# Digitizing Quantum Correlations*

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in collaboration with

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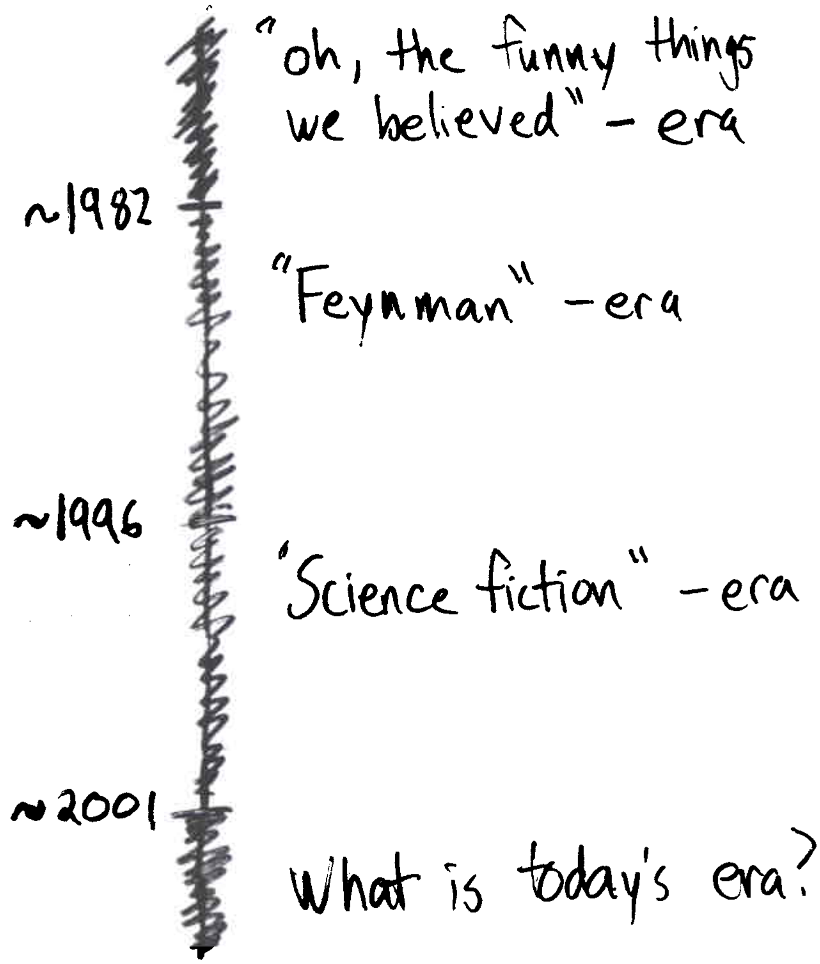
(both at Caltech)

Quantum Algorithms and Complexity

Banff 2002

*an Ocamm-free talk

# ERAS


"oh, the funny things we believed" - era
~1982
"Feynman" - era
~1996
"Science fiction" - era
~2001
What is today's era?

# Obligatory Feynman Quote *old school!

"... trying to find a computer simulation of physics seems to me to be an excellent program to follow out... and I'm not happy with all the analyses that go with just classical theory, because nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make $\boxed{\text{it}}$ quantum mechanical, and by golly it's a wonderful problem because it doesn't look easy!" Feynman, IJTP, **21**, 486 (1982)

## the mysterious $\boxed{\text{it}}$

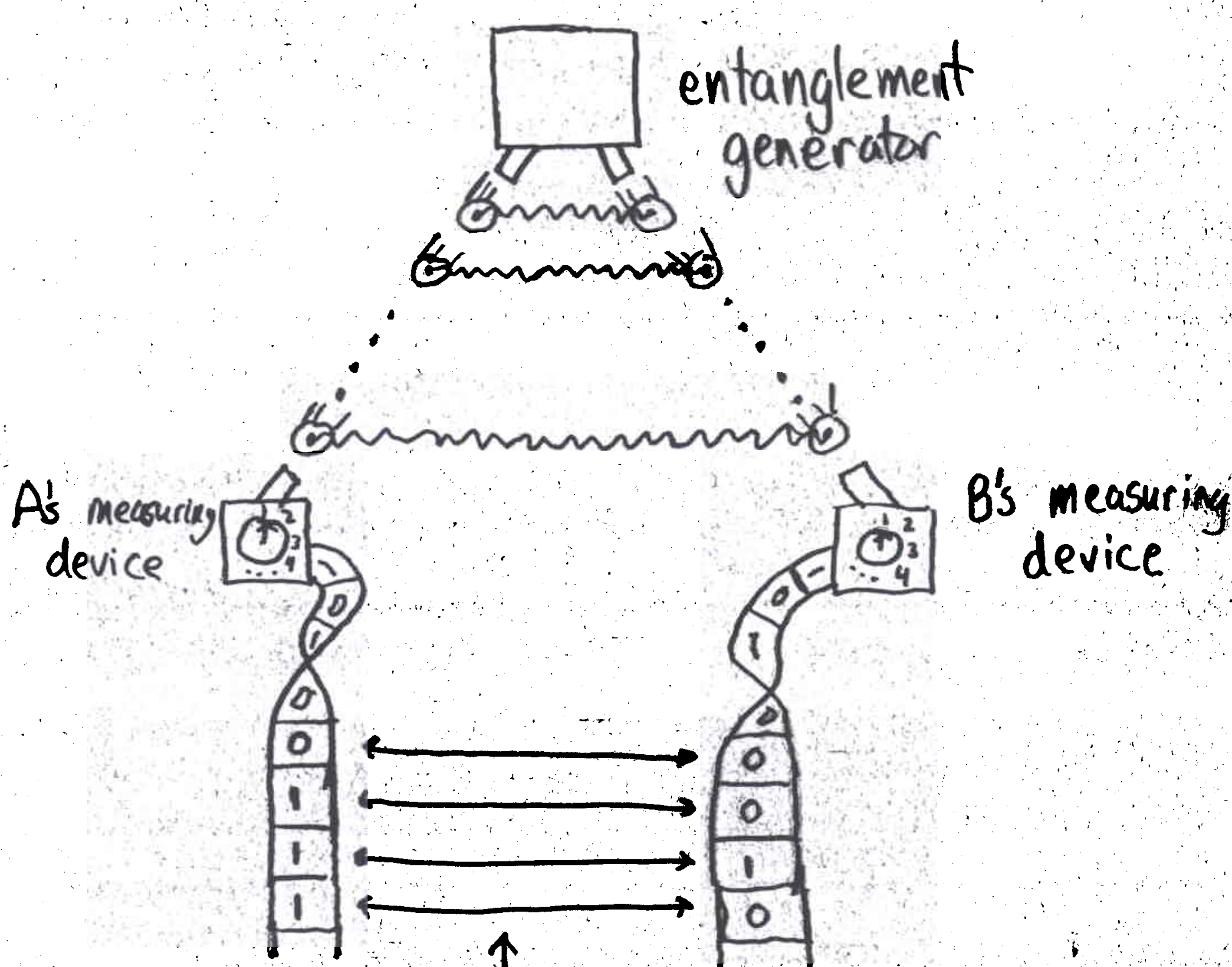
$\boxed{\text{it}}$ = the actual simulation ⇒ quantum computation

What can Physics reveal about Computer Science?

$\boxed{\text{it}}$ = nature ⇒ classical simulation of quantum systems

What can Computer Science reveal about Physics?

# Quantum Correlations




correlated outputs

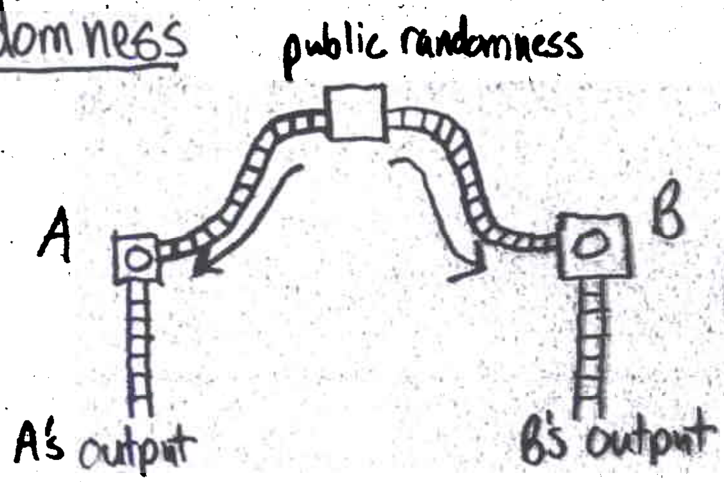
## Goal

Compare and contrast quantum correlations with other methods of creating correlated outputs....... QUANTIFY YOUR RESOURCES!

