HYPERPLANE ARRANGEMENTS + FREE RESOLUTIONS

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TEXAS A&M

ALEX SUCIU
NORTHEASTERN

1. COMBINATORICS + TOPOLOGY, H^*, π_1, LCS
2. RESONANCE VARIETIES, LIN. SYZYGIES, GRAPHICS
3. BERNSTEIN - GELFAND - GELFAND CORR.
4. SPECIAL BONUS !!

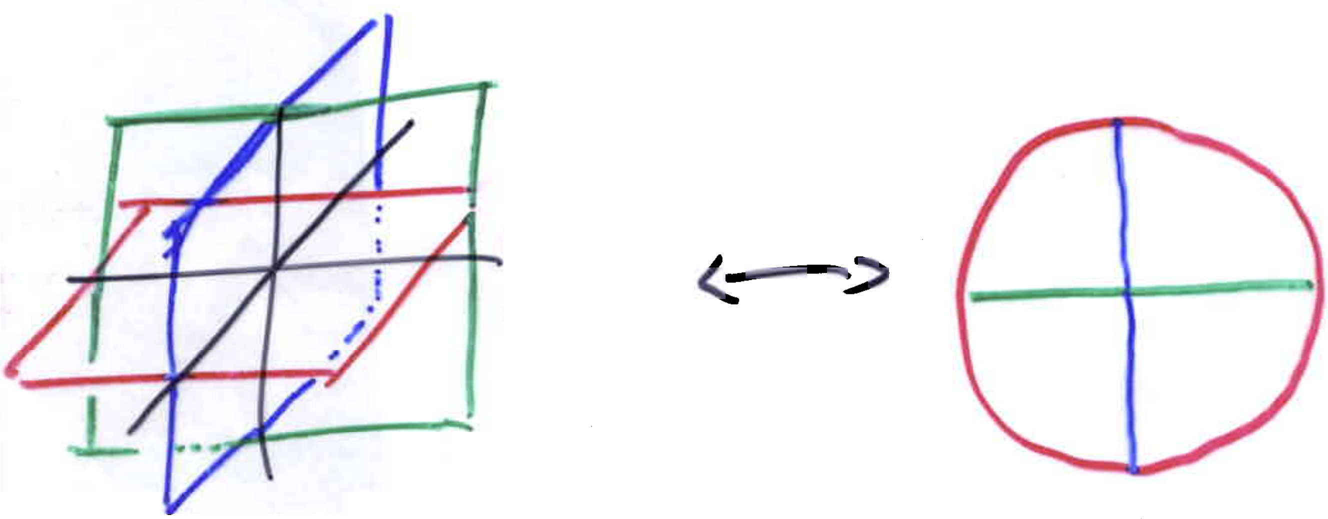
SETTING: V V.S. $/ k$ (THINK $\mathbb{C}$ OR $\mathbb{R}$)

A HYPERPLANE ARRANGEMENT $\mathcal{A} = \bigcup_{i=1}^{d} H_i$

IS A FINITE COLLECTION OF HYPERPLANES IN V.

- $\mathcal{A}$ IS <u>CENTRAL</u> IF ALL H_i PASS THRU 0
- $\mathcal{A}$ IS <u>ESSENTIAL</u> if $\bigcap_{i}^{d} H_i = 0$

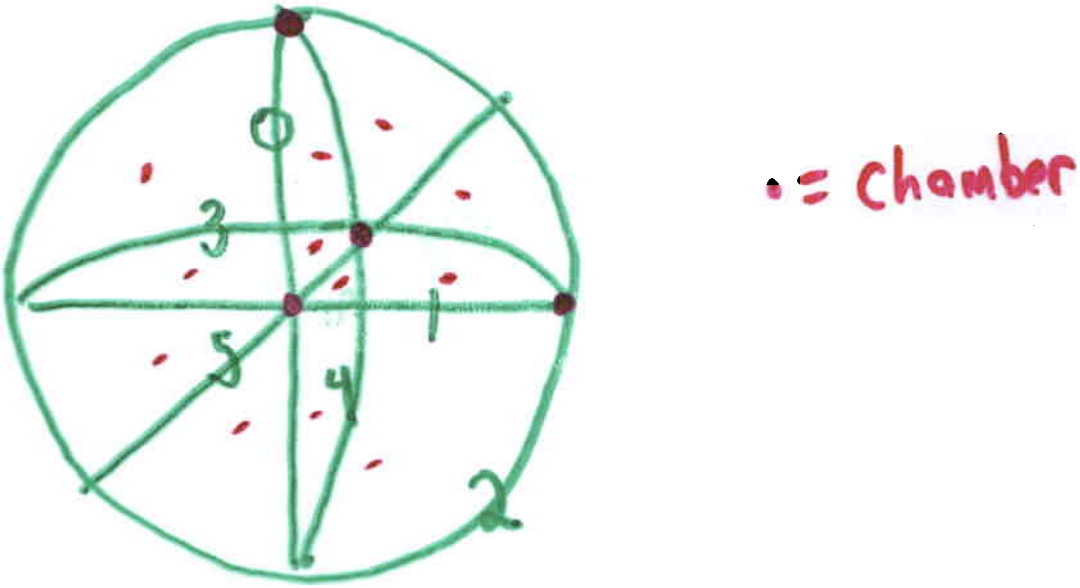
WLOG ABOVE HOLD. WE'LL DRAW PROJECTIVE PICTURES

