! := T^{\cdot}(V^*)/\langle J_2^{\perp}\rangle$ for $J_2^{\perp} \subset V^* \otimes V^*$.)

The motivating example

$$\begin{aligned} A &= k[x_1, \ldots, x_n] \\ &= \text{a (commutative) polynomial algebra} \\ &= k\langle x_1, \ldots, x_n \rangle / \langle x_i x_j - x_j x_i : i < j \rangle \end{aligned}$$

$$\text{Hilb}(A, t) = \frac{1}{(1-t)^n}$$

$$\begin{aligned} A^! &= k\langle y_1, \ldots, y_n \rangle / \langle y_i y_j + y_j y_i, y_i^2 : i < j \rangle \\ &= \bigwedge(y_1, \ldots, y_n) \\ &= \text{an exterior algebra} \end{aligned}$$

$$\text{Hilb}(A^!, t) = (1+t)^n$$

Note that

$$\text{Hilb}(A^!, -t)\text{Hilb}(A, t) = (1-t)^n \frac{1}{(1-t)^n} = 1$$

The linear minimal resolution for k is the usual Koszul complex for $(x_1, \ldots, x_n)$:

$$\begin{array}{ccccccc} \cdots \to & A \otimes A^!_2 & \to & A \otimes A^! & \to A & \to k \to 0 \\ & 1 \otimes (y_1 \wedge y_2) & \mapsto & x_1 y_2 - x_2 y_1 & & \end{array}$$

II. Combinatorial examples/questions

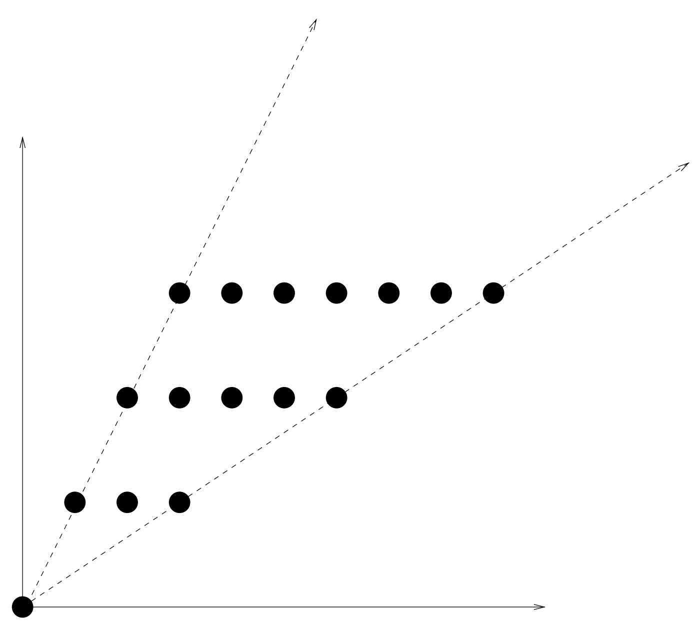
- Is a certain class of combinatorial rings Koszul?
- If so, what can be said about their Hilbert series?

Example 1. Affine semigroup rings

Let Λ be a finitely generated subsemigroup of $\mathbb{N}^d$,

$$\begin{aligned} A &:= k[\Lambda] \text{ its semigroup algebra} \\ &= k[t^\alpha : \alpha \in \mathcal{A}] \subset k[t_1, \ldots, t_d] \\ &\cong k[x_\alpha : \alpha \in \mathcal{A}]/I_{\mathcal{A}} \end{aligned}$$

A is graded if and only if
all α in $\mathcal{A}$ lie on a hyperplane in $\mathbb{N}^d$.