# Possible Scenarios

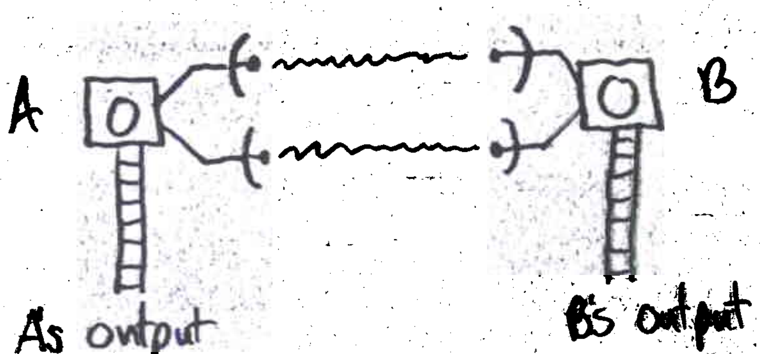
**QUANTIFICATIO**

## shared randomness




random bits per output bit (worst case or expected value)

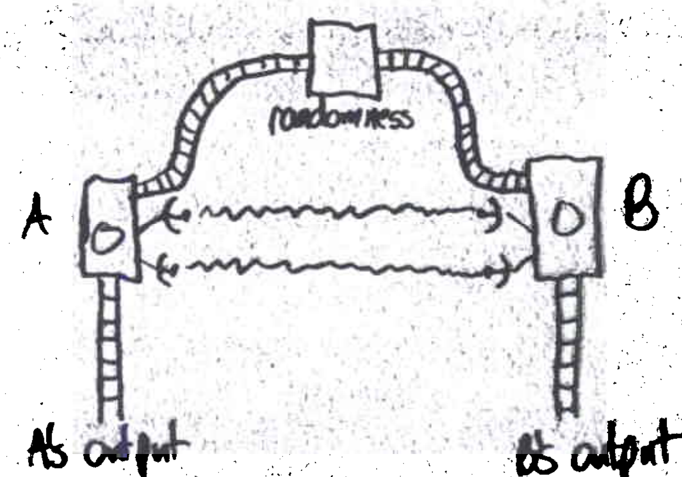
## communicating parties




bits communicated per output bit (worst case or expected value)

subcategories: 1-way versus 2-way

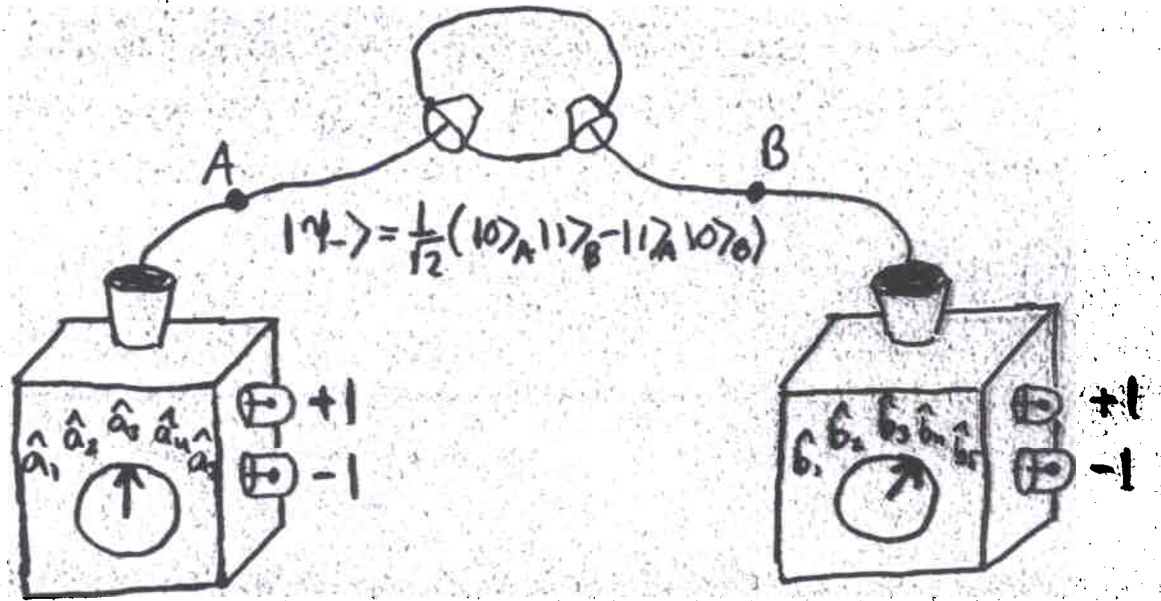
## shared randomness and communication




bits communicated per output bit and/or random bits per output bit (worst case or expected value)

EXACT versus APPROXIMATE correlated outputs.

# For Whom The Bell Experiment Tolls




$m_A$ measurements
$\{\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots, \hat{a}_{m_A}\}$
corresponding to projective measurements
$$\mathbb{P}_{+1}(\hat{a}_i) = \tfrac{1}{2}(\mathbb{I} + \hat{a}_i \cdot \hat{\sigma})$$
$$\mathbb{P}_{-1}(\hat{a}_i) = \tfrac{1}{2}(\mathbb{I} - \hat{a}_i \cdot \hat{\sigma})$$
call outcomes $A \in \{+1, -1\}$

$\hat{a}_i, \hat{b}_i \in \mathbb{R}^3$
$|\hat{a}_i| = |\hat{b}_i| = 1$

$m_B$ measurements
$\{\hat{b}_1, \hat{b}_2, \hat{b}_3, \ldots, \hat{b}_{m_B}\}$
corresponding to projective measurements
$$\mathbb{Q}_{+1}(\hat{b}_i) = \tfrac{1}{2}(\mathbb{I} + \hat{b}_i \cdot \hat{\sigma})$$
$$\mathbb{Q}_{-1}(\hat{b}_i) = \tfrac{1}{2}(\mathbb{I} - \hat{b}_i \cdot \hat{\sigma})$$
call outcomes $B \in \{+1, -1\}$

<u>probabilities for measurements $(\hat{a}_i, \hat{b}_j)$</u>

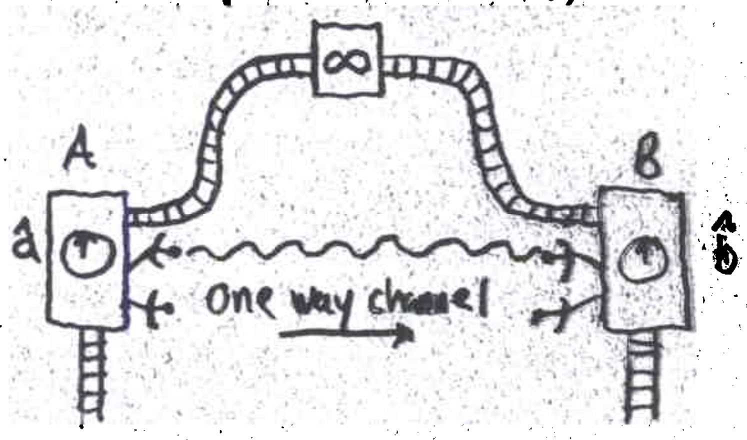
$$\Pr[A=+1] = \Pr[A=-1] = \Pr[B=+1] = \Pr[B=-1] = \boxed{0.5}$$
$$\Pr[A=+1, B=+1] = \Pr[A=-1, B=-1] = \boxed{\tfrac{1}{4}(1 - \hat{a}_i \cdot \hat{b}_j)}$$
$$\Pr[A=+1, B=-1] = \Pr[A=-1, B=+1] = \boxed{\tfrac{1}{4}(1 + \hat{a}_i \cdot \hat{b}_j)}$$

$$\langle(\hat{a}_i \cdot \hat{\sigma}) \otimes \mathbb{I}\rangle = \langle \mathbb{I} \otimes (\hat{b}_j \cdot \hat{\sigma})\rangle = 0 \qquad \langle(\hat{a}_i \cdot \hat{\sigma}) \otimes (\hat{b}_j \cdot \hat{\sigma})\rangle = -\hat{a}_i \cdot \hat{b}_j$$

<u>It Tolls For $\hat{a} \cdot \hat{b}$!</u>

# 8, 6, ..... ?

public randomness




- unlimited shared randomness
- exact creation of Bell experiment probabilities
- one way communication
- worst case bits communicated per output round

Naive: A communicates to B $\hat{a}$.
A outputs shared random bit b. $b = 50\%\ {+1},\ 50\%\ {-1}$
B calculates $\hat{a}\cdot\hat{b}$. B outputs $b$ with probability $\frac{1}{2}(1-\hat{a}\cdot\hat{b})$
B outputs $\bar{b}$ with probability $\frac{1}{2}(1+\hat{a}\cdot\hat{b})$
requires <u>$\infty$ bits</u> to exactly produce Bell experiment

Brassard, Cleve, Tapp (1999): (Amazingly!)
protocol using <u>8 bits</u> of communication.

Csirik (2002):
protocol using <u>6 bits</u> of communication.

# Symmetries

$$Pr[A,B|\hat{a},\hat{b}] = \frac{1}{4}(1 - AB\,\hat{a}\cdot\hat{b})$$

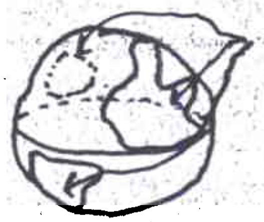
$\boxed{1.}$ If $\hat{a}=\hat{b}$, $Pr[A=B|\hat{a},\hat{b}] = 0$ anticorrelated

If $\hat{a}=-\hat{b}$, $Pr[A=B|\hat{a},\hat{b}] = 1$ correlated

$\Rightarrow$ as B learns info about A's $\hat{a}$, B must

−(A's output) if $\hat{b}$ is in region of A's $\hat{a}$

+(A's output) if $\hat{b}$ is in − region of A's $\hat{a}$



given communicated bits $\hat{a}$ could be in these regions

$\boxed{2.}$ Rotational invariance

$$Pr[A,B|\hat{a},\hat{b}] = Pr[A,B|R(\hat{a}),R(\hat{b})]$$

$R(\hat{a})$ = 3D rotation of $\hat{a}$

**note:** same rotation on both sides.

$\Rightarrow$ we need only consider rotationally invariant protocols

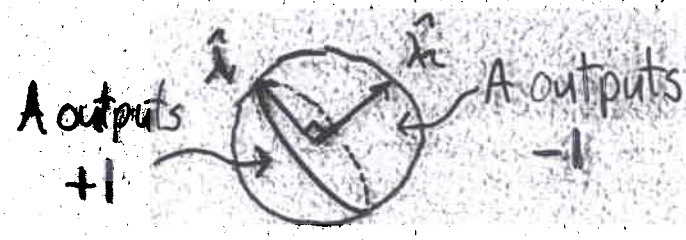
# A 1 Bit Protocol

A and B share random variables $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathbb{R}^3$, $|\hat{\lambda}_1| = |\hat{\lambda}_2| = 1$

