Problems in Cohomology Calculations

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Let G be a finite group and k a field of characteristic $p > 0$. Look at $\mathrm{H}^*(G, k)$ or $\mathrm{Ext}^*(M, N)$ for M and N kG-modules.

Objectives:

Compute some complicated examples.
Test some conjectures.
Discover truth.

Suppose that G is a finite p-group and that $k = \mathbb{F}_p$. Let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$$

be a minimal projective resolution of k. So then

$$\mathrm{Hom}_{kG}(P_n, k) \cong \mathrm{H}^n(G, k) \cong \mathrm{Ext}^*_{kG}(k.k).$$

A cohomology element $\zeta \in \mathrm{H}^n(G, k)$ is represented by a chain map

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & & \cdots \\
 & & \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_0} & \searrow^{\zeta} & & & \\
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0.
\end{array}$$

Cup product is composition of chain maps.

Example

Suppose that $p = 2$ and G is the group of order 128 generated by $g_1, \ldots, g_7$ with relations

$$\begin{array}{lll} g_1^2 = g_5 g_6 g_7, & g_2^2 = g_4, & g_3^2 = g_6 g_7, \\ g_5^2 = g_7, & g_2^{g_1} = g_2 g_3, & g_3^{g_2} = g_3 g_5, \\ g_4^{g_1} = g_4 g_5 g_6 g_7, & g_4^{g_3} = g_4 g_6 g_7, & g_5^{g_2} = g_5 g_6, \\ g_5^{g_4} = g_5 g_7, & g_6^{g_2} = g_6 g_7. & \end{array}$$

Then $\mathrm{H}^*(G, k) \cong k[z, y, x, w, v, u, t, s]/\mathcal{I}$ where the degrees of the variables are $1, 1, 2, 2, 2, 3, 3, 4$ and the ideal $\mathcal{I}$ is generated by the elements

$$zy, \quad y^2, \quad z^3 + yx + zv, \quad zx + yx, \quad yw, \quad z^2w + zt,$$
$$x^2 + xv, \quad yu, \quad yt, \quad yxv + xu + xt, \quad z^2u + xu + vu,$$
$$xw^2 + zwu + ut, \quad w^2v + t^2, \quad zvt + z^2s + u^2 + ut.$$

The Sylow 2-subgroup of Higman-Sims

The Higman-Sims group has

$$44352000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$$

elements . Its Sylow subgroup has order 512. It has the structure

$$4^3 : D_8$$

It's cohomology ring is —

$$H^*(sy2, \mathbb{F}_2) \cong \mathbb{F}_2[z, y, x, w, v, u, t, s, r, q, p, n, m, k, j, i, h]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by the following relations. Note that there are 79 relations generating $\mathcal{I}$ and that the given relations are minimal in the sense that no collection of fewer than 79 elements will generate the ideal.

$yx,$
$zx,$
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$yq + wv,$
$yr + xr + xq + xp + ut,$
$yr + wt,$
$ys + yq,$
$y^2u + yq + xq + wu,$
$zp + yq + xq + v^2 + vu,$
$zr,$
$zs + zq + xq,$
$tq,$
$ts,$
$vr,$
$vs + vq,$
$ws + wq,$
$x^2r + xn + wp + vp + up,$
$x^2r + xn + wr + ur,$
$yu^2 + yn + wp,$
$yvu + ws,$
$y^2r + wr,$
$z^2p + zvu + zn + ws + vs + us,$
$z^3u + z^2p + zvu + zu^2 + zn + zm + x^2r + xu^2 + xn + xm,$
$rq,$
$sp + qp,$
$sq + q^2,$
$sr,$
$s^2 + sq,$
$xuq + vu^2 + vn + sp,$
$ytr + x^3p + x^2n + xup + xtp + r^2,$
$ytr + x^3p + xuq + xtr + xk + r^2,$

$yur + x^3p + xuq + xk + tn + rp,$

$y^2n + yvp + yuq + wn,$

$z^3p+z^2n+zvq+zuq+y^2m+ytr+ytp+yk+x^3p+xk+vn+vm+tn+tm+p^2,$

$z^3p + zvq + zup + yuq + xuq + vu^2 + sq,$

$z^3q + z^3p + zuq + zk,$

$sn + qn,$

$ytn + xtm + xp^2 + rn,$

$yvn + yqp + vuq + vup + qn,$

$y^5v+y^4q+y^4p+y^3n+y^3m+y^2vq+y^2tp+y^2k+yum+ytn+yr^2+yrp+yj+wk,$

$zvm + zq^2 + zj + yvm + yum + yr^2 + yrp + x^3u^2 + x^3m + vup + vk + u^2q + u^2p + uk + sm + rn + rm + pn,$