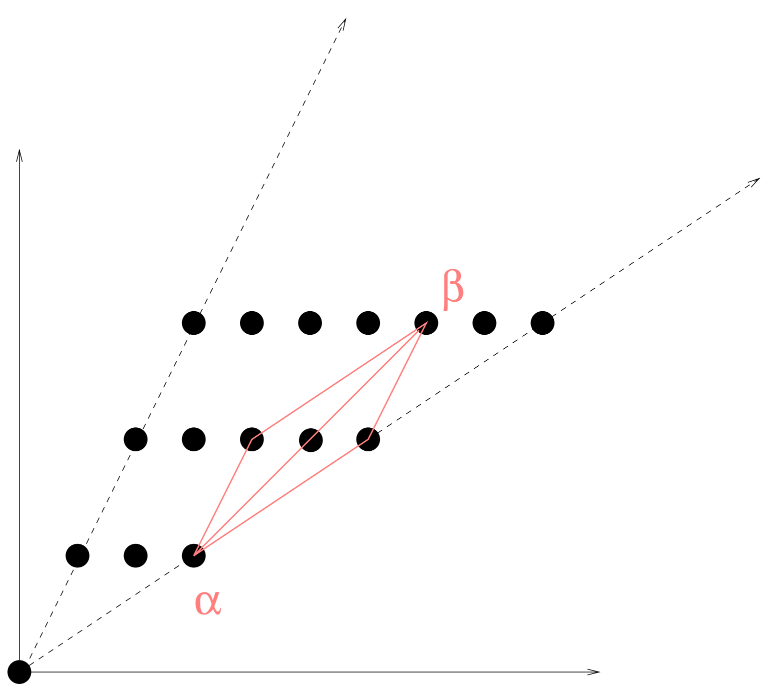
Computing $\mathrm{Tor}^A(k,k)$ via bar resolution of k yields

PROPOSITION (Peeva-R.-Sturmfels):

$A = k[\Lambda]$ is Koszul $\Longleftrightarrow$
Λ is a Cohen-Macaulay poset (over k) when ordered by divisibility.

That is, α divides β implies

$\tilde{H}_i(\Delta(\alpha,\beta);k) = 0$ for $i < \deg(\beta) - \deg(\alpha) - 2$.



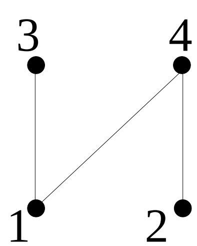
Some known Koszul families of $k[\Lambda]$, (via quadratic initial ideals, yielding homotopy type of intervals in the poset Λ):

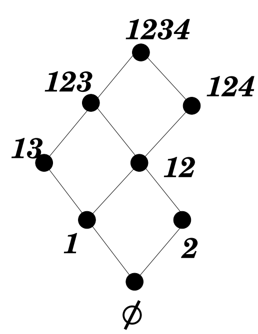
- Veronese subalgebras:
  $A = k[t^\alpha : \deg(\alpha) = r]$

- Segre subalgebras:
  $A = k[s_i t_j]_{\substack{i=1,\ldots,d \\ j=1,\ldots,e}}$

- Hibi ring of a poset P:

$$A = k[t_0 t^I]_{I \in J(P)}$$
$$= k\left[t_0,\, t_0t_1,\, t_0t_2,\, t_0t_1t_2,\, t_0t_1t_3,\, t_0t_1t_2t_3,\, t_0t_1t_2t_4,\, t_0t_1t_2t_3t_4\right]$$

P=


J(P) =


OPEN PROBLEM:

Given vectors $v_1, \ldots, v_n$,
spanning a vector space V,
the associated matroid basis ring is

$$k[\Lambda_B] := k[t^B]_{\{v_i : i \in B\} \text{ a basis for } V}$$

Q: Is $k[\Lambda_B]$ Koszul?
Q: Does it have a quadratic initial ideal?

THEOREM(N. White 1977) $k[\Lambda_B]$ is normal.

(Generalized recently to discrete polymatroids by Herzog and Hibi.)

More Koszul algebras from matroids ...

Given $v_1, \ldots, v_n$ spanning V as before, define the Orlik-Solomon algebra

$$A := \bigwedge(x_1, \ldots, x_n)/I$$

where I is spannned by

$$\sum_{s=1}^{r} (-1)^s x_{i_1} \wedge \cdots \wedge \widehat{x_{i_s}} \wedge \cdots \wedge x_{i_r}$$

for all circuits (= minimal dependent subsets) $\{v_{i_1}, \ldots, v_{i_r}\}$.

OPEN PROBLEM(Yuzvinsky) :

Q: When is the Orlik-Solomon algebra A Koszul? If and only if it has a quadratic initial ideal?

(Equivalently, if and only if the matroid is supersolvable)?