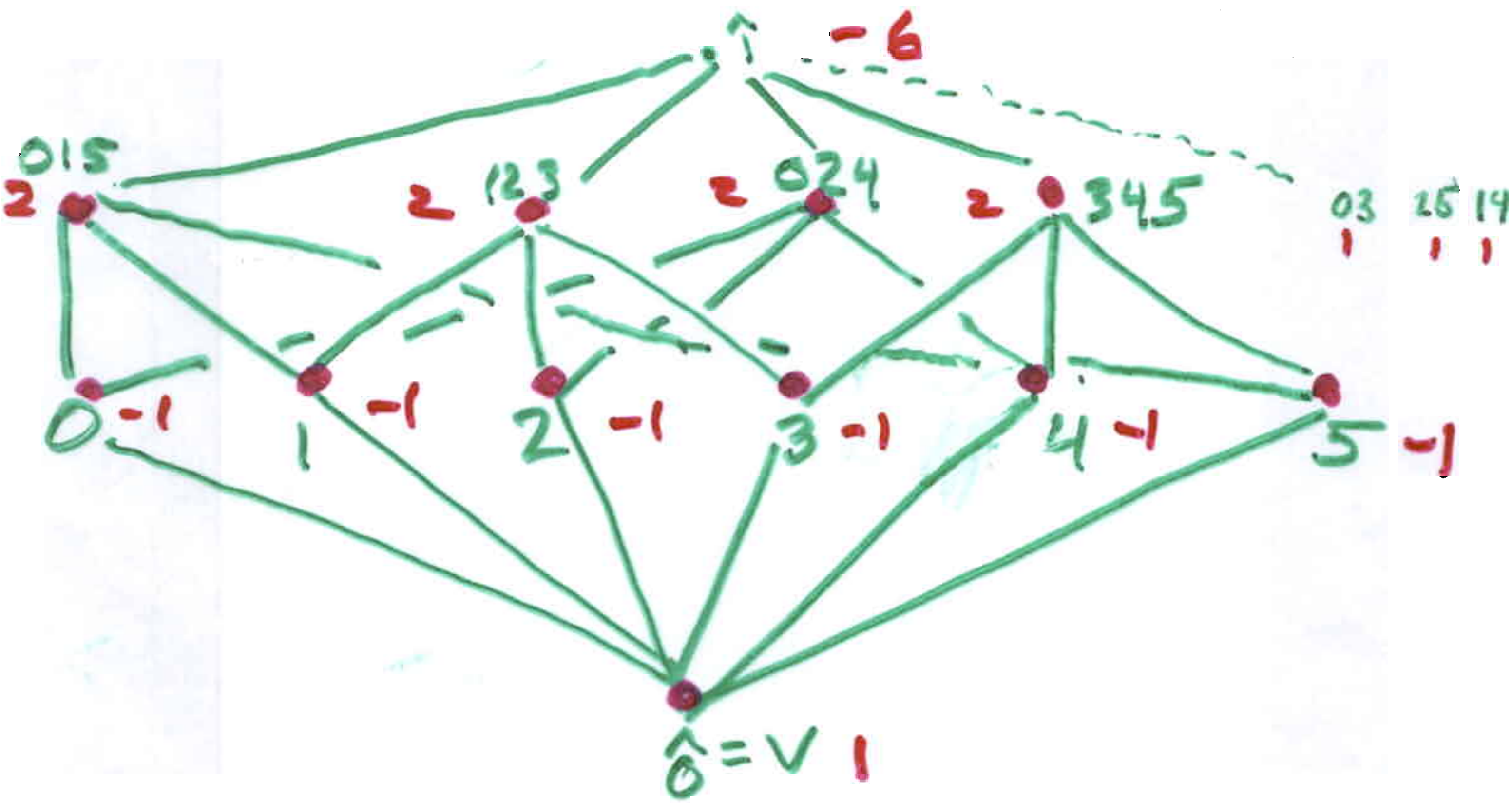


$x=0$ $y=0$ $z=0$

COMBINATORICS: ASSOCIATED TO $\mathcal{A}$ IS THE INTERSECTION LATTICE $L(\mathcal{A})$