$z^2vq + zvm + zq^2 + zqp + zj + yvn + yqp + vuq + u^2s,$

$z^3u^2 + zu^3 + zum + yvn + yqp + x^3u^2 + x^3n + xu^3 + xum + xtn + xj + vuq + u^2s,$

$z^5u+z^3m+zu^3+zum+x^3n+x^3m+xu^3+xum+xtn+xj+wk+vk+u^2s+u^2q+uk,$

$y^4m + y^3k + y^2vn + y^2vm + y^2p^2 + yt^2p + ytk + yrm + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xu^2p + xtk + xrm + xpn + tr^2 + tj + rk,$

$y^5p + y^3k + y^2vn + y^2vm + yrm + ypm + yi + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xt^2p + xqm + wun + wum + wj + tr^2 + sk + qk,$

$zpn + ypn + xpn + wun + vun + vp^2 + uqp + n^2,$

$z^2q^2 + zuk + zqn + zqm + y^4m + y^3vq + y^3tp + y^3k + y^2vm + y^2tm + y^2p^2 + yqm + ypn + x^2un + xuk + xrm + wun + wj + vum + vp^2 + vj + up^2 + qk,$

$z^3k + z^2vm + z^2q^2 + zi + y^3tp + y^2tm + yuk + yt^2p + ytk + yqm + x^3k + x^2u^3 + x^2un + x^2tn + xt^2p + xi + wum + vun + vum + urp + up^2 + tr^2 + tj + sk + rk + qk,$

$z^3k + z^2vm + zqn + zpn + zi + y^5q + y^4n + y^2q^2 + yqm + x^2u^3 + xu^2p + xuk + xrm + wj + vq^2 + vp^2 + u^2n + uq^2 + uj + sk,$

$z^4u^2 + z^2u^3 + z^2un + z^2um + z^2q^2 + zqn + y^5q + y^4n + y^2q^2 + yqm + wun + vun + vq^2 + vp^2 + vj + uq^2 + sk,$

$z^5p + z^3k + z^2un + z^2j + zqn + zi + y^3vq + y^2vn + yuk + yrm + ypn + x^2u^3 + xuk + xqm + wum + vp^2 + u^2n + urp + uj + sk,$

$y^5n + y^3q^2 + y^2vk + y^2qm + yvqp + yup^2 + yuj + yt^2n + yrk + yqk + ynm + wuk + wqn + wpn + wi + trm + sj + r^2p + rp^2 + qj,$

$z^2pn + zuq^2 + zuj + zn^2 + znm + y^6p + y^5m + y^3vm + y^3tm + y^2qn + y^2qm + y^2i + yvqp + yvj + yt^2n + yt^2m + ytj + yrk + ypk + ynm + wqn + wpm + wi + vpm + u^3q + u^2k + urn + trn + q^3 + pj + nk,$

$z^2qn + zuj + zn^2 + znm + yup^2 + yt^2n + yrk + ynm + xpk + wpn + wpm + vuk + vqm + u^3q + u^2k + upn + trm + sj + rp^2 + rj + qj,$

$z^3un + z^2qn + z^2pn + zvj + zuq^2 + zuj + znm + sj + qj,$

$y^2tk + y^2qm + yup^2 + yt^2m + ytp^2 + ytj + yqk + ynm + x^3tn + xrk + xn^2 + wuk + wqn + wpm + t^3r + trn + tpm + ti + sj + r^2p + rp^2 + qj,$

$z^3vn + z^3un + z^2vk + z^2pn + zvj + y^5n + y^4vq + y^3vm + y^2qn + y^2pn + yvq^2 + yvqp + yup^2 + yn^2 + xnm + wuk + vqm + vpm + vi + u^3q + u^3p + u^2k + upn + q^3,$

$z^3vn + z^2qn + z^2pn + zvj + y^5n + y^2vk + y^2tk + y^2qn + yvqp + yup^2 + yt^2n + yt^2m + ytp^2 + ytj + yrk + yn^2 + x^3tn + wpn + wpm + wi + vuk + vqm + urn + t^3r + trm + tpm + ti + sj + r^2p,$