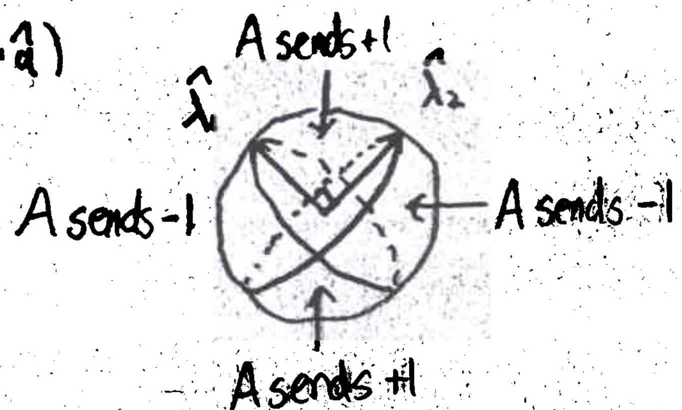
$\hat{\lambda}_1$ sampled uniformly over sphere

$\hat{\lambda}_2$ sampled uniformly over circle defined by $\hat{\lambda}_1 \cdot \hat{\lambda}_2 = 0$

$\boxed{1.}$ A outputs $-\mathrm{sgn}(\hat{\lambda}_2 \cdot \hat{a})$




$\boxed{2.}$ A sends to B $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{a})\mathrm{sgn}(\hat{\lambda}_2 \cdot \hat{a})$




$\boxed{3a.}$ If B receives +1, then

if $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})\mathrm{sgn}(\hat{\lambda}_2 \cdot \hat{b}) = +1$, B outputs $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})$

if $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})\mathrm{sgn}(\hat{\lambda}_2 \cdot \hat{b}) = -1$, B outputs $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})$ with prob $\frac{(\hat{\lambda}_1 \cdot \hat{b})^2}{(\hat{\lambda}_1 \cdot \hat{b})^2 + (\hat{\lambda}_2 \cdot \hat{b})^2}$

B outputs $-\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})$ otherwise.

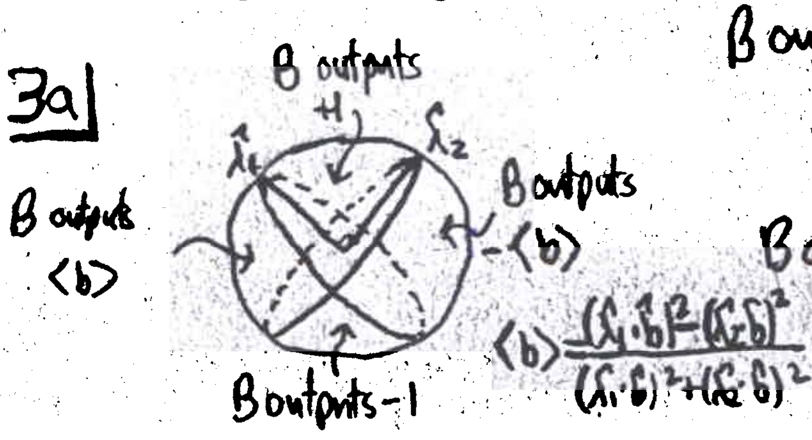
$\boxed{3b.}$ If B receives -1, then

if $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})\mathrm{sgn}(\hat{\lambda}_2 \cdot \hat{b}) = -1$, B outputs $-\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})$

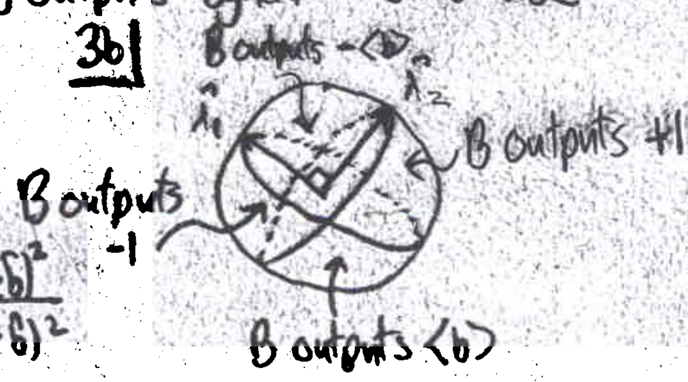
if $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})\mathrm{sgn}(\hat{\lambda}_2 \cdot \hat{b}) = +1$, B outputs $-\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})$ with prob. $\frac{(\hat{\lambda}_1 \cdot \hat{b})^2}{(\hat{\lambda}_1 \cdot \hat{b})^2 + (\hat{\lambda}_2 \cdot \hat{b})^2}$

B outputs $\mathrm{sgn}(\hat{\lambda}_1 \cdot \hat{b})$ otherwise

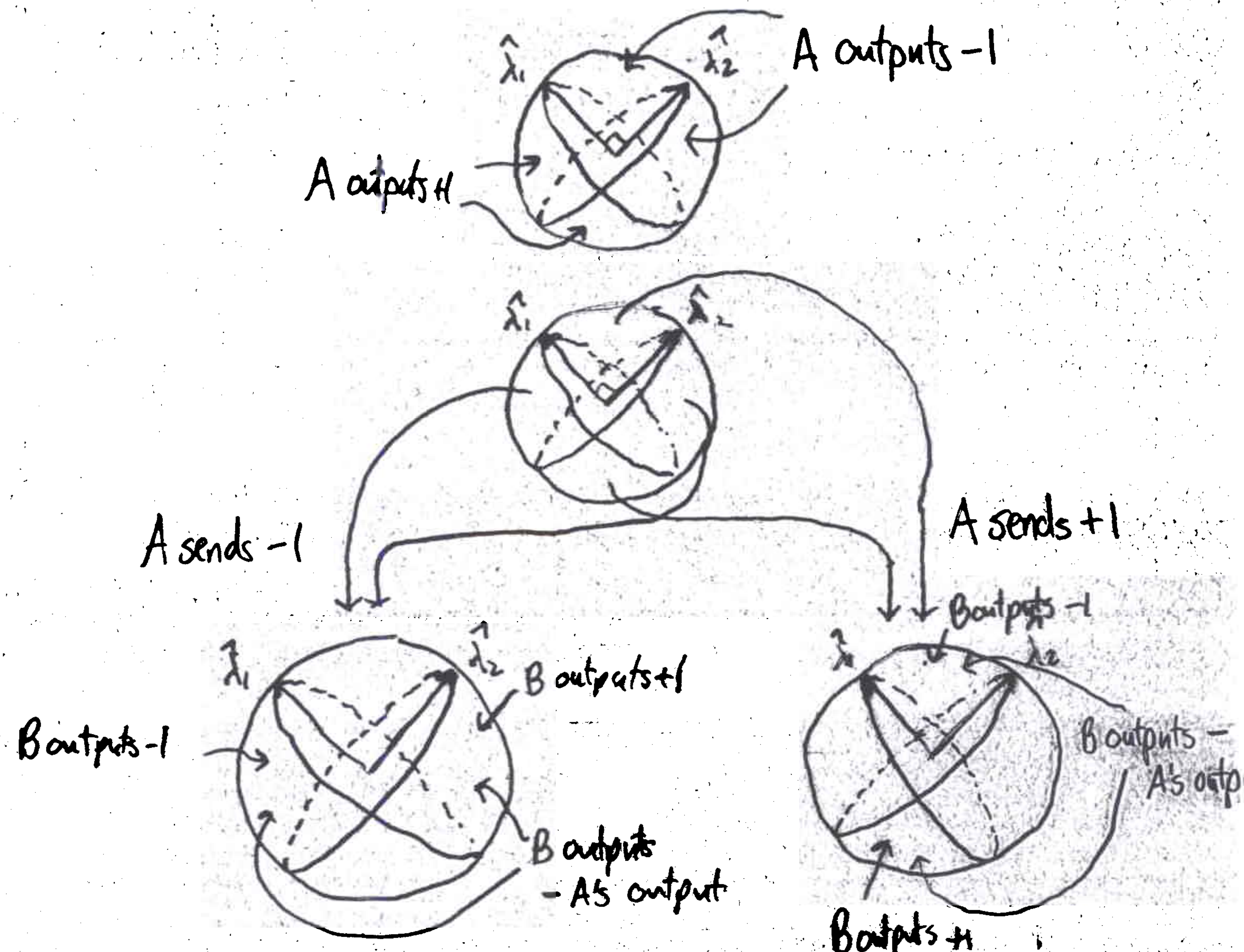
3a

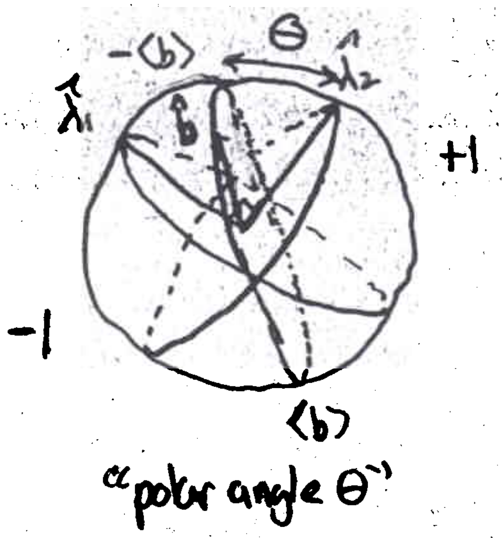


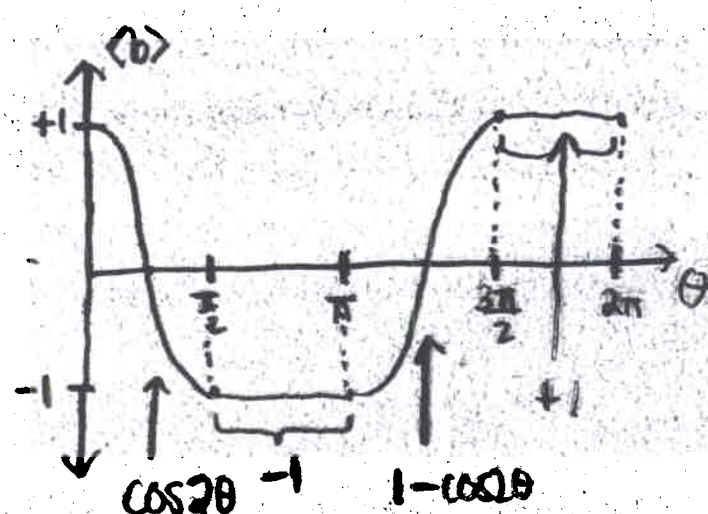

3b



# Schematic


$\hat{\lambda}_1$
$\hat{\lambda}_2$
A outputs -1
A outputs +1
A sends -1
A sends +1
B outputs -1
B outputs +1
B outputs - A's output
B outputs -1
B outputs - A's outp
B outputs +1