$\mathcal{A} = V(xyz(y-z)(x-z)(x-y))$





THE MÖBIUS FN: $\mu(\hat{0}) = 1$

$$\mu(x) = -\sum_{y<x} \mu(y)$$

THE POINCARÉ POLYNOMIAL

$$\pi(A,t) = \sum_{x \in L(A)} \mu(x)(-t)^{rk(x)}$$

FOR THE EXAMPLE

$$1 + 6t + 11t^2 + 6t^3$$

<u>TOPOLOGY</u>: $k = \mathbb{R}$, then $\pi(A,1) = |\text{chamb } A|$

(MORE, ZASLAVSKY, BAMS ~75)

$k = \mathbb{C}$, then $\pi(A,t) = \sum H^i(A^c, \mathbb{C}) t^i$

(BRIESKORN, BOURBAKI, ~80)

THE COHOMOLOGY RING OF A^c:

$$\cong \Lambda(\mathbb{C}^d) \big/ \partial DS$$

ORLIK-SOLOMON
INV. '85

WHERE $\{H_1 \dots H_k\}$ IS A DEPENDENT SET

IF CODIM $\bigcap_i^k H_i < k$

$\partial(123) = 12 - 13 + 23$
$(1-2)(2-3)$

FOR THE EXAMPLE, $DS = \{123, 024, 015, 345, \dots\}$

CHECK ON BRIESKORN:

	0	1	2	3	
dim $(\Lambda\, \mathbb{C}^6)$	1	6	15	20	....
dim $\partial(DS)$	.	.	4	.	..
dim H^i	1	6	11	.	....

MOTIVATION:

"THE TOPOLOGY OF THE COMPLEMENT OF A SET OF LINES IN $\mathbb{P}^2$ IS VERY INTERESTING, THE INVESTIGATION OF THE FUNDAMENTAL GROUP VERY DIFFICULT"

HIRZEBRUCH, '83

- FOR AN ARRANGEMENT COMPLEMENT X, $\pi_1(X)$ IS <u>NOT</u> COMBINATORIAL (RYBNIKOV AG.98...)

- BACK OFF: FOR ANY GROUP G, TAKE CHAIN OF NORMAL S.G.

$$\begin{array}{ccccccc} G & > & [G,G] & > & [G,G_1] & > & [G,G_2] \cdots \\ \| & & \| & & \| & & \| \\ G_0 & & G_1 & & G_2 & & G_3 \end{array}$$

$L(G) = \bigoplus G_k / G_{k+1} \otimes \mathbb{Q}$ IS A <u>GRADED</u> <u>LIE</u> ALGEBRA

ϕ_K = RANK OF K^{TH} GRADED PIECE.

FOR THE EXAMPLE (BRAID)

K	0	1	2	3	4	5
ϕ_K	1	6	4	10	21	54

ENCODE THE ϕ_K AS A POWER SERIES

$$\prod_{1}^{\infty} (1-t^K)^{-\phi_K} = 1 + 6t + 25t^2 + 90t^3 + 301t^4 \ldots$$

$$\pi(A,-t) = 1 - 6t + 11t^2 - 6t^3$$

MULTIPLYING WE GET: $1 + 0t + 0t^2 + 0t^3 + \ldots$

$= 1$!

THIS IS THE "LOWER CENTRAL SERIES" (LCS)

FORMULA
$$\prod_{1}^{\infty}(1-t^k)^{-\phi_k}\cdot \pi(A,-t)=1$$

- KOHNO (INV 85) FOR BRAID ARRANGEMENTS
- FALK · RANDELL (INV 85) FOR "FIBER TYPE"

TERAO = "SUPERSOLVABLE"

(COMPLEMENT IS FORMAL, LHS = HILB SERIES OF $U(g)$, P.B.W).

RECENT ('97 JLMS) REFORMULATION BY SHELTON - YUZVINSKY:

"LCS IS A CONSEQUENCE OF KOSZUL DUALITY"

- $A = T(V)/I$ IS QUADRATIC IF $I \subseteq V^{\otimes 2}$. PUT $A^! = T(V^*)/I^\perp$

- AN ALGEBRA A (GRADED, $\subseteq T(V)$) IS KOSZUL IF THE FREE RESOLUTION OF K OVER A IS LINEAR.