$y^3t^2p + y^2t^2m + yt^3p + yt^2k + yr^2p + yrp^2 + x^2u^2n + x^2t^2n + xrj + t^3n + t^2rp +$

$t^2j + trk + r^2n + r^2m + rpm + ri,$

$z^2uq^2 + z^2qk + zpj + y^5vq + y^4j + y^3vk + y^3t^2p + y^3qn + y^3qm + y^3pm + y^3i + y^2vq^2 + y^2vj + y^2t^2m + y^2tp^2 + y^2tj + y^2pk + y^2h + yvpn + yupn + yt^3p + yt^2k + yr^2p + yqp^2 + yqj + x^2t^2n + xu^3p + xtpn + xrj + xpj + wqk + wpk + wnm + vup^2 + upk + t^3n + t^2p^2 + t^2j + trk + r^2n + r^2m + rpn + rpm + q^2n + p^2n + pi + k^2,$

$zuqn + y^7q + y^4q^2 + y^3vk + y^3qm + y^2vq^2 + y^2vj + yvqm + yvpn + yvi + yupn + yqj + x^2u^2n + x^2nm + xu^3p + xpj + wqk + wpk + vuj + upk + un^2 + tn^2 + r^2n + q^2n + qpn + qpm + nj,$

$z^3uk + zuqm + zupm + zmk + y^3vk + y^3qn + y^3pm + y^2tp^2 + y^2qk + yvpn + yvi + yupn + yupm + yr^2p + yq^2p + yqj + ypj + x^2u^2n + xpj + wpk + vup^2 + upk + t^2rp + rpn + qpm + p^2n + p^2m + pi,$

$z^3uk + z^2u^2n + z^2uj + z^2qk + zupm + zui + zpj + y^3qn + y^3qm + y^3pn + y^2vq^2 + y^2nm + yq^2p + x^3uk + x^2u^4 + x^2nm + xu^3p + xui + xrj + wpk + wnm + vuj + usk + upk + un^2 + tn^2 + si + r^2n + qpn + nj,$

$z^4vm + z^4j + z^3vk + zuqn + zuqm + zupm + zpj + y^7q + y^4q^2 + y^3vk + y^3qn + y^3pn + y^2vj + y^2nm + yvpn + yvi + yq^2p + yqj + x^2u^2n + x^2nm + vup^2 + u^3n + usk + un^2 + unm + qpn + qi,$

$z^2u^2k + z^2uqm + z^2q^3 + z^2qj + zvqk + zu^3n + zu^2j + zupk + zun^2 + y^6vq + y^4qm + y^4pn + y^3tp^2 + y^2qj + yvpk + yvn^2 + yvnm + yupk + yq^2n + yqpm + yqi + x^2u^2k + x^2nk + xunm + xtpk + xtnm + xri + xnj + wupm + wpj + wnk + vq^2p + vpj + urp^2 + unk + t^2rn + t^2pm + trp^2 + tp^3 + tpj + rpk + qpk + qnm + pnm + ni,$

$z^2vqm + z^2u^2k + z^2uqm + z^2q^3 + zvqk + zu^3n + zu^2j + zun^2 + y^9v + y^6vq + y^6k + y^5vm + y^5tm + y^5q^2 + y^4qn + y^4qm + y^3tp^2 + y^3tj + y^3pk + y^3nm + y^3m^2 + y^3h + y^2vi + y^2qj + y^2nk + yvpk + yvnm + yvm^2 + yvh + yupk + yuh + ytm^2 + x^2mk + xunm + xtnm + xqi + wupm + wpj + wnk + wmk + vmk + urj + uqp^2 + upj + unk + t^2pm + tp^3 + r^2k + rn^2 + qpk + qn^2 + qnm + ni + kj,$

$z^2q^2n + zvqj + zvmk + zupj + zqn^2 + zni + y^8m + y^7vq + y^7tp + y^7k + y^6tm + y^5vk + y^5qn + y^4pk + y^4nm + y^3p^3 + y^2vnm + y^2vh + yvq^3 + yvqj + yvnk + yvmk + yup^3 + yumk + ytrj + ytnk + ytmk + yrpk + yqm^2 + yp^2k + ypnm + yni + ykj + x^2mj + xqm^2 + xkj + wum^2 + wq^2m + wpi + vqpn + vp^2m + u^2qk + u^2pk + u^2n^2 + urpm + uri + uq^2m + up^2n + t^2nm + t^2m^2 + tr^2n + tp^2m + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pnk + j^2,$