NB: When $V = \mathbb{C}^d$, Orlik and Solomon 1980 showed that the hyperplane arrangement,

$$\mathcal{A} = \{v_1^\perp, \ldots, v_n^\perp\}$$

has A as the cohomology ring $H^\cdot(\mathbb{C}^d - \mathcal{A}; k)$ of the complement $\mathbb{C}^d - \mathcal{A}$.

THEOREM (see Yuzvinsky 2001) :
A is Koszul $\Longleftrightarrow \mathbb{C}^d - \mathcal{A}$ is a rational $K(\pi, 1)$.

Partial commutation/annihilation monoids

P a collection of unordered pairs $\{i, j\}$,
S a collection of singletons $\{i\}$,
from $[n] := \{1, 2, \ldots, n\}$.

THEOREM:(Froeberg 1970, Kobayashi 1990)

$$A := k\langle x_1, \ldots, x_n\rangle / \langle x_i x_j - x_j x_i, x_k^2 : \{i, j\} \in P, k \in S\rangle$$

is Koszul.

Consequently,

$$\begin{aligned}\mathrm{Hilb}(A, t) &= \frac{1}{\mathrm{Hilb}(A!, -t)} \\ &= \frac{1}{\sum_{C \subset [n]} (-1)^{|C|} t^C}\end{aligned}$$

where C runs over subsets chosen with repetition from $[n]$ in which every pair of elements of C is in P, and repeats are allowed only on the elements of S.

Generalizes a main result of Cartier and Foata's theory of partial commutation monoids (1969)

Algebras from walks in directed graphs

D a directed graph on $[n]$, that is,
a collection of ordered pairs (i, j)
(with $i = j$ allowed).

$$A_D := k\langle x_1, \ldots, x_n \rangle / \ \langle x_i x_j : (i,j) \notin D \rangle$$

is Koszul
(Froeberg 1970, Kobayashi 1990, Bruns-Herzog-Vetter 1992)

Its Koszul dual $A_D^! := A_{\bar{D}}$
for the complementary digraph $\bar{D}$.

A_D has Hilbert function

$$h(A_D, n) := |\{ \text{ walks of length } n \text{ in } D\}|$$

so that $\text{Hilb}(A_D, t)$ can be computed via the transfer-matrix method.

Studied by
Carlitz-Scoville-Vaughan 1976,
Goulden-Jackson 1988, Brenti 1989,
Bruns-Herzog-Vetter 1992

Stanley-Reisner rings

Δ a simplicial complex on $[n]$
has Stanley-Reisner ring

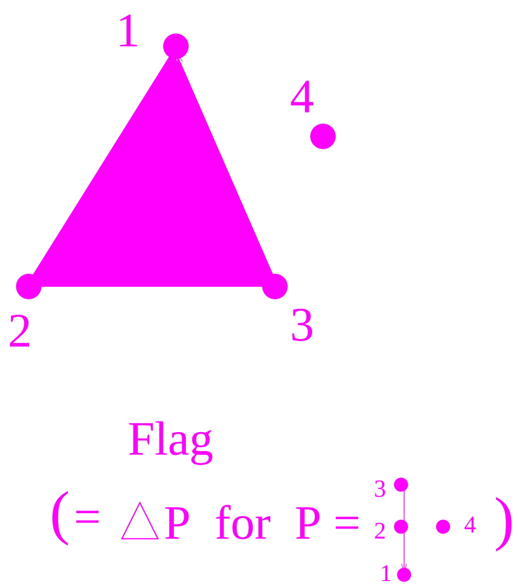
$$k[\Delta] := k[x_1, \ldots, x_n]/I_\Delta.$$

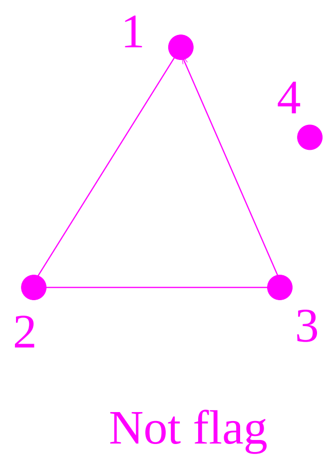
where $I_\Delta := (x^F : F \notin \Delta)$.

Since these are quotients by monomial ideals,
$k[\Delta]$ is Koszul $\iff$
I_Δ is quadratic $\iff$
Δ is a flag (clique, stable/independent set) complex.