- KOSZUL DUALITY : A KOSZUL, THEN $HS(A^!, t) \cdot HS(A, -t) = 1$.

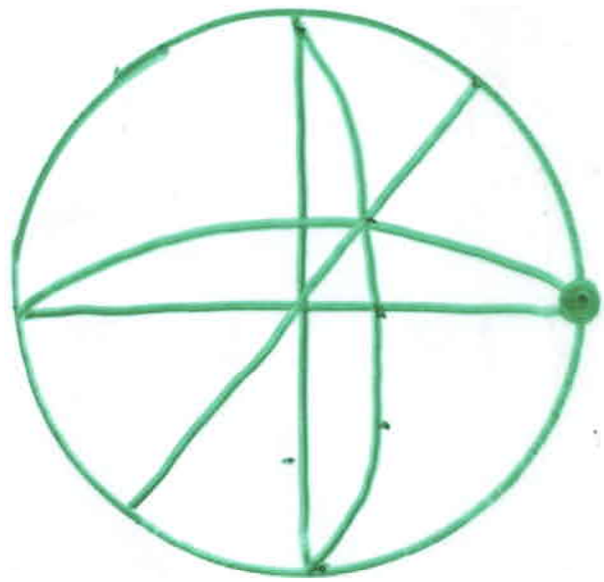
(PRIDDY, LOFWALL, BACKELIN, BGS)

TOY EXAMPLE 3 POINTS IN $\mathbb{P}^2$; MORE POINTS (ASK ALDO!).

POINT: $\prod (1-t^k)^{-\phi_k} = \sum \dim_k \operatorname{Tor}_i^A(k,k)_i$

Braid arrangement

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$



{1,2,3}, {0,2,4}, {0,1,5}, {3,4,5},.....

Resolution of k over OS

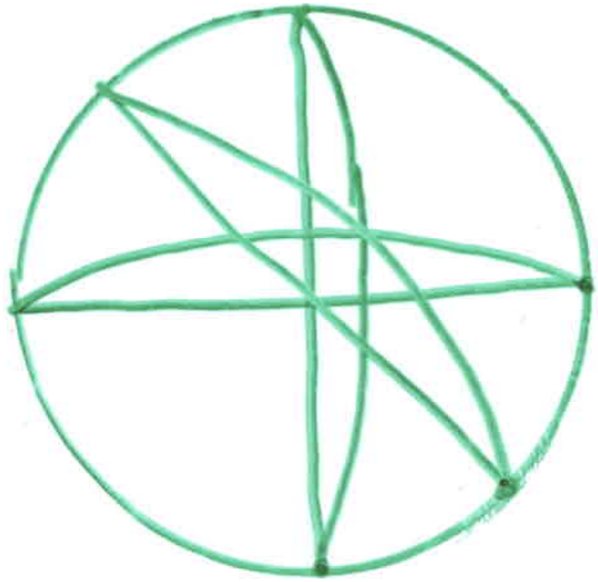
```
total: 1 6 25 90 301 966 3025 9330
    0: 1 6 25 90 301 966 3025 9330
```

Resolution of OS over E

```
total: 1 4 10 21 45 91 170 295
    0: 1 . .  .  .  .   .   .
    1: . 4 10 15 20 25  30  35
    2: . . .  6  25 66 140 260
```

BIG ? #1: DOES KOSZUL $\longrightarrow$ S.S.

Kohno x2 arrangement



```
| 1 0 0  |
| 0 1 0  |
| 0 0 1  |
| 0 1 -1 |
| 1 0 -1 |
| 1 1 0  |
| 1 1 -2 |
```

{{1,2,3}, {0,2,4}, {0,1,5}, {3,4,6}, {2,5,6} ....

Resolution of k over OS

```
total: 1 7 33 134 521 2039 8132
    0: 1 7 33 129 450 1452 4424
    1: . .  .   5  71  587 3683
    2: . .  .   .   .    .   25
```

Resolution of OS over E

```
total: 1 5 25 91 255 595 1220 2275
    0: 1 .  .  .   .   .    .    .
    1: . 5 10 15  20  25   30   35
    2: . . 15 76 235 570 1190 2240
```

$$\prod_{1}^{\infty}(1-t^k)^{-\phi_k} = \sum_{i\geq 0} \dim_{\mathbb{C}} \mathrm{Tor}_i^A(\mathbb{C},\mathbb{C})\, t^i$$

PEEVA (TAMS '02) SHOWS THAT <u>IN GENERAL</u> NO ANALOG OF LCS CAN HOLD. SO HOW CAN WE "GET AT" THE ϕ_k ?