$z^5pm + z^2q^2n + z^2qi + zvqj + zupj + zpm^2 + zkj + y^8m + y^7vq + y^7k + y^6vm + y^5tk + y^5pm + y^4pk + y^3vqn + y^3t^3p + y^3ti + y^2vnm + y^2vh + y^2th + y^2q^2m + y^2k^2 + yvqj + yvnk + yumk + yt^4p + yt^3k + ytnk + ytmk + yrpk + yrn^2 + yq^2k + yqn^2 + yp^2k + ypnm + yni + ykj + x^8u^2 + x^8m + x^7k + x^6u^3 + x^5tk + x^5i + x^4tj + x^2t^2j + x^2mj + xt^2pn + xni + wun^2 + wum^2 + wq^2m + wmj + vum^2 + vqpn + vnj + vmj + u^2qk + uri + uqi + up^2n + upi + uk^2 + t^4n + t^3rp + t^3p^2 + t^3j + t^2nm + t^2m^2 + trpm + tk^2 + r^3p + r^2p^2 + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pmk + ki + j^2,$

$z^{11}u + z^9u^2 + z^9m + z^7vn + z^6vk + z^6pm + z^3vnm + z^3un^2 + z^3nj + z^2unk + z^2pm^2 + z^2mi + z^2kj + zumj + y^7p^2 + y^6vk + y^4pj + y^4nk + y^3qi + y^3pi + y^2vnk + y^2tmk + y^2q^2k + y^2qnm + y^2qh + y^2p^2k + yvq^2m + yvqpm + yvqi + yvk^2 + yup^2n + yup^2m + yumj + yuk^2 + ytri + yr^2p^2 + yr^2j + yq^4 + yq^2j + yqpj + yn^2m + x^6uk + x^5u^2m + x^4ui + x^4mk + x^3nj + xu^4n + xu^2nm + xupi + xumj + xtnj + xpnk + wunk + wq^2k + wmi + vupj + vumk + vqnm + u^5q + u^4k + u^2p^3 + u^2nk + usm^2 + uq^2k + uqn^2 + t^3rn + trnm + tpnm + tpm^2 + q^3n + qp^2n + qnj + p^3m + pnj + nmk + ji,$

$z^{11}p + z^5vi + z^5u^2k + z^4pi + z^3umk + z^2qmk + z^2j^2 + zvqn^2 + zuqnm + zumi + zukj + zq^3n + zm^2k + y^10n + y^9tp + y^8q^2 + y^8j + y^7tk + y^7qn + y^7i + y^6vj + y^6tj + y^6nm + y^6h + y^5vi + y^5ti + y^4vn^2 + y^4vm^2 + y^4mj + y^4k^2 + y^3vqj + y^3tmk + y^3p^2k + y^3mi + y^2vqi + y^2vnj + y^2t^2m^2 + y^2q^2j + y^2n^2m + y^2mh + y^2ki + yvq^2k + yvqm^2 + yvqh + yvpnm + yvmi + yupn^2 + yuph + yuni + yt^3pm + yt^2pj + ytp^2k + ytmi + yr^2pm + yq^3n + yp^2i + yn^2k + x^8u^3 + x^7uk + x^6u^2m + x^6uj + x^4um^2 + x^3mi + x^3kj + x^2u^4n + x^2umj + x^2j^2 + xu^2mk + xrk^2 + xji + wunj + wqnk + wnm^2 + wnh + vumj + vq^2p^2 + vqmk + vn^2m + vj^2 + u^2qi + u^2p^2m + u^2mj + urpj + urmk + uqnk + upmk + un^3 + unm^2 + uki + t^3nm + t^2r^2m + t^2rpm + t^2ri + t^2p^2n + tr^2j + trp^3 + trnk + tp^4 + skj + r^3k + r^2pk + rpn^2 + rni + qmi + pni + mk^2 + i^2$

Detection

Let $\mathcal{H}$ be a collection of subgroups of G. We say that the cohomology of G is detected on $\mathcal{H}$ if the map

$$\prod_{H\in\mathcal{H}} \mathrm{res}_{G,H} : \quad \mathrm{H}^*(G,k) \longrightarrow \prod_{H\in\mathcal{H}} \mathrm{H}^*(H,k)$$

is injective.
In other words, the intersection of the kernels of the restriction maps to the elements of $\mathcal{H}$ is the zero ideal.