Δ is a flag complex if it is
determined by its 1-skeleton
$G(\Delta)$ in the following way:
F is a face of Δ $\iff$
each pair in F is an edge in $G(\Delta)$.

Example: The order complex $\Delta(P)$ of a poset P is always flag.





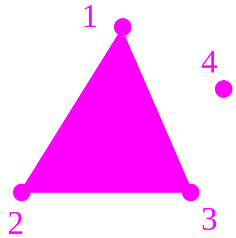
Recall the f-vector
$f_i(\Delta)$ counts i-dimensional faces of Δ.

$$f(\Delta,t) = \sum_{i\geq -1} f_i(\Delta)t^{i+1}$$

$$\begin{aligned}\text{Hilb}(k[\Delta],t) &= f\left(\Delta, \frac{t}{1-t}\right) \\ &= \frac{h(\Delta,t)}{(1-t)^{\dim\Delta+1}}\end{aligned}$$

$$h(\Delta,t) = h_0 + h_1 t + \cdots + h_d t^d$$

where $d = \dim\Delta + 1$ and $(h_0,\ldots,h_d)$ is called the h-vector.



has

$$f(\Delta,t) = 1 + 4t + 3t^2 + t^3$$
$$h(\Delta,t) = 1 + t - 2t^2 - t^3$$

Three open conjectures on real-roots for f-polynomials (or equivalently, h-polynomials) of flag complexes:

Gasharov-Stanley: Δ the flag (clique) complex for a graph G whose complement is claw-free has $f(\Delta, t)$ with only real roots.

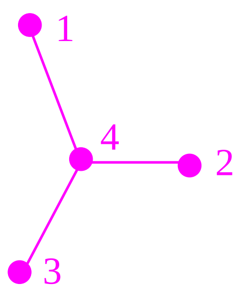
Neggers-Stanley: Δ the order complex of a finite distributive lattice $J(P)$ has $f(\Delta, t)$ with only real roots.

Charney-Davis: Δ a flag simplicial complex triangulating a homology $(d-1)$-sphere with d even has

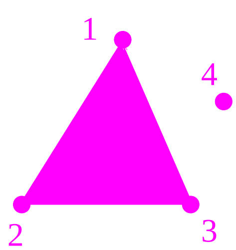
$$(-1)^{\frac{d}{2}} h(\Delta, -1) \geq 0.$$

Gasharov-Stanley is known to hold for order complexes (by a result of Gasharov 1994).

One must avoid the *claw*



because its complement has flag complex



whose f-polynomial has complex roots.

Neggers-Stanley is known only in very special cases.

It motivated Brenti's study of Hilbert functions for algebras of walks in digraphs
(take $D = J(P)$).

The distributive lattice $J(P)$ is shellable, hence $k[\Delta]$ is Koszul and Cohen-Macaulay.
In fact, I_Δ is an initial ideal for the toric ideal of the Hibi ring for P.

Recent work (Welker-R.),
motivated by relation to Charney-Davis
proves unimodality of $h(\Delta(P), t)$
(weaker than real-rooted-ness) when P is graded.

Charney-Davis:
Δ triangulating a flag homology sphere says $k[\Delta]$ is Koszul and Gorenstein.
Hence $h(\Delta, t)$ is symmetric: $h_i = h_{d-i}$.

But how is $(-1)^{\frac{d}{2}} h(\Delta, -1) \geq 0$ related to real roots? It's weaker ...

PROPOSITION:
Suppose $h(t) = \sum_{i=0}^{d} h_i t^i$ with d even

- lies in $\mathbb{N}[t]$,
- is symmetric, and
- has only real roots.

Then $(-1)^{\frac{d}{2}} h(\Delta, -1) \geq 0$.

(Uses the fact that for $h(t)$ symmetric, roots come in pairs $r, \frac{1}{r}$.)

Charney-Davis is

- trivial for 1-spheres,

- proven for homology 3-spheres by Okun-Davis 2000 (but with a lot of work!),

- known under certain geometric hypotheses (local convexity) by Leung-R. 2002 via Hirzebruch signature formula.

- would follow for order complexes by a conjecture of Stanley 1994 on nonnegativity of cd-index for Gorenstein* posets, proven for barycentric subdivisions of convex polytopes.