OUR (—, SUCIU, TAMS '02) IDEA: USE Δ RINGS TO RELATE THE RES $\mathbb{C}/A$ TO THE RES A/E. IN GENERAL, LATTER IS MUCH SMALLER, STUDIED IN EISENBUD - POPESCU - YUZVINSKY AG...

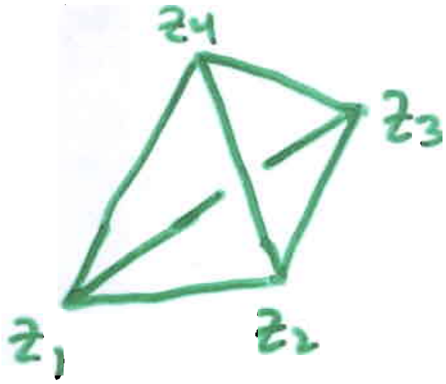
FALK, TAMS '85 $\quad \phi_3 = \dim_{\mathbb{C}} \mathrm{Tor}_2^E(A,\mathbb{C})_3$

—, SUCIU $\quad \phi_4 = \binom{\phi_2}{2} + \dim_{\mathbb{C}} \mathrm{Tor}_3^E(A,\mathbb{C})_4$

$-\delta_4$ = Quadratic Koszul.

11.

GRAPHIC ARRANGEMENTS



$\rightsquigarrow$ $z_i - z_j$ if edge.

(OUR EXAMPLE, THOUGH IT DOES NOT SEEM TO BE, IS OF THIS TYPE).

(—, SUCIU) KOSZUL $\longleftrightarrow$ S.S.

(GRAPHICS) ?

<u>RESONANCE</u>: LET $\lambda = \sum a_i e_i$

$$(*) \quad 0 \longrightarrow H^0(X,\mathbb{C}) \xrightarrow{\wedge\lambda} H^1(X,\mathbb{C}) \xrightarrow{\wedge\lambda} H^2(X,\mathbb{C}) \longrightarrow$$

FOR GENERIC λ, EXACT. FOR SPECIALIZATIONS, GET "RESONANCE". (FALK, AC, '97)

$$R^1(A) \text{ "="} H^1(*) \neq 0.$$

OBSERVE: ELEMENTS OF $R^1(A)$ CORRESPOND TO LINEAR SYZYGIES OF A/E.

COHEN - SUCIU PROVED $R^1(A)$ IS A SUBSPACE ARRANGEMENT; LIBGOBER - YUZVINSKY SHOWED IT CONSISTS OF <u>DISJOINT</u> (PROJECTIVE) SUBSPACES OF DIM ≥ 1.

(EASY TO SEE THE RANK 2 ELEMENTS CORRESPOND TO ELEMENTS OF $R^1(A)$ — THEY STUPIDLY DECOMPOSE

$$\partial(ijk) = ij - ik + jk = (i-j)(i-k)\Big)$$

(—, SUCIU) FOR GRAPHICS, $R^1(A) \longleftrightarrow$ K3 + K4 SUBGRAPHS

(SUFFICES TO DESCRIBE LINEAR STRAND FOR GRAPHICS).

PRIZE!

WE CONJECTURE:

$$\prod_{1}^{\infty} (1-t^k)^{\phi_k} = \prod_{j=1}^{\ell-1} (1-jt)^{\sum_{s=j}^{\ell-1} (-1)^{s-j} \binom{s}{j} k_s}$$

WHERE ℓ = # VERTS, k_s = CLIQUE #S

- CAN'T ALWAYS GET ϕ_k, SO BACK OFF EVEN MORE: SET $G' = [\pi_1, \pi_1]$ and $G'' = [G', G']$. THE <u>CHEN RANKS</u> θ_k ARE THE LCS RANKS FOR G/G'', THE MAXIMAL METABLIAN QUOTIENT.

<u>THM</u> (COHEN-SUCIU, TAMS '99)

THE CHEN RANKS "ARE" THE HILBERT SERIES OF A F'GEND, GRADED S-MODULE.

BGG BASICS

BERNSTEIN · GELFAND · GELFAND CORR IS AN ISO

D^b OF BOUNDED CPX OF COHERENT SHEAVES ON $\mathbb{P}(V^*)$

$+$

D^b OF BOUNDED CPX OF F'GEND, GRADED $\Lambda^{\cdot} V$ MODS.