$-\langle b \rangle$
$\Theta$
$\hat{\lambda}_1$
$\hat{\lambda}_2$
$\hat{b}$
+1
-1
$\langle b \rangle$
"polar angle Θ"


$\langle b \rangle$
+1
-1
$\frac{\pi}{2}$
$\pi$
$\frac{3\pi}{2}$
$2\pi$
$\Theta$
+1
$\cos 2\theta$
-1
$1-\cos 2\theta$

# WWFD?

Expectation of product of A and B's output

A's output    A sent +1    B's output given A sent +

$$\langle AB\rangle = E\Big[-\mathrm{sgn}(\hat{\lambda}_2\cdot\hat{a})\Big[\tfrac{1}{2}\big(1+\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{a})\,\mathrm{sgn}(\hat{\lambda}_2\cdot\hat{a})\big)\, b_{+1}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{b})$$
$$+\tfrac{1}{2}\big(1-\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{a})\,\mathrm{sgn}(\hat{\lambda}_2\cdot\hat{a})\big)\, b_{-1}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{b})\Big]\Big]$$

A sent -1    B's output given A sent -1

$$E[\cdot] = \frac{1}{8\pi^2}\int d\hat{\lambda}_1 \int_{\hat{\lambda}_1\cdot\hat{\lambda}_2=0} d\hat{\lambda}_2\,[\cdot]$$

$$\langle AB\rangle = -E\left[\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{a})\,\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{b})\left(\frac{2(\hat{\lambda}_1\cdot\hat{b})^2}{(\hat{\lambda}_1\cdot\hat{b})^2+(\hat{\lambda}_2\cdot\hat{b})^2}\right)\right]$$

$$= -\frac{1}{4\pi^2}\int_S d\hat{\lambda}_1\,\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{a})\,\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{b})\,(\hat{\lambda}_1\cdot\hat{b})^2\left(\int_{\hat{\lambda}_1\cdot\hat{\lambda}_2=0} d\hat{\lambda}_2\,\frac{1}{(\hat{\lambda}_1\cdot\hat{b})^2+(\hat{\lambda}_2\cdot\hat{b})^2}\right)$$

$$\int d\hat{\lambda}_2\,\frac{1}{(\hat{\lambda}_1\cdot\hat{b})^2+(\hat{\lambda}_2\cdot\hat{b})^2} = \frac{2\pi\,\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{b})}{(\hat{\lambda}_1\cdot\hat{b})}$$

$$\langle AB\rangle = -\frac{1}{2\pi}\int_S d\hat{\lambda}_1\,(\hat{\lambda}_1\cdot\hat{b})\,\mathrm{sgn}(\hat{\lambda}_1\cdot\hat{a}) = -\hat{a}\cdot\hat{b}$$

N.Gisin and B.Gisin
detector inefficiency loop hole
integral

also $\langle A\rangle = 0$, $\langle B\rangle = 0$

easy to see    harder to see

The correlations produced by projective measurements on a Bell pair can be simulated using unlimited shared randomness plus only one bit of communication.

Why care?

Bell pair = basic unit of bipartite entanglement

$\Downarrow$

Build up a theory of simulating bipartite quantum systems using this protocol as fundamental unit.

Infinite number of possible theories which produce correlations: <u>why quantum theory?</u>

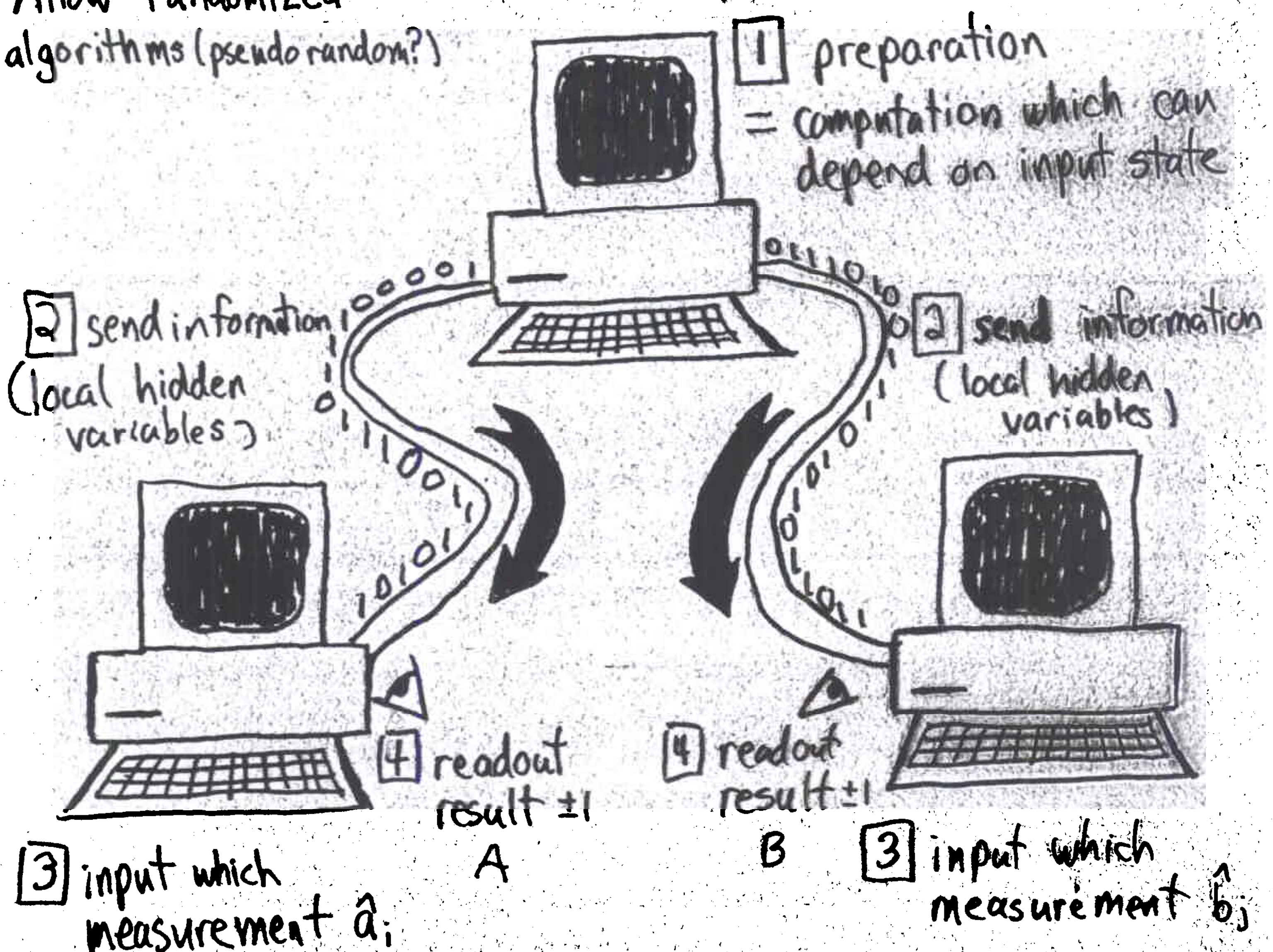
Is

$$\begin{pmatrix}\text{correlations on} \\ \text{1 ebit}\end{pmatrix} = \begin{pmatrix}\text{shared} \\ \text{randomness}\end{pmatrix} + \begin{pmatrix}\text{1 bit}\end{pmatrix}$$

a consequence of a deep principle?

# Through The Eyes Of A Computer Scientist

Allow randomized algorithms (pseudorandom?)

