The Method of Adjoints

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What are Adjoints?

Canonical models and canonical maps are useful in birational geometry (for classification, studying moduli, etc.)

Unfortunately, there are birational classes of algebraic varieties for which there are no canonical maps (those with negative Kodaira dimension), or for which the canonical maps do not give enough information.

For these classes, it is useful to study adjoint maps, depending not only on the birationality class but also on a particular projective embedding. These adjoint maps often give interesting birational equivalent embeddings or interesting fibrations, similar to canonical maps.

Definition (Low-Tech)

Let $X \subset \mathbb{P}^n$ be a projective hypersurface, given by a set of homogeneous polynomial $F(x_0, \ldots, x_n)$ of degree d.

For any $n, m \geq 0$, we define the linear space $V_{n,m}$ as the space of all homogeneous polynomials of degree $n+m(d-n-1)$ vanishing with multiplicity at least $m(r-s)$ at each singularity of X of multiplicity r and codimension s.

Let $v_{n,m} := \dim(V_{n,m})$, and suppose that $\{G_0, \ldots, G_{(v_{n,m}-1)}\}$ is a basis. Then the adjoint map is defined as

$$a_{n,m} : X \mapsto \mathbb{P}^{(v_{n,m}-1)}, p \mapsto (G_0(p) : \cdots : G_{(v_{n,m}-1)}(p))$$

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Definition (High-Tech)

Let $X \subset \mathbb{P}^n$ be projective variety.

Let $\pi : \tilde{X} \to X$ be a resolution of the singularities, i.e. π is a regular birational map and $\tilde{X}$ is projective and nonsingular. It is well-known that such a resolution exists, but it is not unique in general.

Recall that for any effective class of divisors D on $\tilde{X}$, we have an associated rational map

$$m_D : \tilde{X} \to \mathbb{P}^r$$

where $r := \dim(|D|)$.

Definition (High-Tech)

Let $H \in \mathrm{Div}(\tilde{X})$ be the pullback of a hyperplane section.
Let $K \in \mathrm{Div}(\tilde{X})$ be a canonical divisor.

For any $n, m \geq 0$, we define

$v_{n,m} := \dim(|nH + mK|) + 1$,

$a_{n,m} : X \to \mathbb{P}^{(v_{m,m}-1)}$ by the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ & {\scriptstyle nH+mK} \searrow \quad \swarrow {\scriptstyle a_{n,m}} & \\ & \mathbb{P}^{(v_{n,m}-1)} & \end{array}$$

($v_{n,m}$ must be positive, otherwise $a_{n,m}$ is not defined.)

Remark (Still High-Tech)

One can show that $v_{n,m}$ and $a_{n,m}$ are independent on the choice of the resolution.

$$
\begin{array}{ccccc}
\tilde{X}_1 & \xrightarrow{\pi_1} & X & \xleftarrow{\pi_2} & \tilde{X}_2 \\
 & \searrow^{nH_1+mK_1} & \downarrow ! & \swarrow^{nH_2+mK_2} & \\
 & & \mathbb{P}^{(v_{n,m}-1)} & &
\end{array}
$$

Tendencies (Still High-Tech)

- the birational adjoint maps tend to blow down exceptional divisors
- the birational adjoint maps tend to resolve most singularities, but usually leave canonical singularities intact
- if the fibers of an adjoint map are curves, then these tend to have low genus (zero or one)
- the birational adjoint maps tend to simplify the model

These observations make the theory of adjoints interesting to the "minimal model program" of Mori.

History / Presence

Adjoints have been widely used by geometers of the 19th century (Clebsch, Nöther, Cremona, Castelnuovo, Enriques).

They have been used in Lipman's proof of resolution of arithmetic surfaces (1978).

Current authors considering adjoints: Ein, Kawamata, Lazarsfeld, Sommese, Smith, S.

Adjoints appear in Eric Weisstein's encyclopedy of mathematics.

Rational Surfaces (Not)

Recall that a surface is called rational iff it is birationally equivalent to $\mathbb{P}^2$.

The best result would be the following.

Dream: For any rational surface X, there exist integers n, m such that $v_{n,m} = 3$ and $a_{n,m} : X \mapsto \mathbb{P}^r$ is birational.

