ALGEBRAIC SHIFTING

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Surveys:

Gil Kalai & Jürgen Herzog
in "Computational Commutative Algebra & Combinatorics" (ed: Hibi), 2002

Application: (Björner-Kalai 1985)
Characterization of (f, β) of simplicial complexes.

f-vector   topological Betti #s

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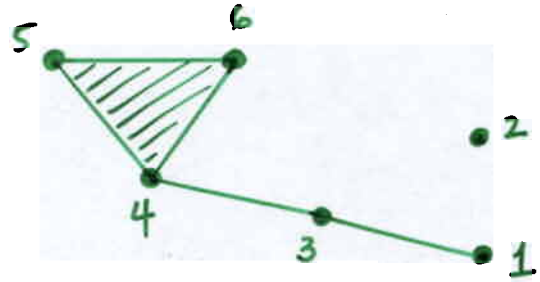
joint work with: Eric Babson (UW)
Isabella Novik

"Symmetric iterated Betti Numbers"
math.CO/0206063

Simplicial Complexes:

Γ simplicial complex on $[n] := \{1, 2, \dots, n\}$



$$\Gamma = \{\underbrace{\{4,5,6\}, \{3,4\}, \{1,3\}, \{2\}}_{\max \Gamma}\} \cup \{\{4,5\}, \{4,6\}, \dots, \emptyset\}$$

- $F \in \Gamma \quad \dim(F) := |F| - 1$
- $\dim(\Gamma) := \max\{\dim(F) : F \in \Gamma\} = d-1$
- $f_k(\Gamma) = \#\{F \in \Gamma : \dim(F) = k\}$
- $f(\Gamma) = (f_0(\Gamma), f_1(\Gamma), \dots, f_{d-1}(\Gamma))$ f-vector of Γ

Stanley-Reisner ideals:

$S = k[x_1, \dots, x_n]$ char $k = 0$

$I_\Gamma = \langle x_{i_1} x_{i_2} \cdots x_{i_k} \mid \{i_1, i_2, \dots, i_k\} \notin \Gamma \rangle \subset S$

$E = \Lambda(S_1)$

$J_\Gamma = \langle x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \mid \{i_1, \dots, i_k\} \notin \Gamma \rangle \subset E$

Shifting:

<u>DEF:</u> Γ is shifted if $\forall F \in \Gamma$, $j \in F$, $i \notin F$ s.t. $i < j$ $(F \setminus \{j\}) \cup \{i\} \in \Gamma$

Γ shifted $\Leftrightarrow$ J_Γ strongly stable w.r.t. $x_1 < x_2 < \cdots < x_n$

$:\Longleftrightarrow$ I_Γ squarefree strongly stable

(Recall: $L \subset S$ strongly stable wrt $x_1 < x_2 < \cdots < x_n$
$:\Leftrightarrow$ $x_\ell x^u \in L \Rightarrow x_k x^u \in L$ $\forall$ $k > \ell$)

<u>DEF:</u> $\Delta : \{\text{simp. cmplxs on } [n]\} \to \{\text{simp. cmplxs on } [n]\}$

$$\Gamma \longmapsto \Delta(\Gamma)$$

is a shifting operator if:

1. $\Delta(\Gamma)$ is shifted
2. Γ shifted $\Rightarrow \Delta(\Gamma) = \Gamma$ $\quad (\Delta(\Delta(\Gamma)) = \Delta(\Gamma))$
3. $f(\Gamma) = f(\Delta(\Gamma))$
4. $\Gamma' \subseteq \Gamma \Rightarrow \Delta(\Gamma') \subseteq \Delta(\Gamma)$

Exterior Shifting: Δ^e

$$J_{\Delta^e(\Gamma)} := \mathrm{Gin}_{RL}(J_\Gamma) := \mathrm{in}_{RL}(g J_\Gamma)$$

- $g \in GL(n,k)$ generic
- $RL: x_1 < x_2 < \cdots < x_n$

Symmetric Shifting: Δ

Compute $\mathrm{Gin}_{RL}(I_\Gamma)$ wrt $x_1 < x_2 < \cdots < x_n$

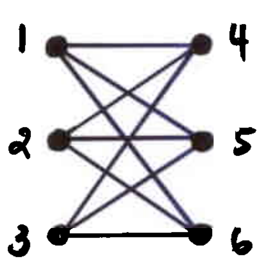
$$\Phi : x_{i_1} x_{i_2} \cdots x_{i_k} \longmapsto x_{i_k} x_{i_{k-1}-1} x_{i_{k-2}-2} \cdots x_{i_1-(k-1)}$$