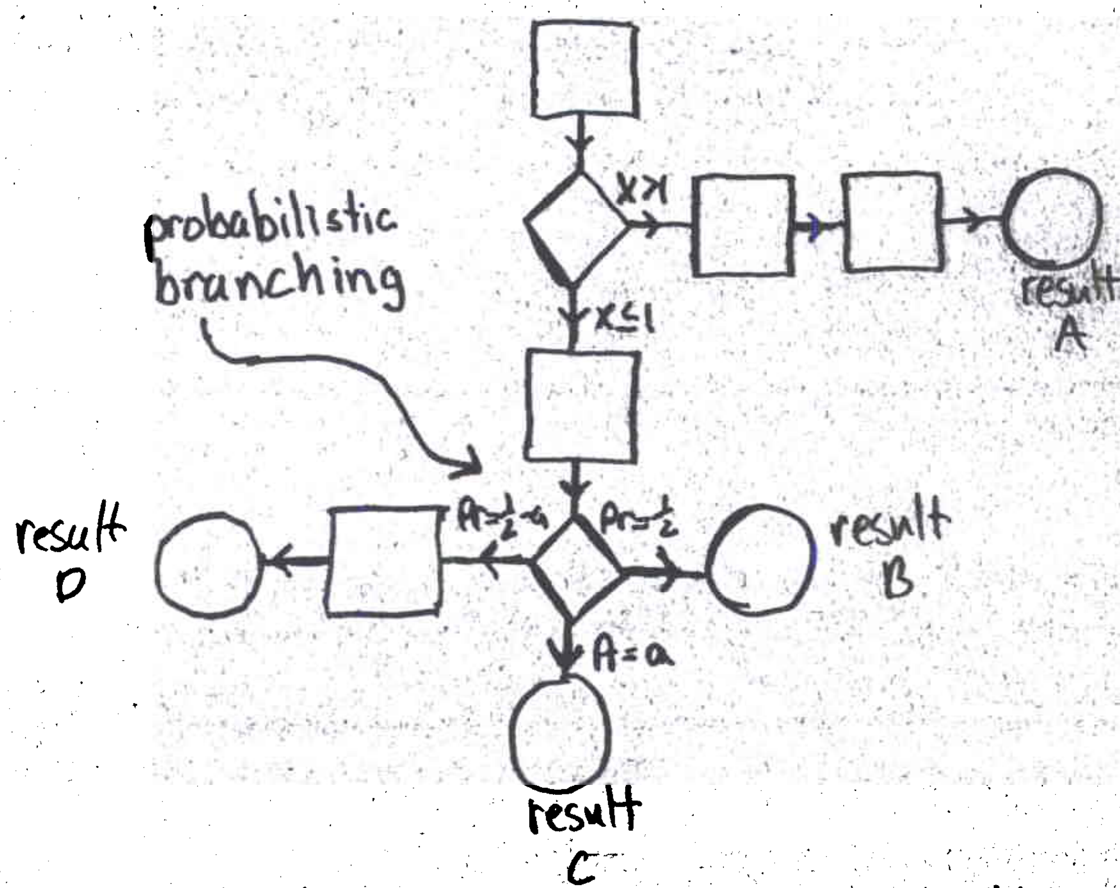



output is characterized by probabilities:

$$\Pr[A, B \mid \{\hat{a}_i\}, \{\hat{b}_j\}]$$

question: what set of $\Pr[A, B \mid \{\hat{a}_i\}, \{\hat{b}_j\}]$ are possible with this setup?

# Where To Stick That Randomness




Running algorithm results in random variables taking on particular values: i.e. with an asignment of value to random variable we get a deterministic algorithm.

$$\text{Probabilistic Algorithm} = \sum_{x} Pr[x] \cdot \left( \text{Deterministic Algorithm} \mid x \right)$$

$x$ — random variables

$\mid x$ — given $x$

# Deterministic Protocols

Probabilities

$$\Pr[A,B \mid \hat{a}_i, \hat{b}_j] \in \mathbb{R}^{4\,\underbrace{(m_A m_B)}_{\text{which measurements}}}$$

$A \in \{\pm 1\},\ B \in \{\pm 1\}$

For fixed measurements, four probabilities

$$p_{++} = \Pr[A=+1, B=+1] \qquad p_{+-} = \Pr[A=+1, B=-1]$$
$$p_{-+} = \Pr[A=-1, B=+1] \qquad p_{--} = \Pr[A=-1, B=-1]$$

deterministic protocols $\rightarrow$ only one outcome

$$(p_{++}, p_{+-}, p_{-+}, p_{--}) \in \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

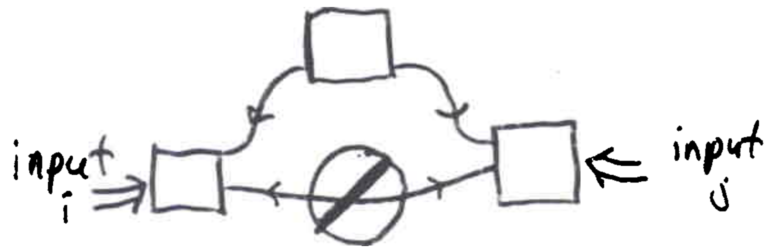
## Example

| | $\hat{b}_1$ | $\hat{b}_2$ | $\hat{b}_3$ |
|---|---|---|---|
| $\hat{a}_1$ | $(1,0,0,0)$ | $(0,0,1,0)$ | $(0,1,0,0)$ |
| $\hat{a}_2$ | $(1,0,0,0)$ | $(0,0,1,0)$ | $(1,0,0,0)$ |
| $\hat{a}_3$ | $(0,0,0,1)$ | $(1,0,0,0)$ | $(1,0,0,0)$ |

deterministic protocols are a finite set of vectors in $\mathbb{R}^{4m_A}$

Accessible region of $\mathbb{R}^{4m_A m_B}$ is exactly the region given by the convex sum of this finite set.

# No Cross Talk




no communication between parties implies they do not know which measurement the other measures.

Probabilities $\Pr[A,B\,|\,\hat{a}_i, \hat{b}_j] \in \mathbb{R}^{4\, m_A m_B}$

basis vectors: $\hat{e}_{i,j,A,B}$

$1 \le i \le m_A$, $1 \le j \le m_B$, $A,B \in \{\pm 1\}$

Deterministic protocols produce probabilities

$$\vec{\eta}_c = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \hat{e}_{i,j,A(i,j),B(i,j)}$$

classical configuration is specification of $A(i,j), B(i,j)$

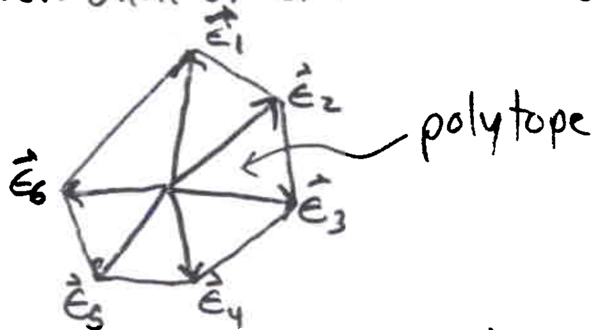
No cross talk

$$\vec{\eta}_c^{\,[1]} = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \hat{e}_{i,j,A(i),B(j)}$$

outcomes do not depend on other party's measurement choice.

# Garden Of Polytopes

Polytope: convex sum of a finite set of extreme vectors $\vec{\epsilon}_i$




assume (simplicity) origin of $\mathbb{R}^d$ is inside the polytope

Two Representations

1) convex sum of extreme vectors

$$\Omega = \{ \vec{x} \mid \vec{x} = \sum_i \lambda_i \vec{\epsilon}_i ,\ \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \}$$

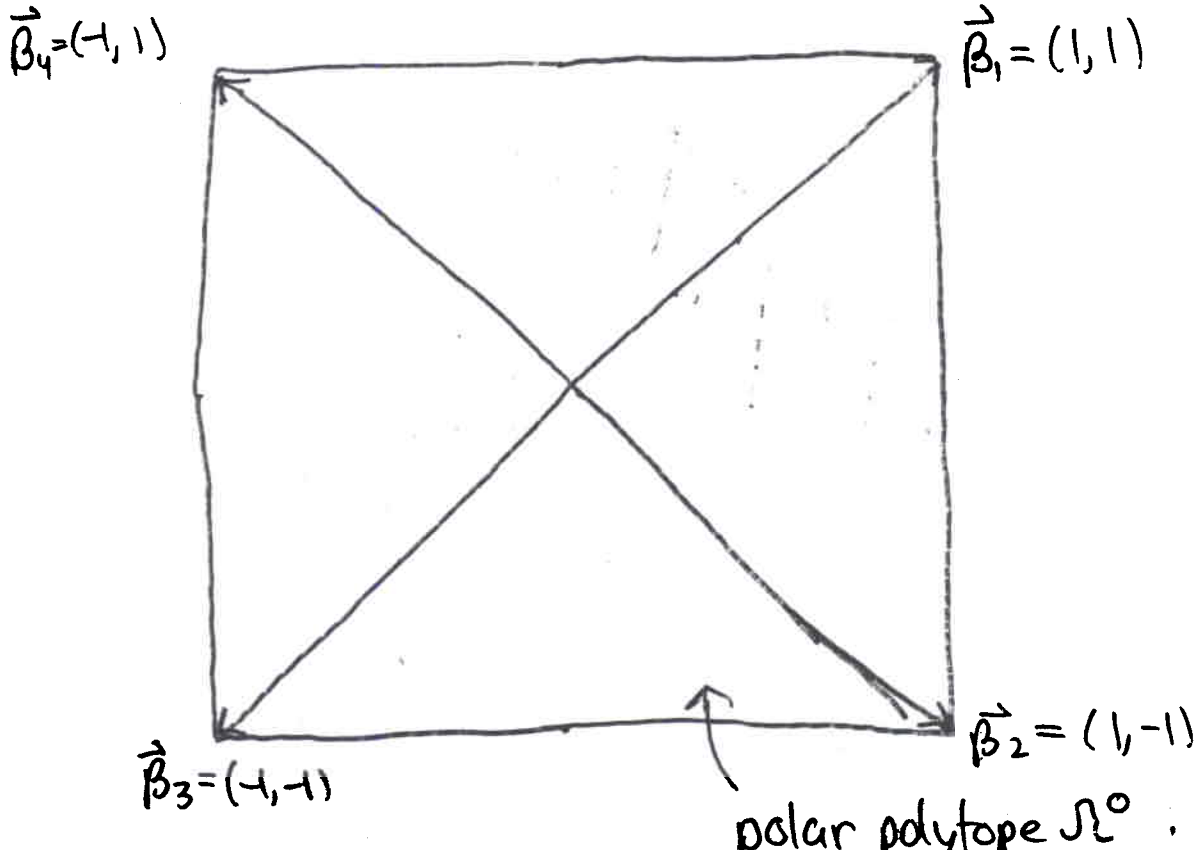
2) intersection of half spaces containing extreme vectors

$$\Omega = \bigcap_{\vec{\beta}} \{ \vec{x} \mid \vec{x} \cdot \vec{\beta} \leq 1,\ \vec{\epsilon}_i \cdot \vec{\beta} \leq 1 \}$$