Theorem 1. *Suppose that the cohomology ring $H^*(G,k)$ has depth d. Then the cohomology is detected on the centralizers of the elementary abelian p-subgroups of rank d.*

The point is that the cohomology ring for the Sylow 2-subgroup of Higman-Sims has depth two.

Families of p-subgroups

Let $P = k[z, y, x, w, v, u, t, s]$ where the variables are in degrees $1, 1, 2, 2, 2, 3, 3, 4$. The for a particular family the cohomology ring has the form $\mathrm{H}^*(G, k) \cong P/\mathcal{I}$ where

a. For the group of order 256, $\mathcal{I} := \langle\ z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

b. For the group of order 128, $\mathcal{I} := \langle\ z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

c. For the group of order 64, $\mathcal{I} := \langle\ z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

d. For the group of order 32, $\mathcal{I} := \langle\ z^2,\ zy,\ y^3+yw,\ zx + yw,\ zw + yw,\ y^2x + yu,\ w^2 + yu,\ zu,\ zt,\ yx^2+xu+wu+wt,\ y^2t+wu,\ x^3+yxt+y^2s+t^2,\ x^2w + yxu + yxt + ut,\ u^2\rangle$.

Groups of order 64

From my web page

http://www.math.uga.edu/~jfc

Demonstration

Computing the cohomology rings of the simple modules for kG where $k = GF(2)$ and $G \cong M_{11}$, using the basic algebra.

Obsolete

Better ideas using noncommutative Groebner bases and localization orderings.

Currentely in MAGMA

Contents of Handbook chapter titled “Algebras”.

- Algebras
- Structure Constant Algebras
- Associative Algebras
- Quaternion Algebras
- Matrix Algebras
- Group Algebras
- Finitely Presented Algebras
- Lie Algebras
- Basic Algebras (in chapter called “Homological Algebra”)

Type	**optomized for**
Structure Constant Algebras	Multiplication
Matrix Algebras	One Representation
Basic Algebras	Representations, Homological
Finitely Presented Algebras	Gröbner Basis

Difficulties

- Can't communicate between types.
- Can't do homological algebra and representation theory over anything except basic algebras (and kG-modules which is NOT a type of algebra).
- Can't compute dimensions of Matrix algebras.
- Can't extract basic algebras.
- Can't define homomorphism of algebras except for FP algebras.
- Can't tell if an assignment of matrices to the generators of an algebra gives a module, except for FP algebras.

Problems

- Groebner bases and term orders.
- Generators and relations for other types of algebras.
- Construct multiplication tables for algebras in which multiplication is not fast.
- Speed issues.

$$H^*(sy2,\mathbb{F}_2) \cong \mathbb{F}_2[z,y,x,w,v,u,t,s,r,q,p,n,m,k,j,i,h]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by the following relations. Note that there are 79 relations generating $\mathcal{I}$ and that the given relations are minimal in the sense that no collection of fewer than 79 elements will generate the ideal.

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$yq + wv,$
$yr + xr + xq + xp + ut,$
$yr + wt,$
$ys + yq,$
$y^2u + yq + xq + wu,$
$zp + yq + xq + v^2 + vu,$
$zr,$
$zs + zq + xq,$
$tq,$
$ts,$
$vr,$
$vs + vq,$
$ws + wq,$
$x^2r + xn + wp + vp + up,$
$x^2r + xn + wr + ur,$
$yu^2 + yn + wp,$
$yvu + ws,$
$y^2r + wr,$
$z^2p + zvu + zn + ws + vs + us,$
$z^3u + z^2p + zvu + zu^2 + zn + zm + x^2r + xu^2 + xn + xm,$
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$ytr + x^3p + x^2n + xup + xtp + r^2,$
$ytr + x^3p + xuq + xtr + xk + r^2,$

