P. Aluffi

http://www.math.fsu.edu/~aluffi/CSM/CSM.html

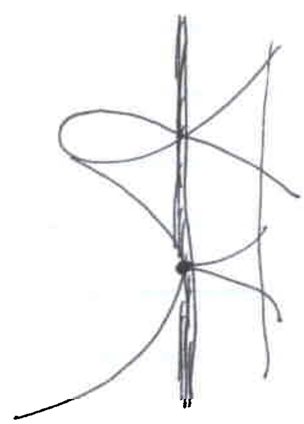
$(xy, xz, yz) \subseteq \mathbb{C}[x,y,z,w]$

$V(\quad) \subseteq \mathbb{P}^3\mathbb{C}$

$\chi_{top}(\quad) = ?$

– "simple-minded" information: Euler characteristic, for $V \subseteq \mathbb{P}^3\mathbb{C}$ Whitney stratification

$y^2 = x^2 z + x^3$



Ingredients:
- Blowup algebra
- Chern-Schwartz-MacPherson class
- Bookkeeping

① Blowup algebras

I : ideal of R

(i) $\mathrm{Rees}_R\, I = \bigoplus_{n \geq 0} I^n$

(ii) $\mathrm{Sym}_R\, I$

(i ½) $A \overset{\varphi}{\twoheadrightarrow} R$ epimorphism

$$\begin{tikzcd}
\mathrm{Sym}_A(\varphi^{-1}(I)) \arrow[r, two heads] \arrow[d] & \mathrm{Sym}_R\, I \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{Rees}_A(\varphi^{-1}(I)) \arrow[r] \arrow[drr, bend right] & \otimes \arrow[dr] & \\
 & & \mathrm{Rees}_R\, I
\end{tikzcd}$$

<u>**Definition**</u>: $q\mathrm{Sym}_{A \to R}(I) =$

$$\mathrm{Sym}_R(I) \otimes_{\mathrm{Sym}_A(\varphi^{-1}(I))} \mathrm{Rees}_A(\varphi^{-1}(I))$$

<u>**Remark**</u>: For $R \xrightarrow[\mathrm{id}]{} R$, $\quad q\mathrm{Sym}_{R \xrightarrow{\mathrm{id}} R}(I) = \mathrm{Rees}_R(I)$

If you are lucky: ($\varphi^{-1}(I)$ of linear type in A)

then $q\mathrm{Sym}_{A \to R}(I) = \mathrm{Sym}_R\, I$

<u>Always:</u> $\quad \mathrm{Sym}_R(I) \twoheadrightarrow q\mathrm{Sym}_{A\to R}(I) \twoheadrightarrow \mathrm{Rees}_R(I)$

Functoriality: if

$$\begin{tikzcd} A \arrow[d] \arrow[dr, two heads] & \\ B \arrow[r, two heads] & R \end{tikzcd}$$

then

$$q\mathrm{Sym}_{A\to R}(I) \twoheadrightarrow q\mathrm{Sym}_{B\to R}(I)$$

<u>Definition:</u> $\quad q\mathrm{Sym}_R(I) = \varprojlim_{A\to R} q\mathrm{Sym}_{A\to R}(I)$

<u>Fact:</u> (for finite-type algebras over a field):
If A is regular and $A \twoheadrightarrow R$, then

$$q\mathrm{Sym}_R(I) = q\mathrm{Sym}_{A\to R}(I).$$

Geometry:

$$\begin{array}{lcll} R & \rightsquigarrow & X & \text{scheme} \\ I & \rightsquigarrow & Y \subseteq X & \text{closed subscheme} \\ A \twoheadrightarrow R & \rightsquigarrow & X \hookrightarrow M & \text{embedding; } M \text{ nonsingular} \end{array}$$

Take Proj of the algebras:

$$\begin{array}{ccccc} & & & & Bl_Y M \\ & & & \subseteq & \\ Bl_Y X & \subseteq & qBl_Y X & & \\ & & \shortparallel & & \\ & & \mathrm{Proj}(q\mathrm{Sym}_R(I)) & \subseteq & qBl_Y X \end{array}$$

qBl: "most economical way to fill this diagram"