Polar: $\Omega^{\circ} = \{ \vec{\beta} \mid \vec{\epsilon}_i \cdot \vec{\beta} \leq 1 \}$

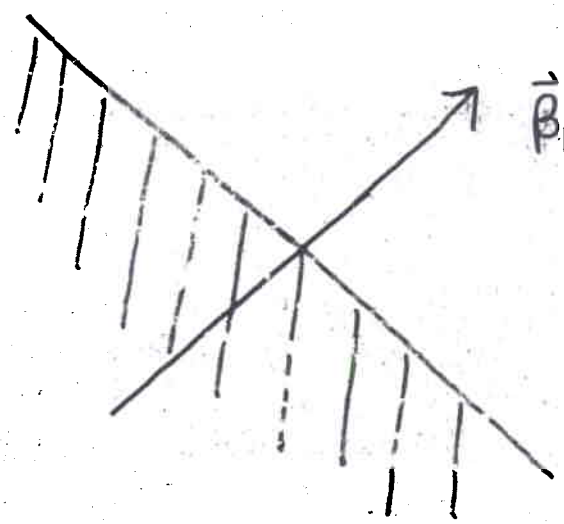
the Polar of a polytope is itself a polytope

Bipolar Theorem: A necessary and sufficient for $\vec{x} \in \Omega$, where $\Omega$ is a polytope, is that $\vec{x} \cdot \vec{\beta}_i \leq 1 \ \forall i$ where $\vec{\beta}_i$ are the extreme points of the polar of $\Omega$, $\Omega^{\circ}$.




To determine if $\vec{x} \in \Omega$ we need only check whether $\vec{x} \cdot \vec{\beta}_i \leq 1$ for the extreme vectors $\vec{\beta}_i$ of the polar $\Omega^\circ$

$\vec{\beta}_1 \cdot \vec{x} \leq 1$ halfspace



$\vec{\beta}_1 = (1,1)$

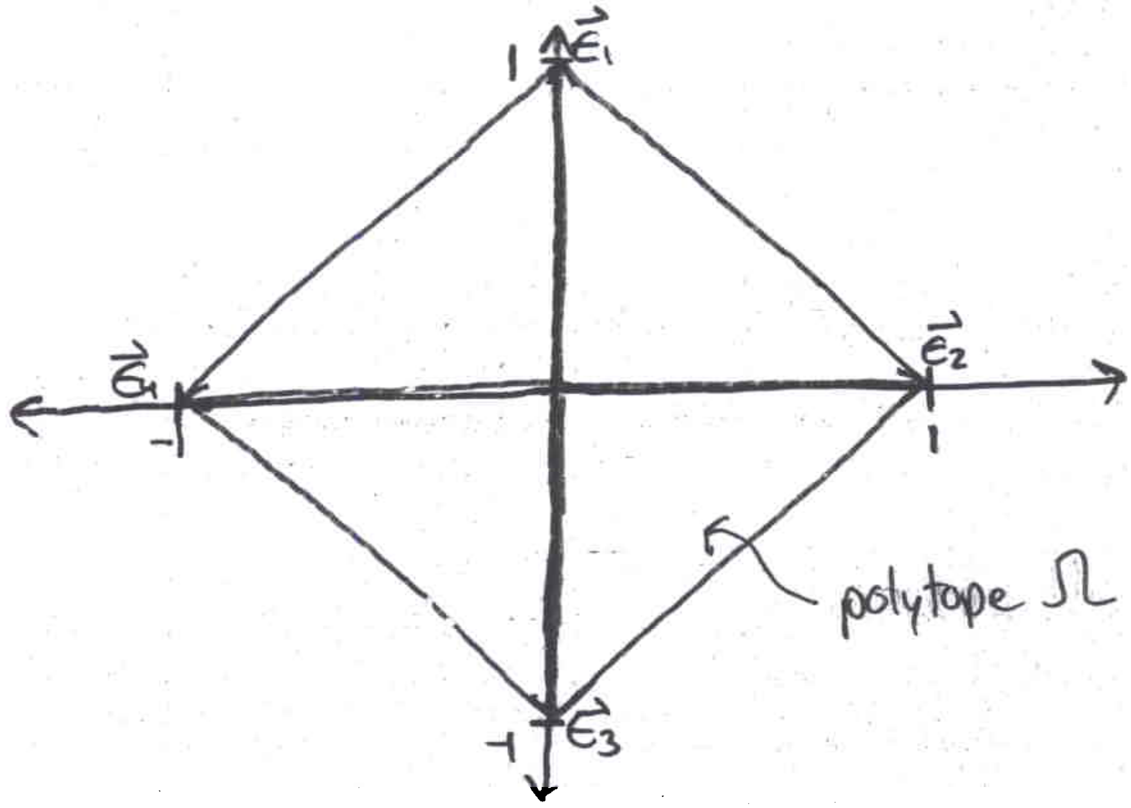
$\vec{\beta}_1 \cdot \vec{\epsilon}_1 = \vec{\beta} \cdot \vec{\epsilon}_2 = 1$

$\vec{\beta}_1 \cdot \vec{\epsilon}_3 = \vec{\beta} \cdot \vec{\epsilon}_4 = -1 \leq$

# A Simple Example

Polytope (gon) with extreme points

$\vec{\epsilon}_1 = (0,1)$ , $\vec{\epsilon}_2 = (1,0)$, $\vec{\epsilon}_3 = (0,-1)$ , $\vec{\epsilon}_4 = (-1,0)$



# Expectation Values

Same arguements hold if we want to explain expectation values of experiment (may be weaker)

probabilities $\in \mathbb{R}^{4 m_A m_B}$

$$Pr[A, B \mid \hat{a}_i, \hat{b}_j]$$

basis $\mathbb{R}^{4 m_A m_B}$

$$\hat{e}_{i,j,A,B}$$

classical configurations with no cross talk

$$\vec{\eta}_c^{[1]} = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \hat{e}_{i,j,A(i),B(j)}$$

classical probabilities as a polytope

$$\vec{K}_{LHV} = \sum_c \lambda_c \vec{\eta}_c^{[1]}$$

$$\sum_c \lambda_c = 1, \quad \lambda_c \geq 0$$

expectation values $\mathbb{R}^{m_A m_B}$

$$O[\hat{a}_i, \hat{b}_j]$$

basis $\mathbb{R}^{m_A m_B}$

$$\hat{e}_{i,j}$$

classical configurations with no cross talk

$$\vec{\omega}_c^{[1]} = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} A(i) B(j)\, \hat{e}_{i,j}$$

classical observables as a polytope

$$\vec{\pi}_{LHV} = \sum_c \lambda_c \vec{\omega}_c^{[1]}$$

$$\sum_c \lambda_c = 1, \quad \lambda_c \geq 0$$

# Bell... At Last!

Observable $\langle AB \rangle = Pr[A=+1, B=+1] + Pr[A=-1, B=-1]$
$\qquad - Pr[A=+1, B=-1] - Pr[A=-1, B=+1]$

## $m_A = m_B = 2$

classical configurations with no cross talk

$$\vec{\omega}_c^{[1]} = \sum_{i=1}^{2} \sum_{j=1}^{2} A(i)\, B(j)\, \hat{e}_{i,j}$$

explicitly

$$\vec{\omega}_1^{[1]} = \hat{e}_{1,1} + \hat{e}_{1,2} + \hat{e}_{2,1} + \hat{e}_{2,2} \qquad \vec{\omega}_5^{[1]} = -\vec{\omega}_1^{[1]}$$
$$\vec{\omega}_2^{[1]} = \hat{e}_{1,1} + \hat{e}_{1,2} - \hat{e}_{2,1} - \hat{e}_{2,2} \qquad \vec{\omega}_6^{[1]} = -\vec{\omega}_2^{[1]}$$
$$\vec{\omega}_3^{[1]} = \hat{e}_{1,1} - \hat{e}_{1,2} + \hat{e}_{2,1} - \hat{e}_{2,2} \qquad \vec{\omega}_7^{[1]} = -\vec{\omega}_3^{[1]}$$
$$\vec{\omega}_4^{[1]} = \hat{e}_{1,1} - \hat{e}_{1,2} - \hat{e}_{2,1} + \hat{e}_{2,2} \qquad \vec{\omega}_8^{[1]} = -\vec{\omega}_4^{[1]}$$

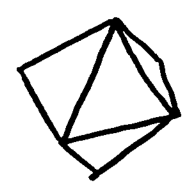
note: not classical configurations with no cross talk

terms like $\hat{e}_{1,1} + \hat{e}_{1,2} + \hat{e}_{2,1} - \hat{e}_{2,2}$

observe: $\vec{\omega}_i^{[1]} \cdot \vec{\omega}_j^{[1]} = 4\delta_{ij} \quad 1 \le i,j \le 4$