Unfortunately, this dream does not come true. The obstruction is that there are many birational maps to $\mathbb{P}^2$, and there is no distinguished one.

More precisely, there are projective embeddings of rational surfaces, such that there is no birational map to $\mathbb{P}^2$ which is stable under the group of projective automorphisms of X.

Rational Surfaces

Theorem (S.): Let X be a rational surface. Then there is an integer l such that $a_{1,l}$ does not exist (i.e. $v_{1,l} = 0$).

Let m be the largest integer such that $a_{1,0}, \ldots, a_{1,m}$ exist. Then one of the following holds.

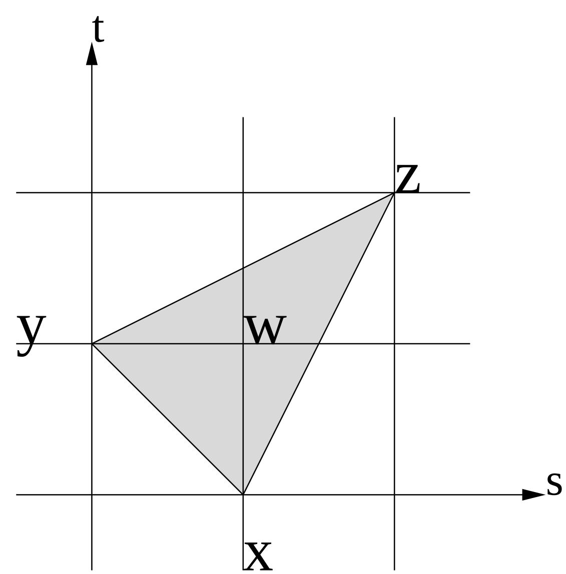
- $a_{1,m}$ is birational onto the image, which is one of the following.
  - $\mathbb{P}^2$ $(v_{1,m} = 3)$
  - a Veronese surface $(v_{1,m} = 6)$
  - a rational normal scroll

- $a_{1,m}$ maps to a rational normal curve, and gives a fibration by rational curves. Moreover, the fibers appear as conics in the model $a_{1,m-1}(X)$ or $a_{2,2m-1}(X)$.

- $a_{1,m}$ maps to a point $(v_{1,m} = 1)$.
  Then $a_{1,m-1}$, or one of its multiples $a_{2,2m-2}, a_{3m-3}$, is birational to a Del Pezzo surface.

Toric Surfaces

Toric surfaces are interesting examples of rational surfaces. We can construct projective embeddings corresponding to convex polygons.