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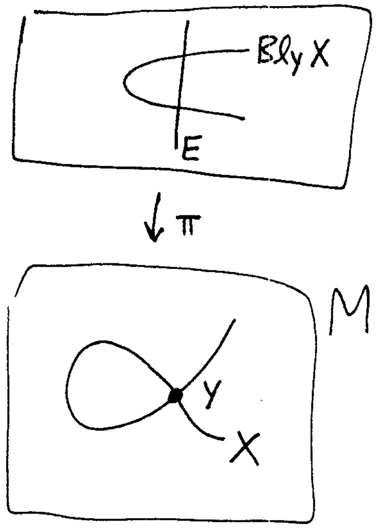
Example:



$Bl_Y M$

$$\pi^{-1} X = \tilde{X} + 2E$$

$$q\, Bl_Y X = \tilde{X} + E$$

$$Bl_Y X = \tilde{X}$$

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② CSM classes

X = variety over $\mathbb{C}$ (or a scheme)

$F(X) = \{\text{cycles on } X\}$ abelian group

$= \{\text{constructible functions } X \to \mathbb{Z}\}$

$$= \left\{ \sum_{V \subseteq X} m_V \mathbb{1}_V \;\middle|\; \begin{array}{l} V \subset X \text{ closed subvariety} \\ m_V \in \mathbb{Z} \\ \mathbb{1}_V = 1 \text{ on } V,\ 0 \text{ outside } V \end{array} \right\}$$

F is a covariant functor for proper morphisms:
if $f: X \to Y$ is proper, then $f_*: F(X) \to F(Y)$
is defined by $\mathbb{1}_V \mapsto f_*(\mathbb{1}_V)$,

$$f_*(\mathbb{1}_V)(p) = \chi_{top}(f^{-1}(p) \cap V)$$

for $V \subset X$, $p \in Y$

Fact: $f_* g_* = (f \circ g)_*$.

Another functor from varieties to abelian groups:

$X \rightsquigarrow AX$ (Chow group)

<u>Question</u> (Grothendieck): Is there is a natural transformation $F \underset{c_*}{\rightsquigarrow} A$?

<u>example:</u>

$$\begin{array}{ccccc} X & & F(X) & \longrightarrow & A(X) \\ \downarrow & & \downarrow & & \downarrow \\ \text{point} & & F(\text{point}) = \mathbb{Z} & = & A(\text{point}) \end{array}$$

$F(X) \ni \mathbb{1}_X \longrightarrow \chi_{top}(X)$

$A(X) \ni c_*(\mathbb{1}_X)$ take degree

So if X is nonsingular, the natural transformation c_* ought to have

$$c_*(\mathbb{1}_X) = c(TX) \cap [X]$$

Chern class of tangent bundle

<u>MacPherson</u> (1974) Annals: $\exists!$ such natural transformation

<u>Definition:</u> $c_{SM}(X) := c_*(\mathbb{1}_X)$ for any X.

(Chern-Schwartz-Macpherson class of X)

Different approaches to this theory (of computation of CSM classes):

Sabbah – Brylinski – Kashiwara – ....

Kennedy

$$\underset{\text{nonsing}}{X \subseteq M} \quad : \text{ construct } \underset{\text{characteristic cycle}}{Ch(X)} \subseteq \mathbb{P}(T^*M)$$

If X is nonsingular, $N_X^* M \subseteq T^*M|_X$

$$\rightsquigarrow \underset{\substack{\| \\ Ch(X)}}{\mathbb{P}(N_X^* M)} \subseteq \mathbb{P}(T^*M)$$

$$\begin{array}{c} Ch(X) \subseteq \mathbb{P}(T^*M) \\ \rightsquigarrow \quad \text{"Shadow"} \\ S_M(X) \end{array}$$

<u>Theorem:</u> (—, others) If $X \subseteq M$ is a hypersurface, M nonsingular, then

$$Ch(X) = [\, q \, Bl_Y X \,]$$

where Y = singularity sub<u>scheme</u> of X (defined by Jacobian ideal).

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③ <u>Inclusion - exclusion</u>

<u>Exercise:</u> $c_{SM}(X \cap Y) = c_{SM}(X) + c_{SM}(Y) - c_{SM}(X \cup Y)$

so we can compute $c_{SM}\big(\underbrace{V(I = (f_1, \dots, f_r)}_{V} \subseteq \mathbb{C}[\underline{x}]\big)\big) \quad \forall I.$

Aluffi ⑥ since $V = \bigcap X_i$, $X_i = V(f_i)$

Whitney: supports of components of $Ch(X)$