3D analogy

extreme points are $\pm$ orthogonal vectors

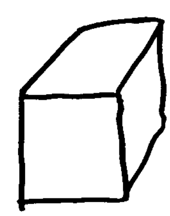


octahedron

Polytope = extreme points $\pm$ orthogonal vectors Polytope $\nearrow$

Polar = extreme points are sum of polytopes extreme points with $\pm 1$ coefficients (normalized)

dual Polytope (polar) $\searrow$



cube

Extreme points of Polar are

$$\vec{\beta}_{s_1,s_2,s_3,s_4} = \frac{1}{4}\left(s_1 \vec{\omega}_1^{[1]} + s_2 \vec{\omega}_2^{[1]} + s_3 \vec{\omega}_3^{[1]} + s_4 \vec{\omega}_4^{[1]}\right)$$

$$s_1, s_2, s_3, s_4 \in \{\pm 1\}$$

explicitly

$$\vec{\beta}_{+1,+1,+1,+1} = -\vec{\beta}_{-1,-1,-1,-1} = +\hat{e}_{1,1}$$

$$\vec{\beta}_{+1,+1,-1,-1} = -\vec{\beta}_{-1,-1,+1,+1} = +\hat{e}_{1,2}$$

$$\vec{\beta}_{+1,-1,+1,-1} = -\vec{\beta}_{-1,+1,-1,+1} = +\hat{e}_{2,1}$$

$$\vec{\beta}_{+1,-1,-1,+1} = -\vec{\beta}_{-1,+1,+1,-1} = +\hat{e}_{2,2}$$

$$\vec{\beta}_{+1,+1,+1,-1} = -\vec{\beta}_{-1,-1,-1,+1} = \frac{1}{2}\left(\hat{e}_{1,1} + \hat{e}_{1,2} + \hat{e}_{2,1} - \hat{e}_{2,2}\right)$$

$$\vec{\beta}_{+1,+1,-1,+1} = -\vec{\beta}_{-1,-1,+1,-1} = \frac{1}{2}\left(\hat{e}_{1,1} + \hat{e}_{1,2} - \hat{e}_{2,1} + \hat{e}_{2,2}\right)$$

$$\vec{\beta}_{+1,-1,+1,+1} = -\vec{\beta}_{-1,+1,-1,-1} = \frac{1}{2}\left(\hat{e}_{1,1} - \hat{e}_{1,2} + \hat{e}_{2,1} + \hat{e}_{2,2}\right)$$

$$\vec{\beta}_{-1,+1,+1,+1} = -\vec{\beta}_{+1,-1,-1,-1} = \frac{1}{2}\left(+\hat{e}_{1,1} - \hat{e}_{1,2} - \hat{e}_{2,1} - \hat{e}_{2,2}\right)$$

Observable give measurement i and j : $\langle AB|ij\rangle$

Inequalities $\vec{x}\cdot\vec{\beta}_i \leq 1$ for iff $\vec{x}\in\Omega$ are therefore

$$|\langle AB|11\rangle| \leq 1, \quad |\langle AB|12\rangle| \leq 1, \quad |\langle AB|21\rangle| \leq 1, \quad |\langle AB|22\rangle| \leq$$

$$\frac{1}{2}\left|\langle AB|11\rangle + \langle AB|12\rangle + \langle AB|21\rangle - \langle AB|22\rangle\right| \leq 1$$

$$\frac{1}{2}\left|\langle AB|11\rangle + \langle AB|12\rangle - \langle AB|21\rangle + \langle AB|22\rangle\right| \leq 1$$

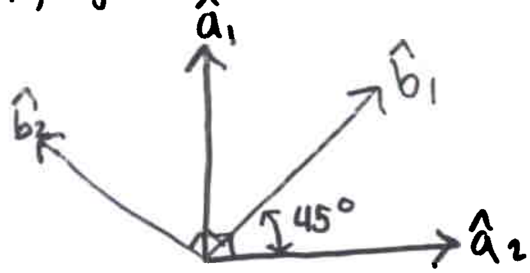
$$\frac{1}{2}\left|\langle AB|11\rangle - \langle AB|12\rangle + \langle AB|21\rangle + \langle AB|22\rangle\right| \leq 1$$

$$\frac{1}{2}\left|-\langle AB|11\rangle + \langle AB|12\rangle + \langle AB|21\rangle + \langle AB|22\rangle\right| \leq 1$$

# Wrong Again CS boy!

Quantum mechanics says

$$\langle AB | \hat{a}_i , \hat{b}_j \rangle = \hat{a}_i \cdot \hat{b}_j$$




$$\langle AB | \hat{a}_1 , \hat{b}_1 \rangle = \langle AB | \hat{a}_1 , \hat{b}_2 \rangle = \langle AB | \hat{a}_2 , \hat{b}_1 \rangle = - \langle AB | \hat{a}_2 , \hat{b}_2 \rangle = \frac{1}{\sqrt{2}}$$

$$\frac{1}{2} \left| \langle AB|11\rangle + \langle AB|12\rangle + \langle AB|21\rangle - \langle AB|22\rangle \right| = \sqrt{2} \nleq 1$$

"It has not yet become obvious to me that there is no real problem ... I have entertained myself always by squeezing the difficulty of quantum mechanics into a smaller and smaller place, so as to get more and more worried about this particular item. It seems almost ridiculous that you can squeeze it to a numerical question that one thing is bigger than another. But there you are — it is bigger..." —Feynman, IJTP, 21 485 (1982)

# Making Bell Inequalities Useful

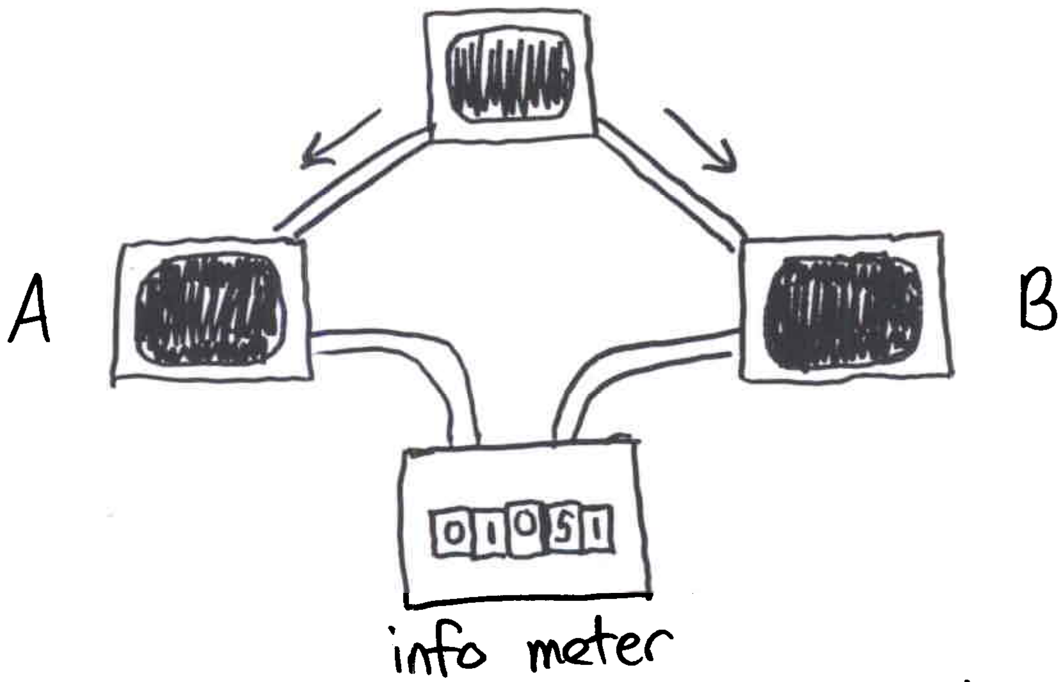
Bell Inequalities test

Quantum Mechanics (is / is not) nonlocal.

<u>not useful</u> as it only tells use one bit of info.

computer scientist asks

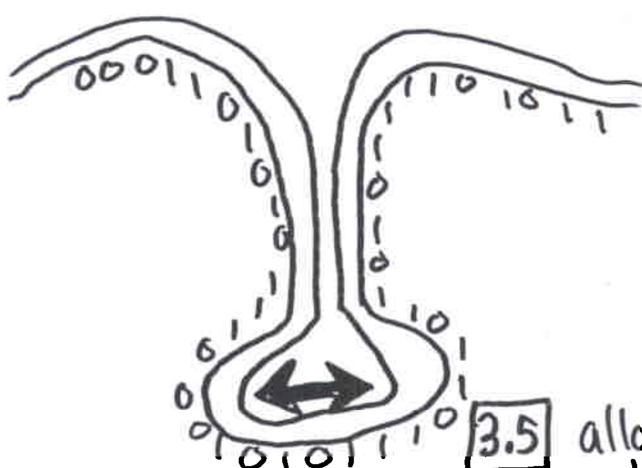
HOW NONLOCAL??




How <u>much</u> information transfer between A and B is needed to produce quantum correlations?

If nature is cheating by sending nonlocal communications, by how much must she cheat??

WARNING!
-ulation.



3.5 allow communication between parties

# Deterministic Protocols With Communication

Deterministic protocols

$$\vec{\eta}_c = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \hat{e}_{i,j,A(i,j),B(i,j)}$$

now allow a bounded amount of communication (possibly two way) between parties.

$$\vec{\eta}_c^{[p]} = \sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \hat{e}_{i,j,A^{[p]}(i,j),B^{[p]}(i,j)}$$

total alphabet size used in protocol

outputs $A^{[p]}(i,j)$, $B^{[p]}(i,j)$ are of a limited form due to boundness of communication.