$yur + x^3p + xuq + xk + tn + rp,$
$y^2n + yvp + yuq + wn,$
$z^3p + z^2n + zvq + zuq + y^2m + ytr + ytp + yk + x^3p + xk + vn + vm + tn + tm + p^2,$
$z^3p + zvq + zup + yuq + xuq + vu^2 + sq,$
$z^3q + z^3p + zuq + zk,$
$sn + qn,$
$ytn + xtm + xp^2 + rn,$
$yvn + yqp + vuq + vup + qn,$
$y^5v + y^4q + y^4p + y^3n + y^3m + y^2vq + y^2tp + y^2k + yum + ytn + yr^2 + yrp + yj + wk,$
$zvm + zq^2 + zj + yvm + yum + yr^2 + yrp + x^3u^2 + x^3m + vup + vk + u^2q + u^2p + uk + sm + rn + rm + pn,$
$z^2vq + zvm + zq^2 + zqp + zj + yvn + yqp + vuq + u^2s,$
$z^3u^2 + zu^3 + zum + yvn + yqp + x^3u^2 + x^3n + xu^3 + xum + xtn + xj + vuq + u^2s,$
$z^5u + z^3m + zu^3 + zum + x^3n + x^3m + xu^3 + xum + xtn + xj + wk + vk + u^2s + u^2q + uk,$
$y^4m + y^3k + y^2vn + y^2vm + y^2p^2 + yt^2p + ytk + yrm + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xu^2p + xtk + xrm + xpn + tr^2 + tj + rk,$
$y^5p + y^3k + y^2vn + y^2vm + yrm + ypm + yi + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xt^2p + xqm + wun + wum + wj + tr^2 + sk + qk,$
$zpn + ypn + xpn + wun + vun + vp^2 + uqp + n^2,$
$z^2q^2 + zuk + zqn + zqm + y^4m + y^3vq + y^3tp + y^3k + y^2vm + y^2tm + y^2p^2 + yqm + ypn + x^2un + xuk + xrm + wun + wj + vum + vp^2 + vj + up^2 + qk,$
$z^3k + z^2vm + z^2q^2 + zi + y^3tp + y^2tm + yuk + yt^2p + ytk + yqm + x^3k + x^2u^3 + x^2un + x^2tn + xt^2p + xi + wum + vun + vum + urp + up^2 + tr^2 + tj + sk + rk + qk,$
$z^3k + z^2vm + zqn + zpn + zi + y^5q + y^4n + y^2q^2 + yqm + x^2u^3 + xu^2p + xuk + xrm + wj + vq^2 + vp^2 + u^2n + uq^2 + uj + sk,$
$z^4u^2 + z^2u^3 + z^2un + z^2um + z^2q^2 + zqn + y^5q + y^4n + y^2q^2 + yqm + wun + vun + vq^2 + vp^2 + vj + uq^2 + sk,$
$z^5p + z^3k + z^2un + z^2j + zqn + zi + y^3vq + y^2vn + yuk + yrm + ypn + x^2u^3 + xuk + xqm + wum + vp^2 + u^2n + urp + uj + sk,$
$y^5n + y^3q^2 + y^2vk + y^2qm + yvqp + yup^2 + yuj + yt^2n + yrk + yqk + ynm + wuk + wqn + wpn + wi + trm + sj + r^2p + rp^2 + qj,$
$z^2pn + zuq^2 + zuj + zn^2 + znm + y^6p + y^5m + y^3vm + y^3tm + y^2qn + y^2qm + y^2i + yvqp + yvj + yt^2n + yt^2m + ytj + yrk + ypk + ynm + wqn + wpm + wi + vpm + u^3q + u^2k + urn + trn + q^3 + pj + nk,$
$z^2qn + zuj + zn^2 + znm + yup^2 + yt^2n + yrk + ynm + xpk + wpn + wpm + vuk + vqm + u^3q + u^2k + upn + trm + sj + rp^2 + rj + qj,$
$z^3un + z^2qn + z^2pn + zvj + zuq^2 + zuj + znm + sj + qj,$
$y^2tk + y^2qm + yup^2 + yt^2m + ytp^2 + ytj + yqk + ynm + x^3tn + xrk + xn^2 + wuk + wqn + wpm + t^3r + trn + tpm + ti + sj + r^2p + rp^2 + qj,$
$z^3vn + z^3un + z^2vk + z^2pn + zvj + y^5n + y^4vq + y^3vm + y^2qn + y^2pn + yvq^2 + yvqp + yup^2 + yn^2 + xnm + wuk + vqm + vpm + vi + u^3q + u^3p + u^2k + upn + q^3,$
$z^3vn + z^2qn + z^2pn + zvj + y^5n + y^2vk + y^2tk + y^2qn + yvqp + yup^2 + yt^2n + yt^2m + ytp^2 + ytj + yrk + yn^2 + x^3tn + wpn + wpm + wi + vuk + vqm + urn + t^3r + trm + tpm + ti + sj + r^2p,$
$y^3t^2p + y^2t^2m + yt^3p + yt^2k + yr^2p + yrp^2 + x^2u^2n + x^2t^2n + xrj + t^3n + t^2rp +$