III. Koszulness and Polya frequency sequences.

Real-rooted-ness of $f(\Delta, t)$ or $h(\Delta, t)$, has an equivalent formulation for power series in t that need not be polynomial or even rational...

Say $H(t) := \sum_{n \geq 0} a_n t^n \in \mathbb{R}[[t]]$ generates a Polya frequency (PF) sequence $(a_0, a_1, a_2, \ldots)$ if the (infinite) Toeplitz matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

has all minor determinants non-negative.

A deep result of Aissen-Schoenberg-Whitney (1952) says that when $H(t) \in \mathbb{N}[t]$, this is equivalent to all real roots.

The right questions?

- Among Koszul algebras, when is the Hilbert function a PF sequence? Say A is PF when this occurs.

- In particularly, which (commutative) Cohen-Macaulay Koszul algebras A are PF, so that $h(A,t)$ has only real roots?

- Even more particularly, which (commutative) Gorenstein Koszul algebras A satisfy the weaker condition that

$$(-1)^{\frac{d}{2}} h(A,-1) \geq 0?$$

Some more instances...

- P. Ho Hai proves that certain quantized symmetric and exterior algebras are
  - Koszul 1997
  - PF 1999

  via representation-theory.

- Brenti 1989 investigated the question of which digraphs D have their algebra of walks PF (without referring to Koszul algebras).

- Heilmann-Lieb 1972 proved the f-polynomial of the (flag) simplicial complex of partial matchings of a graph has only real roots.

Several auspicious features.

- The question respects Koszul duality:

$$A \text{ is PF} \iff A^! \text{ is PF.}$$

  because $H(t)$ generates a PF sequence if and only if $\frac{1}{H(-t)}$ generates a PF sequence.

- All three notions Koszul, PF, Cohen-Macaulay respect several other constructions well:
  - Veronese subalgebras,
  - tensor products of algebras
  - Segre products of algebras,
  - quotients by a linear non-zero-divisor.

A fact well-known to $\text{Tor}^A(k,k)$ experts (but apparently overlooked by the rest of us)*

PROPOSITION: When A Koszul, if $\text{Hilb}(A,t)$ is rational (e.g. if A is commutative) then it has at least one real zero.

(In fact, one only needs $\frac{1}{\text{Hilb}(A,-t)} \in \mathbb{N}[[t]]$.)

This is particularly handy when A is Gorenstein since it often gives two real zeroes!

For example, Charney-Davis for homology 3-spheres (Okun-Davis) already suffices to imply their h-polynomials have only real-roots.

*With thanks to I. Peeva and V. Gasharov for pointing this out.

CHALLENGE: Simplify the Okun-Davis proof by showing more generally (and hopefully more simply!) that a Gorenstein Koszul algebra A with

$$h(A,t) = 1 + h_1 t + h_2 t^2 + h_1 t^3 + t^4$$

has

$$h(A,-1) = h_2 - 2h_1 + 2 \geq 0.$$

IDEA (from conversation with M. Kapranov):

Non-negativity of Toeplitz matrix minors should be modelled by homology concentration of certain chain complexes
(i.e. the minor should be the Euler characteristic of the complex).

True for consecutive superdiagonal minors using the graded components of the bar complex computing $\mathrm{Tor}^A(k,k)$.

$$\text{e.g. } \det \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & a_1 & a_2 \\ 0 & 1 & a_1 \end{bmatrix}$$

is modelled by

$$A_1 \otimes A_1 \otimes A_1 \;\to\; \begin{matrix} A_1 \otimes A_2 \\ A_2 \otimes A_1 \end{matrix} \;\to\; A_3 \;\to\; 0$$

QUESTION: Complexes modelling other Toeplitz minors? What beyond Koszul gives homology concentration?

References:

- R. Fröberg,
  Koszul algebras,
  in "Advances in commutative ring theory" (Fez, Morocco),
  Lecture Notes in Pure and Applied Math **205**, Marcel Dekker, 1999.

- V. R. and V. Welker,
  "On the Charney-Davis and Neggers-Stanley conjectures",
  manuscript in preparation.

- S. Yuzvinsky,
  "Orlik-Solomon algebras in algebra and topology",
  Russian Math. Surveys **56** (2001), 293–364.