Example

if $p=2$ (1 bit of communication), then

$$\begin{array}{c|ccc} i \backslash j & 1 & 2 & 3 \\ \hline 1 & +1 & -1 & -1 \\ 2 & -1 & +1 & +1 \\ 3 & -1 & -1 & -1 \end{array}$$

$A^{[2]}(i,j)$

VALID

$$\begin{array}{c|ccc} i \backslash j & 1 & 2 & 3 \\ \hline 1 & +1 & -1 & -1 \\ 2 & -1 & +1 & -1 \\ 3 & -1 & -1 & +1 \end{array}$$

$A^{[2]}(i,j)$

NOT VALID

Study of the form of valid $A^{[p]}(i,j)$ is the realm of

COMMUNICATION COMPLEXITY

?

# Rip Off

Accessible polytope with $\log_2 p$ bits communication

$$\Omega^{[p]} = \{\vec{x} \mid \vec{x} = \sum_i \lambda_i \vec{\eta}^{[p]}_{c(i)}, \ \sum_i \lambda_i = 1, \ \lambda_i \geq 0\}$$

$$\Omega^{[p]} \subseteq \Omega^{[q]}, \ p<q$$

For the polytope $\Omega^{[p]}$ we can use its dual to find iff conditions for $\vec{x} \in \Omega^{[p]}$

## Bell Inequalities with Communication

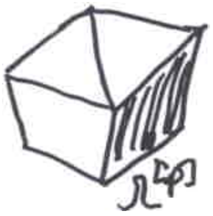
Set of $\vec{\beta}_i^{[p]}$ such that $\vec{x} \cdot \vec{\beta}_i^{[p]} \leq 1$ is an iff condition for producing probabilities $\vec{x}$ with $\log_2 p$ bits of communication.

note if $m_A \lesssim p$ or $m_B \lesssim p$ then all $A^{[p]}(i,j)$, $B^{[p]}(i,j)$ can be produced
$\Rightarrow$ interesting only to consider $m_A, m_B > p$

note Polytope $\Omega^{[p]}$ has faces with at most $K(p)$ vertices associated with each face. Number of random bits needed for the protocols scales like $\log_2 K(p)$.

note $K(p) \geq 4 m_A m_B \Rightarrow$ in the limit of simulating all projective measurements, infinite random bits needed

# Facet Enumeration Is Hard



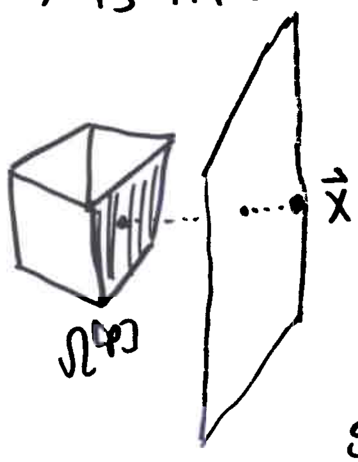

Algorithms for facet enumeration

input: extreme point $\vec{c}_i^{[p]}$

output: facet defining inequalities $\vec{x}\cdot\vec{\beta}_i \leq 1$

In general, NP-hard problem.

Algorithm for determining whether point $\vec{x}$ (produced say by QM) is in $\Omega^{[p]}$?




exists a hyperplane such that $\vec{x}$ is on one side and $\Omega^{[p]}$ is on the other side?

Find $z_0 \in \mathbb{R}$ $\vec{z} \in \mathbb{R}^d$

satisfying $\vec{z}\cdot\vec{c}_i^{[p]} \leq z_0 \quad \forall i$

$\vec{z}\cdot\vec{x} \geq z_0$

or

$$\underbrace{\begin{aligned} &\text{maximize: } \vec{z}\cdot\vec{x} - z_0 \\ &\text{subject to: } \vec{z}\cdot\vec{c}_i^{[p]} - z_0 \leq 0 \quad \forall i \\ &\qquad\qquad \vec{z}\cdot\vec{x} - z_0 \leq 1 \leftarrow \text{to make bounde} \end{aligned}}_{\text{linear programming problem}}$$

<u>easier</u>

# This Is Canada, We Do Hard Things

First nontrivial case, $m_A = m_B = 3$, $p = 2$ (1 bit.)

Look at expectation values $\langle AB(i,j)\rangle$, $i,j \in \{1,2,3\}$

Two Bell Inequalities with 1 bit of Communication

trivial: $|\langle AB(i,j)\rangle| \le 1$

nontrivial: $\left|\sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij} \langle AB(i,j)\rangle\right| \le 1$

$$M_{ij} = \begin{pmatrix} 0 & -1 & +1 \\ -1 & +1 & +1 \\ +1 & +1 & +1 \end{pmatrix} \times \frac{1}{6}$$

or any matrix obtained from this $M_{ij}$ via permuting rows and columns and/or multiplying rows and columns by $\pm 1$

$$M_{ij} = \frac{1}{11}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

EXAMPLE: $M_{ij} = \begin{pmatrix} +1 & -1 & -1 \\ 0 & -1 & +1 \\ +1 & +1 & +1 \end{pmatrix}$

explicitly

$$|-\langle AB(1,2)\rangle + \langle AB(1,3)\rangle - \langle AB(2,1)\rangle + \langle AB(2,2)\rangle + \langle AB(2,3)\rangle + \langle AB(3,1)\rangle + \langle AB(3,2)\rangle + \langle AB(3,3)\rangle| \le 6$$

$$|1\langle AB(11)\rangle + 2\langle AB(12)\rangle - 2\langle AB(13)\rangle + 2\langle AB(21)\rangle + 1\langle AB(22)\rangle + 2\langle AB(23)\rangle - 2\langle AB(31)\rangle + 2\langle AB(32)\rangle + 1\langle AB(33)\rangle| \le 11$$

# The Easy "f" of "Iff"

Look at

$$P_{ij} = \begin{pmatrix} +1 & -1 & +1 \\ -1 & +1 & +1 \\ +1 & +1 & +1 \end{pmatrix} \qquad Q_{ij} = \begin{pmatrix} -1 & -1 & +1 \\ -1 & +1 & +1 \\ +1 & +1 & +1 \end{pmatrix}$$

with associated vectors

$$\vec{\alpha} = \sum_{i=1}^{3}\sum_{j=1}^{3} P_{ij}\,\hat{e}_{i,j} \qquad \vec{\beta} = \sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij}\,\hat{e}_{i,j}$$

note: $\vec{\alpha}$ and $\vec{\beta}$ are not in $\Omega^{[2]}$, but are extreme points of $\Omega^{t}$

Consider

$$\vec{\alpha}\cdot\vec{\eta}_c^{[2]} \quad \text{and} \quad \vec{\beta}\cdot\vec{\eta}_c^{[2]}$$

1st note

$$|\vec{\alpha}\cdot\vec{\eta}_c^{[2]}| \le 7 \qquad \text{(at least one "mis match")}$$

$$|\vec{\alpha}\cdot\vec{\eta}_c^{[2]}| \in \{1,3,5,7\}$$

if $\vec{\eta}_c^{[2]}$ such that $|\vec{\alpha}\cdot\vec{\eta}_c^{[2]}| \le 7$, then $|\overset{\beta}{\vec{\alpha}}\cdot\vec{\eta}_c^{[2]}| \le 5$

such that $|\vec{\alpha}\cdot\vec{\eta}_c^{[2]}| \le 5$, then $|\underset{\beta}{\vec{\alpha}}\cdot\vec{\eta}_c^{[2]}| \le 7$

$$\Rightarrow \left|\tfrac{1}{2}(\vec{\alpha}+\vec{\beta})\cdot\vec{\eta}_c\right| \le 6$$

$$\tfrac{1}{2}(P_{ij}+Q_{ij}) = \begin{pmatrix} 0 & -1 & +1 \\ -1 & +1 & +1 \\ +1 & +1 & +1 \end{pmatrix} = M_{ij}$$

# Useless?

For Bell Pair

$$\langle AB(\hat{a}_i, \hat{b}_j)\rangle = -\hat{a}_i \cdot \hat{b}_j$$

inequality becomes

$$\tfrac{1}{6}\left| \hat{a}_1\cdot\hat{b}_2 - \hat{a}_1\cdot\hat{b}_3 + \hat{a}_2\cdot\hat{b}_1 - \hat{a}_2\cdot\hat{b}_2 - \hat{a}_2\cdot\hat{b}_3 - \hat{a}_3\cdot\hat{b}_1 - \hat{a}_3\cdot\hat{b}_2 - \hat{a}_3\cdot\hat{b}_3 \right| \le 1$$

$$\left|\hat{a}_1\cdot(\hat{b}_2 - \hat{b}_3) + \hat{a}_2\cdot(\hat{b}_1 - \hat{b}_2 - \hat{b}_3) + \hat{a}_3\cdot(-\hat{b}_1 - \hat{b}_2 - \hat{b}_3)\right| \le 6$$

maximize lhs respect to $\hat{a}_i$:

$$\max \text{lhs} = |\hat{b}_2 - \hat{b}_3| + |\hat{b}_1 - \hat{b}_2 - \hat{b}_3| + |-\hat{b}_1 - \hat{b}_2 - \hat{b}_3|$$

use

$$\max_{\vec{a},\vec{b}} |\vec{a}+\vec{b}| + |\vec{a}-\vec{b}| \le 2\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$$

to obtain

$$\max \text{lhs} \le \sqrt{2} + 2\sqrt{3} \simeq 4.878 \le 6$$

similar argument for other inequali

$\Rightarrow$ Bell experiment with three measurements can be simulated with one bit of communication

$m_A = 3$, $m_B = m$ also holds.

What about $m_A = m_B = 4$?

Computer unable to do full facet enumeration in time < 1 week.

# Fin At Last. Fin At Last! Glory, Glory, Fin At Last!

What are the costs of simmulating quantum correlations?

(Projective measurements on a Bell pair) = (shared randomness) + (1 bit of communication)

(One bit of communication) + (shared randomness) > (Simmulation to produce joint expectation value of two outcome measurements with each party selecting one of three measurements)