$t^2j + trk + r^2n + r^2m + rpm + ri,$

$z^2uq^2 + z^2qk + zpj + y^5vq + y^4j + y^3vk + y^3t^2p + y^3qn + y^3qm + y^3pm + y^3i + y^2vq^2 + y^2vj + y^2t^2m + y^2tp^2 + y^2tj + y^2pk + y^2h + yvpn + yupn + yt^3p + yt^2k + yr^2p + yqp^2 + yqj + x^2t^2n + xu^3p + xtpn + xrj + xpj + wqk + wpk + wnm + vup^2 + upk + t^3n + t^2p^2 + t^2j + trk + r^2n + r^2m + rpn + rpm + q^2n + p^2n + pi + k^2,$

$zuqn + y^7q + y^4q^2 + y^3vk + y^3qm + y^2vq^2 + y^2vj + yvqm + yvpn + yvi + yupn + yqj + x^2u^2n + x^2nm + xu^3p + xpj + wqk + wpk + vuj + upk + un^2 + tn^2 + r^2n + q^2n + qpn + qpm + nj,$

$z^3uk + zuqm + zupm + zmk + y^3vk + y^3qn + y^3pm + y^2tp^2 + y^2qk + yvpn + yvi + yupn + yupm + yr^2p + yq^2p + yqj + ypj + x^2u^2n + xpj + wpk + vup^2 + upk + t^2rp + rpn + qpm + p^2n + p^2m + pi,$

$z^3uk + z^2u^2n + z^2uj + z^2qk + zupm + zui + zpj + y^3qn + y^3qm + y^3pn + y^2vq^2 + y^2nm + yq^2p + x^3uk + x^2u^4 + x^2nm + xu^3p + xui + xrj + wpk + wnm + vuj + usk + upk + un^2 + tn^2 + si + r^2n + qpn + nj,$

$z^4vm + z^4j + z^3vk + zuqn + zuqm + zupm + zpj + y^7q + y^4q^2 + y^3vk + y^3qn + y^3pn + y^2vj + y^2nm + yvpn + yvi + yq^2p + yqj + x^2u^2n + x^2nm + vup^2 + u^3n + usk + un^2 + unm + qpn + qi,$

$z^2u^2k + z^2uqm + z^2q^3 + z^2qj + zvqk + zu^3n + zu^2j + zupk + zun^2 + y^6vq + y^4qm + y^4pn + y^3tp^2 + y^2qj + yvpk + yvn^2 + yvnm + yupk + yq^2n + yqpm + yqi + x^2u^2k + x^2nk + xunm + xtpk + xtnm + xri + xnj + wupm + wpj + wnk + vq^2p + vpj + urp^2 + unk + t^2rn + t^2pm + trp^2 + tp^3 + tpj + rpk + qpk + qnm + pnm + ni,$

$z^2vqm + z^2u^2k + z^2uqm + z^2q^3 + zvqk + zu^3n + zu^2j + zun^2 + y^9v + y^6vq + y^6k + y^5vm + y^5tm + y^5q^2 + y^4qn + y^4qm + y^3tp^2 + y^3tj + y^3pk + y^3nm + y^3m^2 + y^3h + y^2vi + y^2qj + y^2nk + yvpk + yvnm + yvm^2 + yvh + yupk + yuh + ytm^2 + x^2mk + xunm + xtnm + xqi + wupm + wpj + wnk + wmk + vmk + urj + uqp^2 + upj + unk + t^2pm + tp^3 + r^2k + rn^2 + qpk + qn^2 + qnm + ni + kj,$