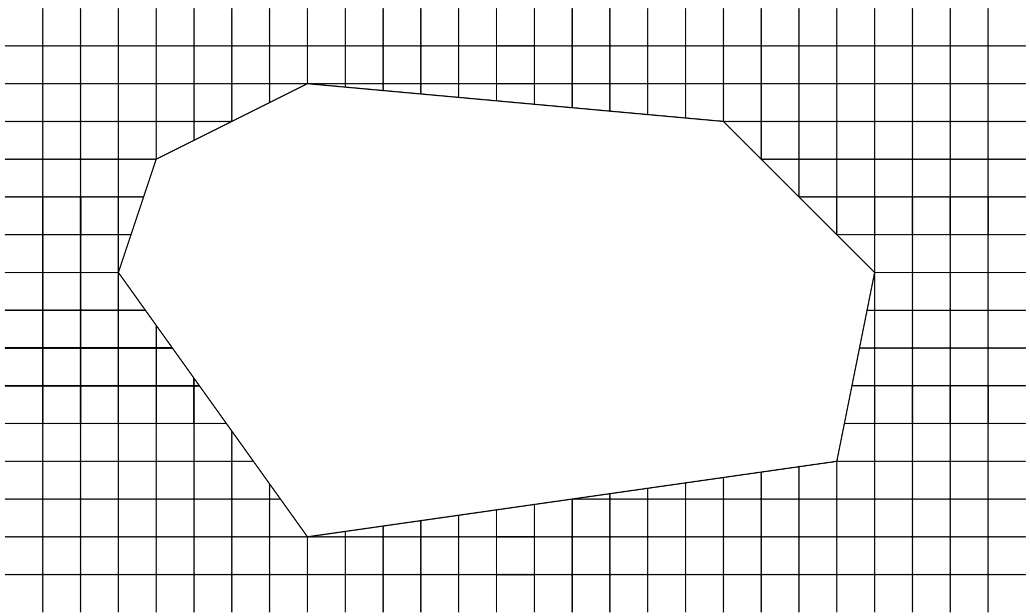


Here is the toric embedding given by the parametrization

$$(x : y : z : w) = (s : t : s^2t^2 : st)$$

For any convex lattice polygon Γ, let $\mathrm{CHI}(\Gamma)$ be the convex hull of the interior points of Γ.

The adjoint maps $a_{1,m}$ of the toric embedding corresponding to Γ are the toric maps corresponding to $\mathrm{CHI}^m(\Gamma)$.



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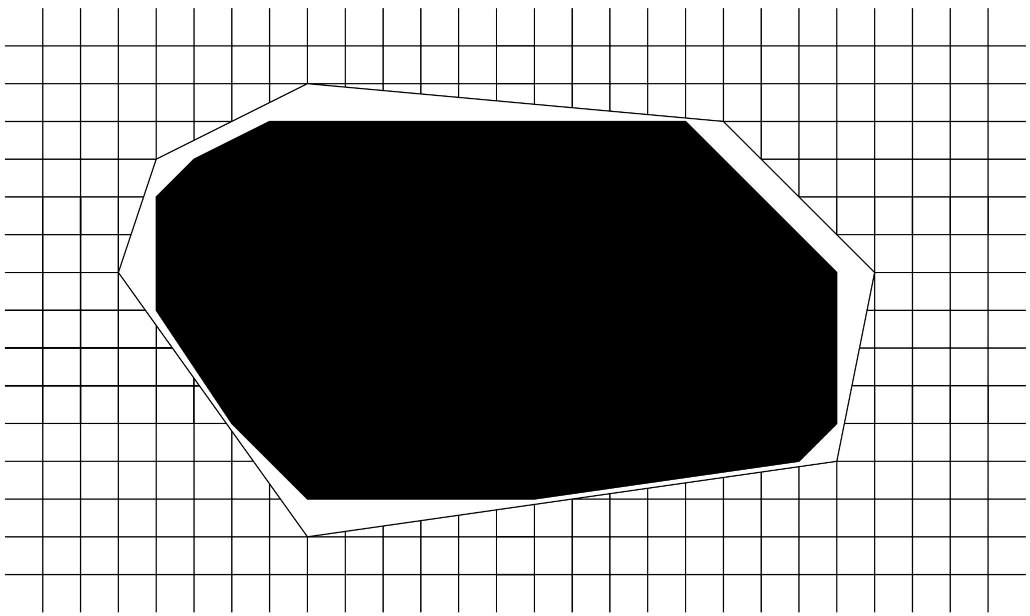

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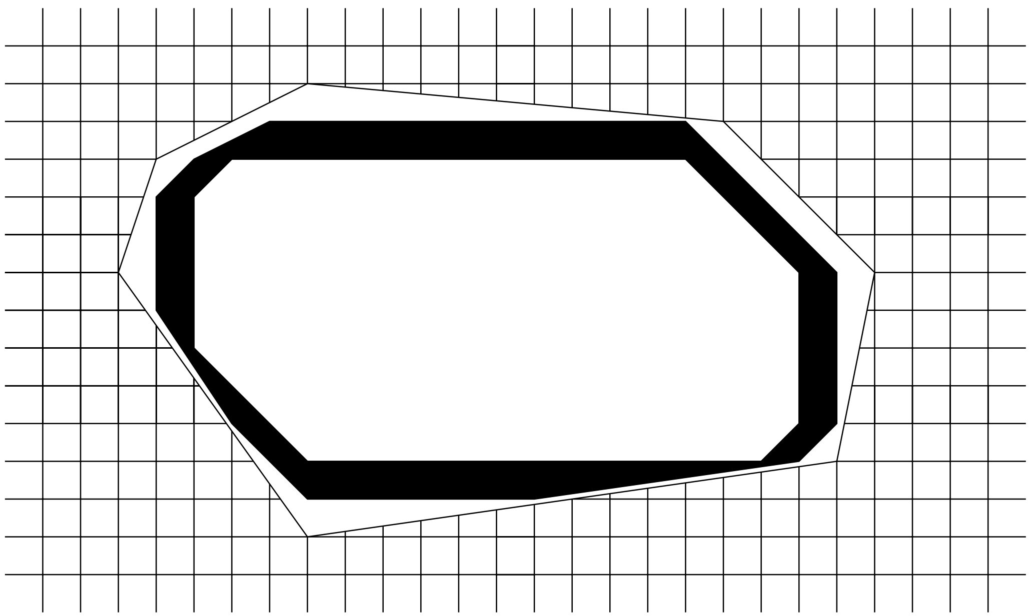

