ADDITIVITY OF ENTANGLEMENT OF FORMATION

$\Updownarrow$?

ADDITIVITY OF HOLEVO CHANNEL CAPACITY

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(1) ENTANGLEMENT OF FORMATION

LET ρ BE A STATE ON $\mathcal{H}_A \otimes \mathcal{H}_B$ FINITE

$$E_f(\rho) := \min\left\{ \sum_i p_i E(\psi_i) : \sum_i p_i \psi_i = \rho \right\}$$

PURE STATES

$$E(\psi_{AB}) := S(\mathrm{Tr}_A \psi_{AB})$$ "ENTROPY OF ENTANGLEMENT"

[INTRODUCED IN B/D/S/W, PRA 54, 3824 (1996)]

"$E_f(\rho)$ IS MINIMAL AMOUNT OF ENTANGLEMENT* NEEDED BETWEEN A & B TO CREATE ρ, USING LOCC"

PROBLEM WITH *: $E(\psi)$ IS "ENTANGLEMENT OF ψ" ONLY ASYMPTOTICALLY (IF YOU INSIST ON MAKING OPERATIONAL SENSE...).

$\Rightarrow$ RATHER LOOK AT ASYMPTOTIC COST OF MAKING $\rho^{\otimes n}$, $n \to \infty$!

THM [H/H/T, J. PHYS. A, 34, 6891 (2001)]

$$E_c(\rho) = \lim_{n\to\infty} \frac{1}{n} E_f(\rho^{\otimes n}).$$

IS THIS DIFFERENT FROM $E_f(\rho)$?

OBSERVE: $E_c(\rho) = E_f(\rho)$

IFF

$\forall n \; E_f(\rho^{\otimes n}) = n \cdot E_f(\rho)$

$\Uparrow$

$$\boxed{\text{ADDITIVITY: } \forall \rho,\sigma \; E_f(\rho\otimes\sigma) = E_f(\rho) + E_f(\sigma)}$$

$$\boxed{\forall \rho, \sigma \quad E_f(\rho \otimes \sigma) = E_f(\rho) + E_f(\sigma)}$$

- "$\leq$" IS TRIVIAL BY DEFINITION
- WIDELY CONJECTURED TO BE TRUE
  - PROVED IN SOME CASES
    - VIDAL, DÜR, CIRAC, PRL 89, 027901 (2002)
    - THIS WORK
    - FAN, quant-ph/0210169

2) HOLEVO CAPACITY

LET $T: \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H}_B)$ BE A CHANNEL (C.P.T.P. LINEAR MAP)

$$C(T) := \max_{\substack{\text{ALL ENSEMBLES} \\ \{p_i, \psi_i\} \text{ ON } \mathcal{H}_0}} \left\{ S\left(\sum_i p_i T(\psi_i)\right) - \sum_i p_i S(T(\psi_i)) \right\}$$

"CLASSICAL CAPACITY* OF T"

*: CAPACITY (OPERATIONALLY) INVOLVES BLOCK CODING. $C(T)$ IS THE CAPACITY ONLY IF THE ENCODING OF MESSAGES USES PRODUCT STATES $\psi_1 \otimes \dots \otimes \psi_n$ [HOLEVO–SCHUMACHER/WESTMORELAND]

THM.

THE CAPACITY OF T (WITHOUT THE PRODUCT RESTRICTION) IS

$$\overline{C}(T) = \lim_{n \to \infty} \frac{1}{n} C(T^{\otimes n}).$$

IS THIS DIFFERENT FROM C(T)?

NOTE: $\bar{C}(T) = C(T)$

IFF

$\forall n \quad C(T^{\otimes n}) = n\,C(T)$

$\Uparrow$

$$\boxed{\text{ADDITIVITY: } \forall T, T' \quad C(T \otimes T') = C(T) + C(T')}$$

- "$\geq$" IS TRIVIAL BY DEFINITION
- PROVED IN A NUMBER OF CASES:
  - $T = id$ (SCHUMACHER/WESTMORELAND)
  - T = UNITAL QUBIT CHANNEL (KING)
  - T = GEN. DEPOLARISING CHANNEL (KING)
  - T = ENTANGLEMENT-BREAKING CHANNEL (SHOR)

(3) A LINK BETWEEN THE PROBLEMS

I. CAN ALWAYS PRESENT T AS A PARTIAL TRACE (STINESPRING, ESSENTIALLY):

$$
\begin{tikzcd}
\mathcal{B}(\mathcal{H}_0) \arrow[r, "T"] \arrow[dr, "\mathcal{U}"'] & \mathcal{B}(\mathcal{H}_B) \\
& \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \arrow[u, "\mathrm{Tr}_A"'] \arrow[ul, phantom, "\equiv"]
\end{tikzcd}
$$

$\mathcal{U}$ (ISOMETRY $\mathcal{H}_0 \hookrightarrow \mathcal{H}_A \otimes \mathcal{H}_B$)

$$\mathcal{K} := \mathcal{U}(\mathcal{H}_0) < \mathcal{H}_A \otimes \mathcal{H}_B$$