$z^2q^2n + zvqj + zvmk + zupj + zqn^2 + zni + y^8m + y^7vq + y^7tp + y^7k + y^6tm + y^5vk + y^5qn + y^4pk + y^4nm + y^3p^3 + y^2vnm + y^2vh + yvq^3 + yvqj + yvnk + yvmk + yup^3 + yumk + ytrj + ytnk + ytmk + yrpk + yqm^2 + yp^2k + ypnm + yni + ykj + x^2mj + xqm^2 + xkj + wum^2 + wq^2m + wpi + vqpn + vp^2m + u^2qk + u^2pk + u^2n^2 + urpm + uri + uq^2m + up^2n + t^2nm + t^2m^2 + tr^2n + tp^2m + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pnk + j^2,$

$z^5pm + z^2q^2n + z^2qi + zvqj + zupj + zpm^2 + zkj + y^8m + y^7vq + y^7k + y^6vm + y^5tk + y^5pm + y^4pk + y^3vqn + y^3t^3p + y^3ti + y^2vnm + y^2vh + y^2th + y^2q^2m + y^2k^2 + yvqj + yvnk + yumk + yt^4p + yt^3k + ytnk + ytmk + yrpk + yrn^2 + yq^2k + yqn^2 + yp^2k + ypnm + yni + ykj + x^8u^2 + x^8m + x^7k + x^6u^3 + x^5tk + x^5i + x^4tj + x^2t^2j + x^2mj + xt^2pn + xni + wun^2 + wum^2 + wq^2m + wmj + vum^2 + vqpn + vnj + vmj + u^2qk + uri + uqi + up^2n + upi + uk^2 + t^4n + t^3rp + t^3p^2 + t^3j + t^2nm + t^2m^2 + trpm + tk^2 + r^3p + r^2p^2 + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pmk + ki + j^2,$

$z^{11}u + z^9u^2 + z^9m + z^7vn + z^6vk + z^6pm + z^3vnm + z^3un^2 + z^3nj + z^2unk + z^2pm^2 + z^2mi + z^2kj + zumj + y^7p^2 + y^6vk + y^4pj + y^4nk + y^3qi + y^3pi + y^2vnk + y^2tmk + y^2q^2k + y^2qnm + y^2qh + y^2p^2k + yvq^2m + yvqpm + yvqi + yvk^2 + yup^2n + yup^2m + yumj + yuk^2 + ytri + yr^2p^2 + yr^2j + yq^4 + yq^2j + yqpj + yn^2m + x^6uk + x^5u^2m + x^4ui + x^4mk + x^3nj + xu^4n + xu^2nm + xupi + xumj + xtnj + xpnk + wunk + wq^2k + wmi + vupj + vumk + vqnm + u^5q + u^4k + u^2p^3 + u^2nk + usm^2 + uq^2k + uqn^2 + t^3rn + trnm + tpnm + tpm^2 + q^3n + qp^2n + qnj + p^3m + pnj + nmk + ji,$

$z^{11}p + z^5vi + z^5u^2k + z^4pi + z^3umk + z^2qmk + z^2j^2 + zvqn^2 + zuqnm + zumi + zukj + zq^3n + zm^2k + y^10n + y^9tp + y^8q^2 + y^8j + y^7tk + y^7qn + y^7i + y^6vj + y^6tj + y^6nm + y^6h + y^5vi + y^5ti + y^4vn^2 + y^4vm^2 + y^4mj + y^4k^2 + y^3vqj + y^3tmk + y^3p^2k + y^3mi + y^2vqi + y^2vnj + y^2t^2m^2 + y^2q^2j + y^2n^2m + y^2mh + y^2ki + yvq^2k + yvqm^2 + yvqh + yvpnm + yvmi + yupn^2 + yuph + yuni + yt^3pm + yt^2pj + ytp^2k + ytmi + yr^2pm + yq^3n + yp^2i + yn^2k + x^8u^3 + x^7uk + x^6u^2m + x^6uj + x^4um^2 + x^3mi + x^3kj + x^2u^4n + x^2umj + x^2j^2 + xu^2mk + xrk^2 + xji + wunj + wqnk + wnm^2 + wnh + vumj + vq^2p^2 + vqmk + vn^2m + vj^2 + u^2qi + u^2p^2m + u^2mj + urpj + urmk + uqnk + upmk + un^3 + unm^2 + uki + t^3nm + t^2r^2m + t^2rpm + t^2ri + t^2p^2n + tr^2j + trp^3 + trnk + tp^4 + skj + r^3k + r^2pk + rpn^2 + rni + qmi + pni + mk^2 + i^2$