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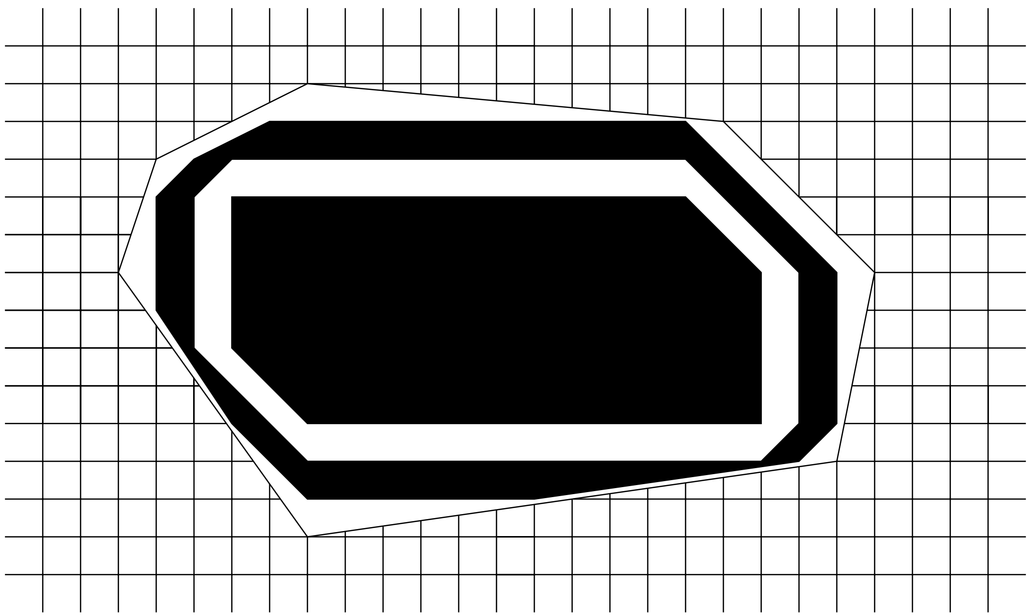

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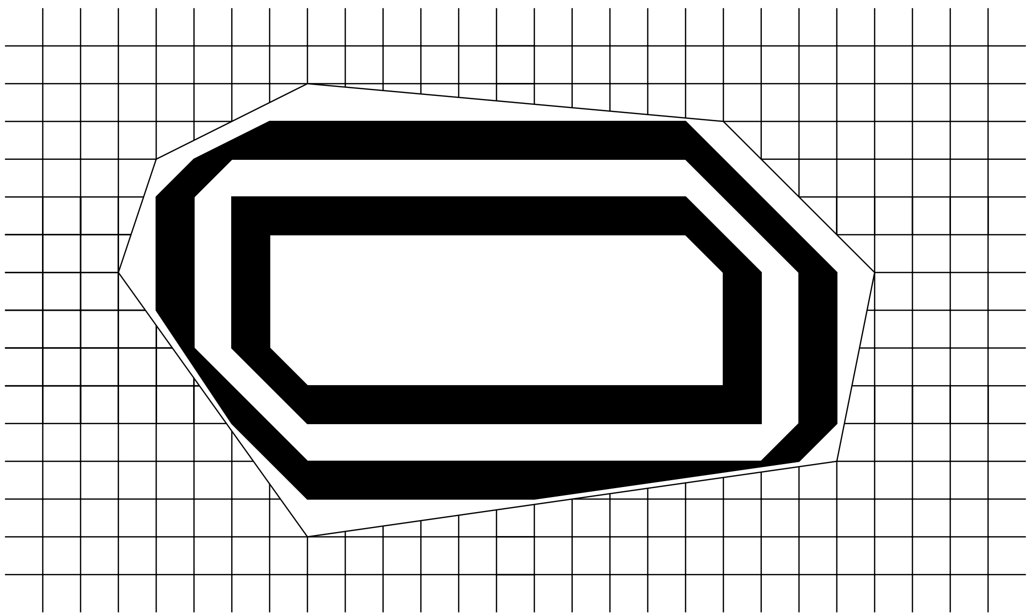