"square-free operator" $\Phi(x_3^2 x_4^2) = x_4 x_3 x_1 x_0 x_{-1}$

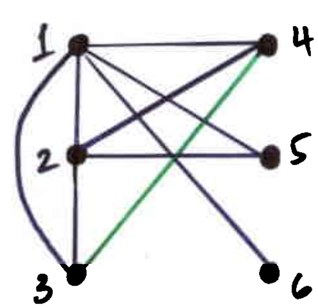
$$I_{\Delta(\Gamma)} := \Phi\, \mathrm{Gin}_{RL}(I_\Gamma) = \langle \Phi(m) : m \text{ mingen of } \mathrm{Gin}_{RL}(I_\Gamma) \rangle$$

Example:

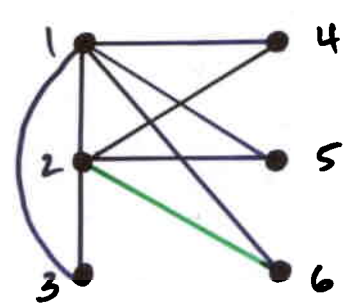
$K_{3,3}$



$\Delta^e(K_{3,3})$



$\Delta(K_{3,3})$



Q: What are the connections between $\Delta(\Gamma)$ and $\Delta^e(\Gamma)$?

Symmetric Iterated Betti Numbers

Def:

$$b_{i,r}(\Gamma) := \dim H^0_{m}\left(S/J_i(I_\Gamma) \right)_r \qquad 0 \le i, r \le n$$

$J_0(I_\Gamma) = g\, I_\Gamma \qquad g \in GL(n,k)$ generic

$J_i(I_\Gamma) = \langle x_i \rangle + \left(J_{i-1}(I) : m^\infty \right)$

Theorem: $b_{i,r}(\Gamma) =$

$$\begin{cases} \left| \{ F \in \max \Delta(\Gamma) : |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F \} \right| & \text{when } r \le i \ (\le d) \\ 0 & \text{otherwise} \end{cases}$$

$i \backslash r$	0	1	2	3
0	0			
1	0	0		
2	0	0	2	
3	1	4	8	1

(b-triangle of Γ)

$\max \Delta(\Gamma) =$

$\{$ 123, 124, 125, 126, 127, 134, 135, 136, 137, 145, 146, 147, 156, 234, 57, 67 $\}$

$$\beta_{i-1}(\Gamma) = b_{i,i}(\Gamma)$$

$$\| \quad \left| \{ F \in \max \Delta(\Gamma) : |F| = i,\ 1 \notin F \} \right|$$

Theorem: {extremal Betti #s of I_Γ} = {extremal entries in b-triangle of Γ}

	0	1	2	3
0	0			
1	0	0		
2	0	0	2	
3	1	4	8	1

b-triangle of Γ

	1	21	49	42	15	2
0	1	·	·	·	·	·
1	·	·	·	·	·	·
2	·	21	49	42	14	2
3	·				1	

from resol$^{\underline{n}}$ of S/I_Γ

<u>uses:</u>

Alexander dual

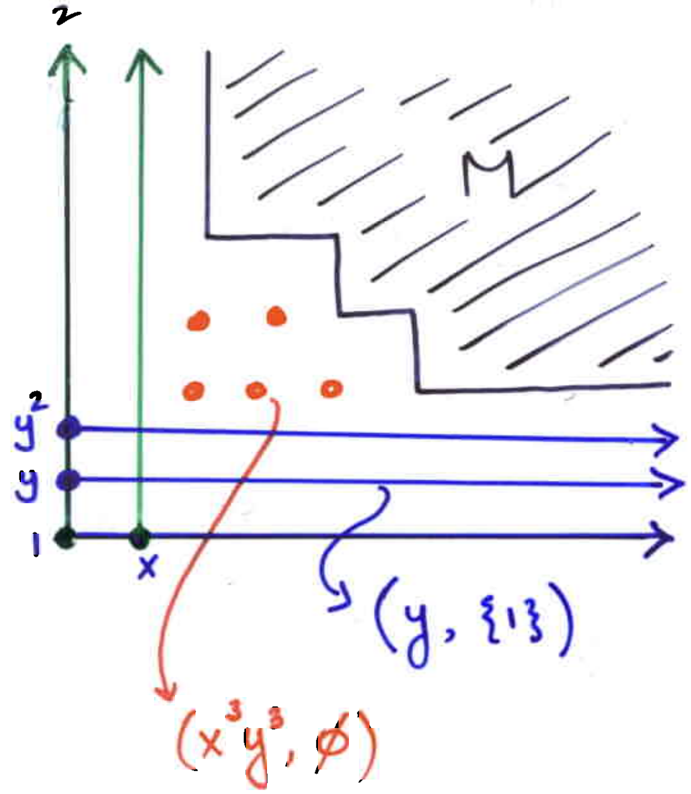
$$\Gamma^* = \{ F \subseteq [n] : [n] \setminus F \notin \Gamma \}$$