I.E. $T \cong \mathrm{Tr}_A \big|_{\mathcal{B}(\mathcal{K})}$

II. INPUT ENSEMBLES $\{p_i, \psi_i\}$ ON $\mathcal{H}_0 / \mathcal{K}$ ARE DECOMPOSITIONS OF $\rho = \sum_i p_i \psi_i$

III. $$C(T) = \max_{\rho = \sum_i p_i \psi_i} \left\{ S(\mathrm{Tr}_A \rho) - \sum_i p_i S(\mathrm{Tr}_A \psi_i) \right\}$$

$$= \max_{\rho} \left\{ S(\mathrm{Tr}_A \rho) - E_f(\rho) \right\}$$

THM. $T = \mathrm{Tr}_{A}|_{\mathcal{K}}$

$$C(T) = \max_{\rho} \left\{ S(\mathrm{Tr}_A \rho) - E_f(\rho) \right\}$$

ρ (SUPPORTED) ON $\mathcal{K} < \mathcal{H}_A \otimes \mathcal{H}_B$

COR. IF $C(T \otimes T') = C(T) + C(T')$ AND OPTIMAL ρ, ρ' FOR T, T', RESP., THEN

$$E_f(\rho \otimes \rho') = E_f(\rho) + E_f(\rho').$$

[OTHERWISE, IF "<":

$$C(T) + C(T') = S(\mathrm{Tr}_A \rho) - E_f(\rho) + S(\mathrm{Tr}_A \rho') - E_f(\rho')$$

$$< S(\mathrm{Tr}_A(\rho \otimes \rho')) - E_f(\rho \otimes \rho')$$

$$\leq C(T \otimes T') \quad \lightning$$
]

... MANY EXAMPLES, INCLUDING SOME FROM VIDAL/DÜR/CIRAC (WHO USE T = ENT.-BREAKING!)

EXAMPLE:

UNITAL QUBIT CHANNEL (USING NORMAL FORM OF KING/RUSKAI)

$$T : \underset{|\phi\rangle\langle\phi|}{\overset{\in B(\mathbb{C}^2)}{\alpha}} \longmapsto \sum_{i=0,x,y,z} p_i \sigma_i \alpha \sigma_i^*$$

$$\left[2p_0+2p_z-1 \geq |2p_0+2p_x-1|, |2p_0+2p_y-1|\right]$$

$$\begin{array}{ccc} \alpha = |\phi\rangle\langle\phi| & \longmapsto & \sum_i p_i \sigma_i \alpha \sigma_i^* \\ & \searrow^{\mathcal{U}} & \big\updownarrow \mathrm{Tr}_{\mathbb{C}^4} \\ & & \mathcal{U}|\phi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^4 \end{array}$$

ISOMETRY OF BLOCK FORM

$$\mathcal{U} = \begin{bmatrix} \sqrt{p_0}\,\mathbb{1} \\ \sqrt{p_x}\,\sigma_x \\ \sqrt{p_y}\,\sigma_y \\ \sqrt{p_z}\,\sigma_z \end{bmatrix}$$

OPTIMAL ρ: EQUAL MIXTURE OF

$$|\psi_T\rangle = |0\rangle \otimes (\sqrt{p_0}|0\rangle + \sqrt{p_z}|z\rangle) + |1\rangle \otimes (\sqrt{p_x}|x\rangle + i\sqrt{p_y}|y\rangle)$$

$$|\psi_T^\perp\rangle = |0\rangle \otimes (\sqrt{p_x}|x\rangle - i\sqrt{p_y}|y\rangle) + |1\rangle \otimes (\sqrt{p_0}|0\rangle - \sqrt{p_z}|1\rangle)$$

$$\Rightarrow E_f(\rho_T) = E_c(\rho_T) = H(p_0+p_z, 1-p_0-p_z)$$

COR. IF WE COULD PROVE THAT AN OPTIMAL STATE FOR $T \otimes T'$ IS $\rho \otimes \rho'$, ADD. OF E_f WOULD IMPLY ADD. OF C !

[PROBLEM: IN GENERAL THERE ARE MANY OPTIMAL INPUT STATES — SOME ENTANGLED...]

(OPTIMAL) OUTPUT STATE $T(\rho) = \mathrm{Tr}_A \rho$! $\omega(T)$ HOWEVER, IS UNIQUE (SEE E.G. SCHUMACHER/WESTMORELAND)

(ESSENTIALLY STRICT CONCAVITY OF S ...)

$\Longrightarrow$ PROMISING FIRST STEP (?):

PROVE $\omega(T \otimes T') = \omega(T) \otimes \omega(T')$

(4) THOUGHTS ON UNIFICATION: A STRONGER CONJECTURE

RECALL THAT FOR E_f WE NEED TO PROVE $E_f(\rho \otimes \rho') \geq E_f(\rho) + E_f(\rho')$

WHY RESTRICT TO PRODUCT STATES ??

<u>CONJ.</u>: FOR ANY STATE $\rho = \rho_{ABA'B'}$!

$$E_f(\rho) \geq E_f(\rho_{AB}) + E_f(\rho_{A'B'})$$

STATED BY VOLLBRECHT & WERNER, PRA <u>64</u>, 062307, 2001.

(NOTE THAT IT SUFFICES TO PROVE THIS FOR <u>PURE</u> $\rho = |\psi\rangle\langle\psi|$, WHEN $E_f(\rho) = S(\mathrm{Tr}_{AA'}|\psi\rangle\langle\psi|)$.)

<u>THM.</u> IF TRUE, THIS CONJECTURE (CALLED <u>SUPERADDITIVITY OF E_f</u>) IMPLIES ADDITIVITY OF BOTH E_f & OF C. ALSO, FOR CAPACITY WITH COST CONSTRAINTS.

<u>NOTE:</u> PROVED IN SOME CASES!