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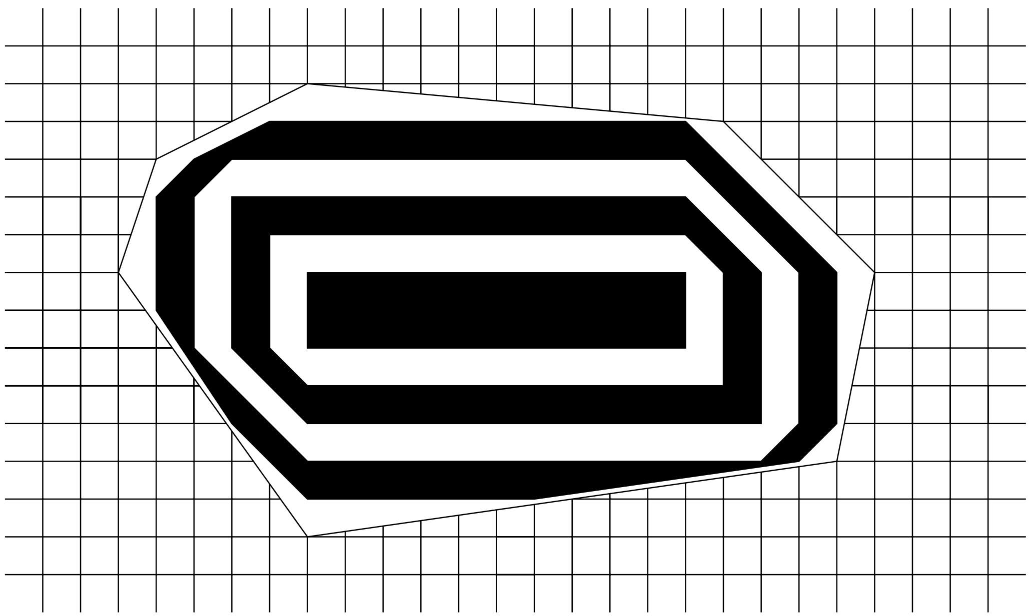