<u>BCP</u>: I_Γ & I_{Γ^*} have same extremal Betti #s

$$\beta_{i,i+j}\left(I_{\Delta(\Gamma^*)}\right) = \sum_r \binom{n-r-j}{i} b_{n-j,\,n-r-j}(\Gamma)$$

Herzog - Hibi
Herzog - Reiner - Welker
Duval ....
Babson, Novik, T. -

Standard Pair Decomposition: (Sturmfels Trung Vogel)



(m, σ)

monomial $\quad \subseteq [n]$

standard pair of M if

① $\operatorname{supp}(m) \cap \sigma = \emptyset$

② $m \cdot x_\sigma^*$ standard

③ $(m,\sigma) \equiv \{m \cdot x_\sigma^*\}$ inclusion maximal

(STV:)

① $(*, \sigma)$ std pair of $M \iff$
$\quad P_\sigma := \langle x_j : j \notin \sigma \rangle \in \operatorname{Ass}(M)$

② $(1,\sigma)$ std pair $\iff P_\sigma \in \{\text{min primes}(M)\}$

③ $\operatorname{mult}_M(P_\sigma) :=$ length of largest ideal of finite length in $S_{P_\sigma}/MS_{P_\sigma}$
$\quad = \#$ std pairs $(*, \sigma)$

④ geom-deg$(M) = \#$ std pairs $(1,\sigma)$

⑤ arithmetic-deg$(M) = \#$ std pairs of M.

Standard Pairs of $\mathrm{Gin}_{RL}(I_\Gamma)$:

all of the form: $(*, [i])$

(all ass primes: $P_{[i]} = \langle x_j : j > i \rangle$)

$A_i(\Gamma) := \{ m : (m, [i]) \text{ std pair of } \mathrm{Gin}_{RL}(I_\Gamma) \}$

$A_{i,r}(\Gamma) := (A_i(\Gamma))_r$

Theorem:

① $b_{i,r}(\Gamma) = |A_{i,r}(\Gamma)|$

② $\exists$ bijection between

$$A_{i,r}(\Gamma) \longleftrightarrow \{ F \in \max \Delta(\Gamma) : |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F \}$$

$$m \longleftrightarrow [i-r] \cup \operatorname{supp}(\Phi(m))$$

monomial b-triangle of Γ:

$i \backslash r$	0	1	2	3
0	$\emptyset$			
1	$\emptyset$	$\emptyset$		
2	$\emptyset$	$\emptyset$	$\{g^2, gf\}$	
3	$\{1\}$	$\{g, f, e, d\}$	$\{ge, gd, f^2, fe, fd, e^2, ed, d^2\}$	$\{d^3\}$

$A_{i,r}(\Gamma) \rightsquigarrow |A_{i,r}(\Gamma)| \rightsquigarrow$ b-triangle

Properties of $\Delta^{(e)}$:

① $\Delta^{(e)}$ preserve topological Betti numbers

$$\beta_{i-1}(\Gamma) = \beta_{i-1}(\Delta^{(e)}(\Gamma))$$
$$= \left|\{F \in \max \Delta^{(e)}(\Gamma) : |F| = i,\ 1 \notin F\}\right|$$

T shifted $\Rightarrow$ T near-cone with apex 1

$(:\Leftrightarrow \forall F \in T,\ 1 \notin F \Rightarrow \forall j \in F\ (F \setminus \{j\}) \cup \{1\} \in T)$

$$T = (1 * \underbrace{T'}_{\mathrm{lk}_T(1)}) \,\dot\cup\, \underbrace{B(T)}_{\{F \in T :\ F \cup \{1\} \notin T\} \subseteq \max(T)}$$