REFS: DECKER - EISENBUD M2 BOOK
EIS - FLOYSTADT - SCHREYER AG...

CAN EXTRACT FUNCTORS

$\mathbb{R}$: F'GEN'D, GRADED $S(V^*)$ MODS
$\longrightarrow$ LIN, FREE $E = \Lambda^{\cdot} V$ CPX.

$\mathbb{L}$: " E MODS
$\longrightarrow$ " S CPX.

$\mathbb{R}$ + $\mathbb{L}$ ARE EASY TO DESCRIBE

P - GRADED E - MOD

$$\mathbb{L}(P) \qquad \begin{array}{ccc} p & \longrightarrow & \sum x_i \otimes e_i p \\ P_i & \longrightarrow & P_{i+1} \\ \otimes & & \otimes \\ S & & S \end{array}$$

$\mathbb{R}(M)$ ALMOST SAME (TWIST)

EXAMPLE: $E = \Lambda^{\cdot} k^3$

$$E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3$$

BLACKBOARD

$$M = k[x_0, x_1] / x_0^2, x_0 x_1$$

BLACKBOARD

BLACK BOARD

BGG 1

$E = \Lambda k^3$

$$S \otimes E_0 \longrightarrow S \otimes E_1 \longrightarrow S \otimes E_2 \longrightarrow S \otimes E_3$$

E_1: e_0, e_1, e_2

E_2: $e_0 \wedge e_1$, $e_0 \wedge e_2$, $e_1 \wedge e_2$

E_3: $e_0 \wedge e_1 \wedge e_2$

$1 \to \sum x_i e_i \cdot p$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$e_0 \longrightarrow x_1 e_1 e_0 + x_2 e_2 e_0$

$e_1 \longrightarrow x_0 e_0 e_1 + x_2 e_2 e_1$

$e_2 \longrightarrow x_0 e_0 e_2 + x_1 e_1 e_2$

$01 \to x_2 e_2\, e_0 e_1$

~~$12 \to x_0 e_0\, e_1 e_2$~~

$02 \to x_1 e_1\, e_0 e_2$

$12 \to x_0 e_0\, e_1 e_2$

$\mathbb{L}(P)$

$$\begin{bmatrix} -x_1 & x_0 & 0 \\ -x_2 & 0 & x_0 \\ 0 & -x_2 & x_1 \end{bmatrix} \qquad \begin{bmatrix} x_2 & -x_1 & x_0 \end{bmatrix}$$

The Koszul complex !!

$M = k[x_0, x_1] / x_0^2, x_1 x_0$ $\qquad m \otimes 1 \to \sum x_j m \otimes e_j$

$M =$ 1 $\quad x_0, x_1$ $\quad x_1^2$ $\quad x_1^3 \ldots$

$$E \to E^2 \to E \to E \to$$

$\begin{bmatrix} e_0 \\ e_1 \end{bmatrix}$ $\quad [0, e_1]$ $\quad [e_1] \ldots$

$x_0 \to x_0 x_0 \otimes e_0 + x_1 x_0 \otimes e_1$

$x_1 \to x_0 x_1 \otimes e_0 + x_1 x_1 \otimes e_1$

NOTICE: WHEN IS $\mathbb{R}(M)$ exact? At position 2.

REG $(M) = 1.$

THM

THM (EIS - FLØYSTADT - SCHREYER)

$\mathbf{R}(M)_{\geq i}$ IS EXACT $\forall\, i \geq S$

IFF $S > \operatorname{reg} M$.

THM (EIS - SCHREYER)

$\mathbf{L}(P)$ IS A FREE RES OF M

$\longleftrightarrow$ $\mathbf{R}(M)$ IS A INJ. RES OF P.

THM (EIS - POPESCU - YUZVINSKY)

FOR $P = A$ THE OS ALG,

$$0 \longrightarrow A_0 \otimes S \longrightarrow A_1 \otimes S \xrightarrow{d_1} \cdots \longrightarrow A_n \otimes S \longrightarrow F(A) \longrightarrow 0$$

IS A FREE RES OF $F(A)$.

NOTICE: THIS IS EXACTLY THE COMPLEX USED TO COMPUTE RESONANCE !!