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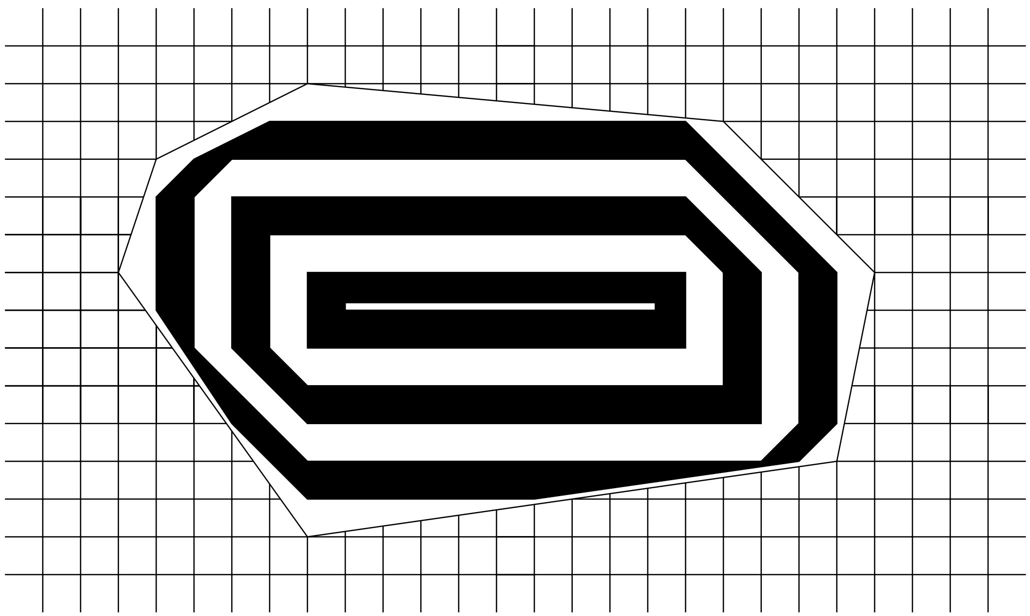
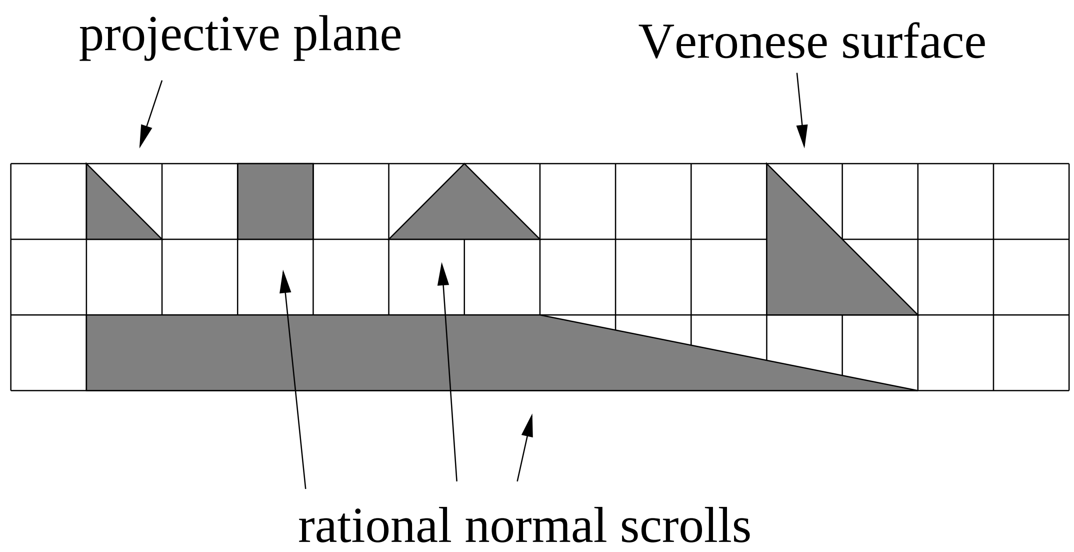