(Björner-Kalai) $\beta_{i-1}(\Delta^e(\Gamma)) = f_{i-1}(B(\Delta^e(\Gamma)))$

T' shifted on $\{2,\dots,n\}$ $\Rightarrow$

$$T = 1 * ((2 * T'') \,\dot\cup\, B(T')) \,\dot\cup\, B(T)$$

$\dots$ $T^{(r)} = ((r+1) * T^{(r+1)}) \cup B_r$

(Duval, Rose 2000) exterior iterated Betti #s:

$$\boxed{b^e_{i,r}(\Gamma)} := \beta_{r-1}\left(\Delta^e(\Gamma)^{(i-r)}\right)$$
$$= f_{r-1}(B_{i-r})$$
$$\boxed{= \left|\{F \in \max \Delta^e(\Gamma) : |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F\}\right|}$$

② Δ^e preserves exterior iterated Betti #s

MAIN RESULT: Symmetric iterated Betti #s

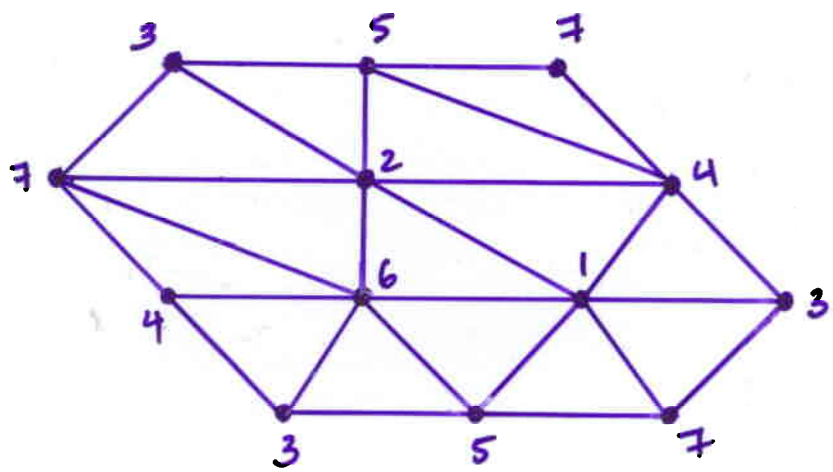
③ $\Delta^{(e)}$ preserve Cohen-Macaulayness:

Γ CM $\iff \Delta^{(e)}(\Gamma)$ CM $\iff \Delta^{(e)}(\Gamma)$ pure

④ I_Γ, $I_{\Delta(\Gamma)}$ & $I_{\Delta^e(\Gamma)}$ have same extremal Betti #s

(symm case: Bayer - Charlambous - Popescu
ext case: Aramova - Herzog)

Γ



S/I_Γ

	1	21	49	42	15	2
0	1	·	·	·	·	·
1	·	·	·	·	·	·
2	·	21	49	42	14	2
3	·	·	·	·	1	

$S/I_{\Delta(\Gamma)}$

	1	22	53	48	18	2
0	1	·	·	·	·	·
1	·	·	·	·	·	·
2	·	21	50	45	17	2
3	·	1	3	3	1	

Std pairs of $Gin_{RL}(I_\Gamma)$	$\Phi(m) \to$ supp $\Phi(m)$	$\max \Delta(\Gamma)$ $[i-r] \cup \mathrm{supp}(\Phi(m))$
$(1, \{1,2,3\})$	$1 \to \emptyset$	$\{1,2,3\}$
$(g, \{1,2,3\})$	$g \to \{7\}$	$\{1,2,7\}$
$(f^2, \{1,2,3\})$	$fe \to \{5,6\}$	$\{1,5,6\}$
$\vdots$	$\vdots$	$\vdots$
$(d^3, \{1,2,3\})$	$dcb \to \{2,3,4\}$	$\{2,3,4\}$
$(g^2, \{1,2\})$	$gf \to \{6,7\}$	$\{6,7\}$
$(gf, \{1,2\})$	$ge \to \{5,7\}$	$\{5,7\}$

$$\mathrm{mult}_{Gin(I_\Gamma)}(P_{[i]}) = \sum_r b_{i,r}(\Gamma)$$

$$= |\{F \in \max \Delta(\Gamma) : |F| = i\}|$$

$$\text{geom-deg}(Gin\, I_\Gamma) = \sum_r b_{d,r}(\Gamma)$$

$$= |\{F \in \max \Delta(\Gamma) : |F| = d\}|$$ 14

$$\text{arith deg}(Gin(I_\Gamma)) = \sum_{i,r} b_{i,r}(\Gamma)$$ 16

$$= |\max \Delta(\Gamma)|$$

	0	1	2	3
0	0			
1	0	0		
2	0	0	2	
3	1	4	8	1