THE MODULE M WHICH APPEARED IN THE THM. OF COHEN-SUCIU ON THE CHEN RANKS IS PRECISELY

$$M = S \otimes A^1 / \operatorname{im} d_1^t.$$

THUS, WE GET

$$0 \longrightarrow Ext^2(F(A), S) \to M \to m \to 0$$

A THEOREM OF EISENBUD - FLOYSTADT - SCHREYER GIVES THE PIECES OF THE RES OF A/E IN TERMS OF LOCAL COHOMOLOGY OF F(A):

Example 1 The braid arrangement

```
i2 : EPY K4
o2 = coker | a_3+a_4+a_5 -a_1-a_2     -a_1-a_2 ...
           | -a_3-a_4     a_1+a_2+a_5 -a_3-a_4 ...
           | 0            a_1         a_1      ...
           | 0            0           0        ...
           | a_4          -a_2        0        ...
           | a_3          0           a_3      ...

i4 : scan(6, i->print hilbertFunction(i-3,Ext^3(FA,S1)))
1 0 0 0  0  0
i5 : scan(6, i->print hilbertFunction(i-3,Ext^2(FA,S1)))
0 0 4 10 15 20
i6 : scan(6, i->print hilbertFunction(i-3,Ext^1(FA,S1)))
0 0 0 0  0  6

i7 : matAEij(K4,5,5)
Resolution of OS over E
total: 1 4 10 21 45 91
    0: 1 .  .  .  .  .
    1: . 4 10 15 20 25
    2: . .  .  6 25 66
```

PONCHLINE: "KNOWING"

M MEANS KNOWING $\mathrm{Ext}^2(FA, S)$

" " $\mathrm{Tor}_i^E(A,k)_{i+1}$

COHEN·SUCIU CONJECTURE A FORMULA FOR THE CHEN RANKS. BGG, EFS AND OUR WORK ON GRAPHICS LETS US PROVE IT (FOR FREE!) IN THIS CASE. IN FACT, (MSRI, FEB 03) WE RECENTLY PROVED

$$\Theta_k \geq \sum h_r \Theta_k(F_r)$$

h_r = # res comp of (affine) dim r

REFS (ALL ON AG)

SURVEYS

A. SUCIU (CONT. MATH)

S. YUZVINSKY (RMS)

RESONANCE

COHEN · SUCIU (TAMS)

FALK (ANN COMP

LIBGOBER-YUZVINSKY (COMP)

BGG

DECKER - EISENBUD

EISENBUD - SCHREYER

EISEN. - POPESCU - YUZVINSKY

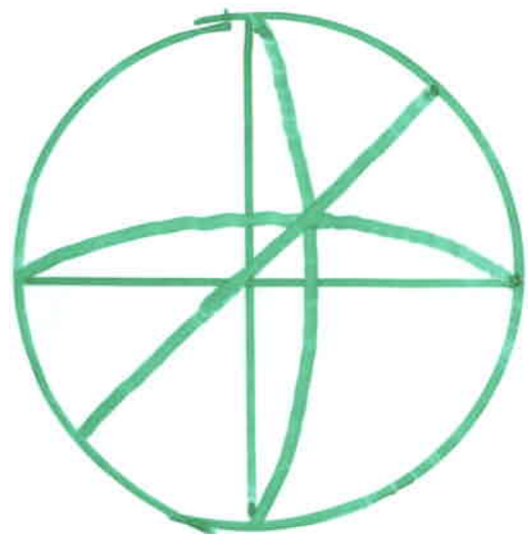
S - SUCIU (see)

ORLIK - TERAO (THE BOOK)