Possible Last Adjoints



Case 1: last adjoint is birational

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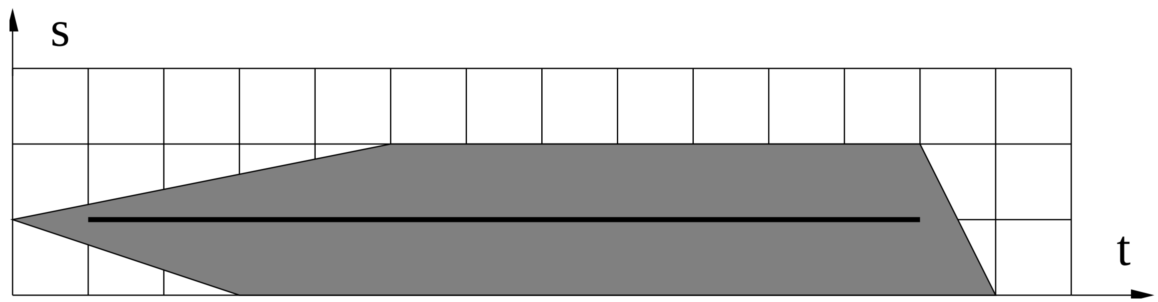
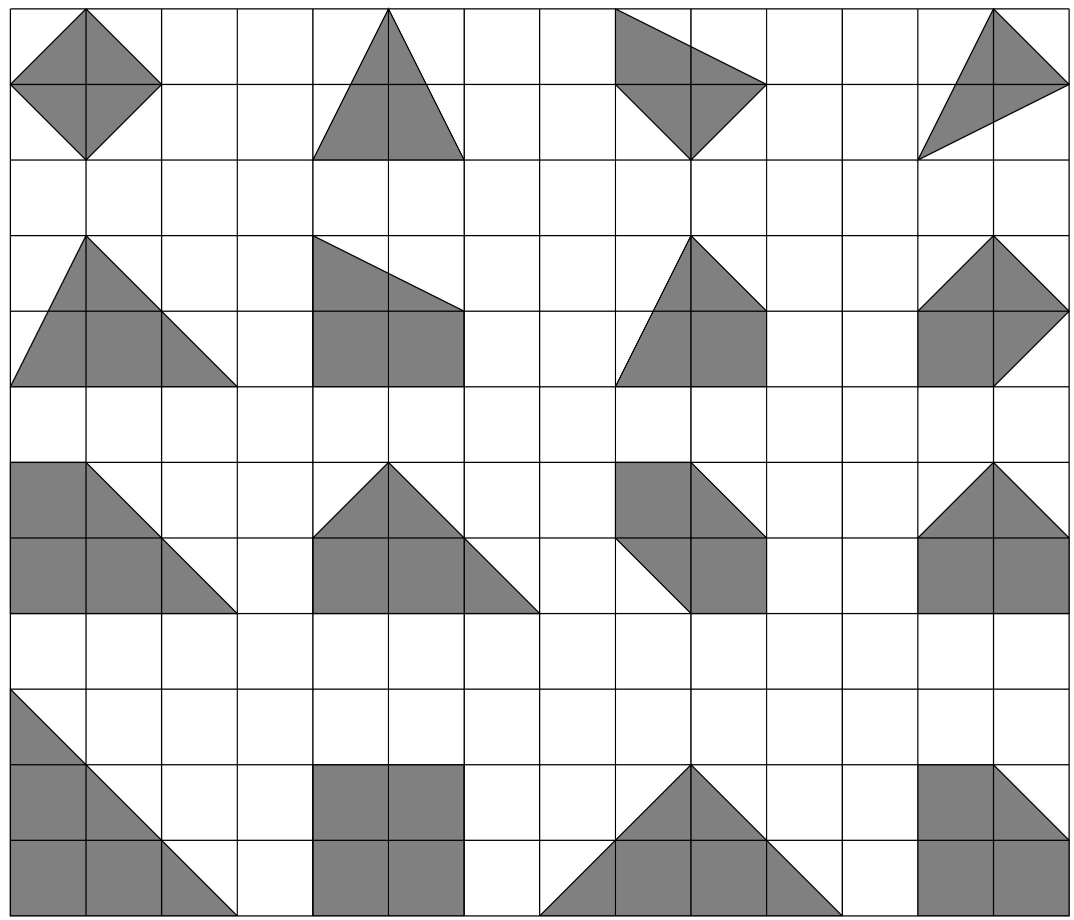


Case 2: last adjoint maps to a rational normal curve. The fibers "$t =$const." appear as degree 2 curves in the model given by the last but one adjoint map.



Case 3: last adjoint maps to a point. The last but one corresponds to a lattice polygon with 1 inner point. There are 16 of these, by a result of Rabinowitz.

Digression

Is there only a finite number of polygons with a fixed number p of inner points?

If $p = 0$, no. If $p \geq 1$, yes. This follows from a result of Scott:

$$\text{Area} \leq 2p + 5/2$$

Adjoints for Toric Varieties of Higher Dimension

If Γ is a lattice polytop, then $a_{1,m}$ is the toric map given by the points in the set

$$\bigcap_r \{p \mid \langle r,p\rangle \geq m + \min_{q\in\Gamma}\langle r,q\rangle\},$$

where r ranges over all primitive lattice vectors. This set is always a polytope, but in general not a lattice polytop.