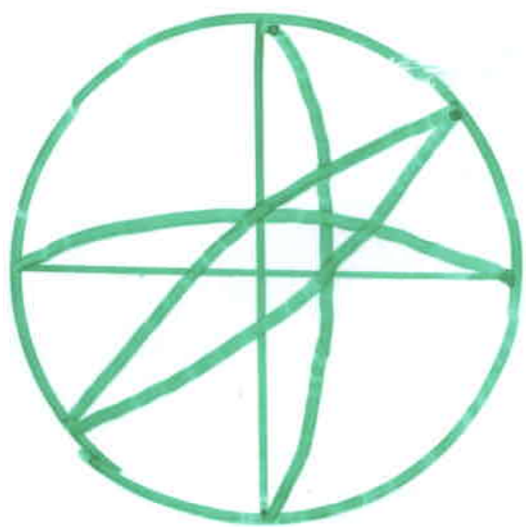
$Q = \Pi(l_i)$, J = Jacobian Q

Example 2 Terao's conjecture

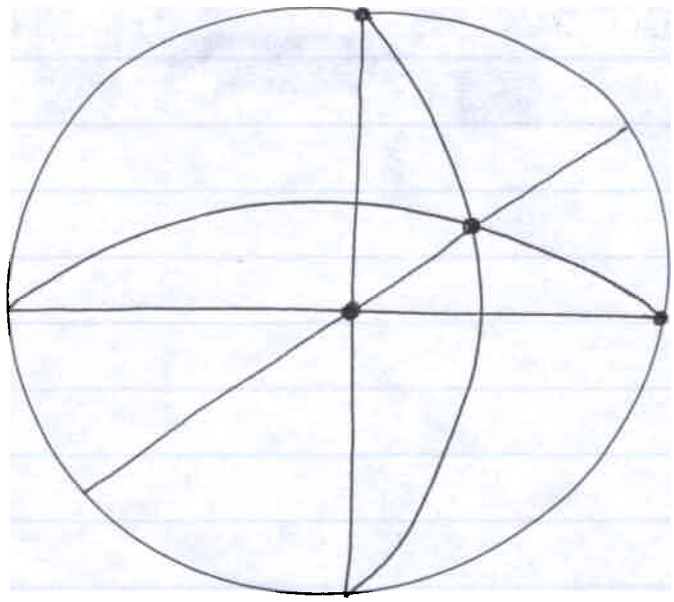
```
i2 : freeinfo braid
res of R/J
o2 = total: 1 3 2
        -6: 1 . .
        -5: . . .
        -4: . . .
        -3: . . .
        -2: . 3 .
        -1: . . 1
         0: . . 1
```



```
i3 : freeinfo X2
res of R/J
o3 = total: 1 3 3 1
        -7: 1 . . .
        -6: . . . .
        -5: . . . .
        -4: . . . .
        -3: . . . .
        -2: . 3 . .
        -1: . . . .
         0: . . 1 .
         1: . . 2 1
```



- S-S,
- YUZVINSKY



1 6 25 90 301 966 3025

1

4 10 15 20 25

6 25 66

$$\begin{bmatrix} \Sigma & \cdot & \cdot & \cdot \\ \cdot & \Sigma & \cdot & \cdot \\ \cdot & \cdot & \Sigma & \cdot \\ \cdot & \cdot & \cdot & \Sigma \\ 1+2+3 & \cdot & \cdot & \cdot \\ \cdot & 4+5+6 & \cdot & \cdot \\ \cdot & \cdot & 2+4+6 & \cdot \\ \cdot & \cdot & \cdot & 3+4+5 \\ 4 & -4 & \cdot & \cdot \\ 5 & \cdot & +5 & \cdot \\ 6 & \cdot & \cdot & -6 \\ \cdot & 3 & 3 & \cdot \\ \cdot & 2 & \cdot & -2 \\ \cdot & \cdot & 1 & 1 \end{bmatrix}$$

$\Sigma = x_1 + x_2 + \cdots + x_6$

R= ZZ/101 [$x_1 \ldots x_{10}$,
Skew Commutative
=> true]

Schenck
1

$E_0 = \text{span}(1)$
$E_1 = \text{span}(e_1, e_2, e_3)$
$E_2 =$

$E = \Lambda(k^3)$

$$\begin{array}{ccccccc} S & & S & & S & & S \\ \otimes & & \otimes & & \otimes & & \otimes \\ E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 \end{array}$$

$$1 \longmapsto \sum_{i=1}^{3} x_i e_i$$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} x_2 & -x_1 & x_0 \end{bmatrix}$$

$$\begin{aligned} e_0 &\longmapsto -x_1 e_1 e_0 - x_2 e_2 e_0 \\ e_1 &\longmapsto x_0 e_0 e_1 + x_2 e_2 e_1 \\ e_2 &\longmapsto x_0 e_0 e_2 + x_1 e_1 e_2 \end{aligned}$$

$$\begin{bmatrix} -x_1 & x_0 & 0 \\ -x_2 & 0 & x_0 \\ 0 & -x_2 & x_1 \end{bmatrix}$$

= Koszul complex.

$k[x_0, x_1] / (x_0^2, x_1 x_0)$

$$\begin{array}{ccccc} 1 & \begin{matrix} x_0 \\ x_1 \end{matrix} & x_1^2 & x_1^3 & x_1^4 \end{array}$$

exact ... exact ....

$$\overset{*}{E} \to \overset{*}{E^2} \longrightarrow E \longrightarrow E \longrightarrow E \longrightarrow \cdots$$

$$\begin{bmatrix} e_0 \\ e_1 \end{bmatrix} \quad \begin{bmatrix} 0 & e_1 \end{bmatrix} \quad \begin{bmatrix} e_1 \end{bmatrix} \quad \begin{bmatrix} e_1 \end{bmatrix}$$

*: not exact here

$\text{Reg}(M) = 1$

Schenck
2