Applications

The structure theorem on adjoints for rational surface can be used to

- parametrize an implicitely given rational surface
- simplify a parametrically given surface by reparametrization

The first requires a resolution of the singularities, the second requires a resolution of the base points of the given parametrization.

This has potential applications in CAD/CAM, because industrial standards require algebraic curves/surfaces to be represented by rational parametrizations.

Deciding Rationality

Theorem (Castelnuovo): A surface is rational iff the following two numbers vanish:

- p_a, the arithmetic genus;
- P_2, the second plurigenus.

These numbers can be computed with adjoints:

$$P_2 = v_{0,2}$$

$$p_a = d + 2v_{1,1} - v_{2,1} - 1 = 3v_{1,1} - 3v_{2,1} + v_{3,1} - 1$$

($v_{n,1}$ is a quadratic polynomial with leading coefficient $d/2$ for $n \geq 1$, we need $v_{0,1}$)

Quantitative Theory

$X \subset \mathbb{P}^n$ is a rational surface,
$P : \mathbb{P}^2 \to X$ is a birational parametrization.

- d is the degree of X
- the parametric degree is the degree of the polynomials defining P
- the intrinsic parametric degree d_p is the parametric degree of the smallest possible parametrization
- the sectional genus p_1 is the genus of a generic hyperplance section
- m is the smallest number such that $a_{2,m+1}$ does not exist

Obvious Relations

- $d \leq d_p^2$ (Bezout's formula)
- $p_1 \leq (d-1)(d-2)/2$ (genus formula)
- $p_1 \leq (d_p-1)(d_p-2)/2$ (genus formula in parameter space)
- $p_1 = v_{1,1}$ (Riemann-Roch)

More Relations

- $d_p \leq 2m + 2v_{2,m}$ (complexity analysis of parametrization)
- $d_p \geq 3m/2 + v_{2,m}$ (complexity analysis of simplification)
- $v_{1,i+1} < v_{1,i}$ or $v_{1,i+2} - v_{1,i+1} < v_{1,i+1} - v_i$, for $1 \leq i \leq m-1$ (lemma by Castelnuovo)
- $d_p \leq 8p_1^2$ if $p_1 \geq 1$ (from above)
- $d_p \leq 2d^4$ (from above)

Open Problems

- Can we reduce the exponent in $d_p \leq 2d^4$?
- For real algebraic surfaces, is Castelnuovos criterion sufficient for the existence of a real parametrization?
- If X is a ruled surface, is there an adjoint map computing the ruling as a fibration?
- Devise an algorithm for simplification of parametric 3-folds

Part 2: Computation

One needs some sort of resolution of the singularities.

There are easy situations, e.g. when we have singularities that come a finite projection. Then it suffices to compute the normalization.

Villamayor's Algorithm

while the largest multiplicity is greater than one **do**
  compute a "hypersurface of maximal contact"
  {contains all points of highest multiplicity}
  define a resolution problem in this hypersurface {one dimension less}
  resolve this problem, applying the algorithm recursively
  compute the induced sequence of blowing ups of the original space
  {at this point, the largest multiplicity must have dropped}
end while

(gross oversimplification, just enough for this discussion)

A similar algorithm has been given by Bierstone/Milman.

Villamayor's Algorithm

It is not necessary to compute a primary decomposition of the singular set. The blowing up center arises as an intersection of hypersurfaes of maximal contact.

This also means that the center is given by a regular sequence, which again allows a simpler blowing up computation.

Performance

Implementations by Bodnár/S. with help by S. Encinas, in Maple (very slow) and in Singular (much faster, but still slow).

With the newest implementation, we could not resolve most 3-fold singularities we tried, just surface singularities if they are not too complicated.

The output gets quite big. For instance, the resolution of the Whitney umbrella has 140 affine charts.

Resolution by Toric Varieties

Theorem (Jung/Hirzebruch/Abhyankar): Let X be a hypersurface such that its discriminant is a normal crossing divisor. Let Y be the normalization of X. Then the singularities of Y are analytically isomorphic to toric varieties.

By resolving the discriminant and subsequent normalization, we can compute a resolution by a locally toric variety. This is as good as a resolution by nonsingular variety, because we can compute adjoints for toric varieties easily.