Outline

1. Diagonal Qubit Channels.

2. Diagonal 3D Channels.

3. Depolarizing Channel.

4. Open Questions.

Qubit channels.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Pauli matrices

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

State:

$$\rho = \frac{1}{2}\left(I + \sum_{i=1}^{3} w_i \sigma_i \right)$$

$$\sum_{i=1}^{3} w_i^2 \leq 1 \quad ; \; w_i \in \mathbb{R}$$

Diagonal Qubit Channel:

$$\Phi : [\lambda_1, \lambda_2, \lambda_3] \qquad \lambda_i \in \mathbb{R}$$

$$\Phi : \rho = \frac{1}{2}\left(I + \sum_{i=1}^{3} w_i \sigma_i \right)$$

$$\longmapsto \frac{1}{2}\left(I + \sum_{i=1}^{3} \lambda_i w_i \sigma_i \right)$$

Φ is linear and unital : $\Phi(I) = I$

Conditions for C.P.:

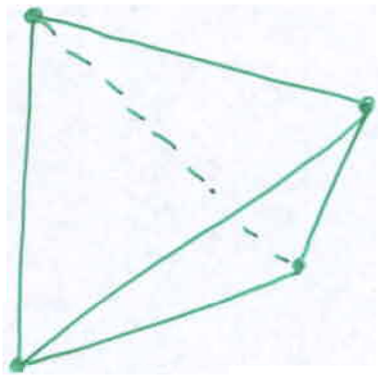
$$1 + \lambda_1 + \lambda_2 + \lambda_3 \geq 0$$

$$1 + \lambda_1 - \lambda_2 - \lambda_3 \geq 0$$

$$1 - \lambda_1 + \lambda_2 - \lambda_3 \geq 0$$

$$1 - \lambda_1 - \lambda_2 + \lambda_3 \geq 0$$

$\Longleftrightarrow (\lambda_1, \lambda_2, \lambda_3)$ in tetrahedron in $\mathbb{R}^3$



<u>Corners:</u>

$[1, 1, 1] \qquad \rho \longmapsto \rho$

$[1, -1, -1] \qquad \rho \longmapsto \sigma_3 \rho \sigma_3$

$[-1, 1, -1] \qquad \rho \longmapsto \sigma_2 \rho \sigma_2$

$[-1, -1, 1] \qquad \rho \longmapsto \sigma_1 \rho \sigma_1$

<u>Edges:</u> $[\lambda, \lambda, 1]$

$$\rho \longmapsto \lambda \rho + (1-\lambda) \rho_{diag}$$

"phase-damping channel".

Diagonal action extends to product channel:

state ρ_{12} on $\mathbb{C}^2 \otimes \mathbb{C}^K$:

$$\rho_{12} = I \otimes A_0 + \sum_{j=1}^{3} \sigma_j \otimes A_j$$

If $\Phi = [\lambda_1, \lambda_2, \lambda_3]$

Ω = CP channel on $\mathbb{C}^{K \times K}$:

$$(\Phi \otimes \Omega)(\rho_{12}) = I \otimes A_0 + \sum_{j=1}^{3} \lambda_j \sigma_j \otimes A_j$$

$\Rightarrow$ can compute capacity to carry classical information via possibly entangled input states & entangled measurements.

Results.

$[\lambda_1, \lambda_2, \lambda_3]$ "behaves like" symmetric binary channel with transition matrix

$$P_{ij} = P(o/p = j \mid i/p = i) = \lambda \delta_{ij} + \frac{1-\lambda}{2}$$

where $\lambda = \max_i |\lambda_i|$.

Additional benefit:

any 2D unital channel Φ can be written as

$$\Phi = \Gamma_U \circ [\lambda_1, \lambda_2, \lambda_3] \circ \Gamma_V$$

conjugate by unitary

$\Rightarrow$ compute capacity of Φ also.

3 dimensions?

back to 2D : "derive" Pauli matrices

$\|\varphi_0\rangle$	$\|\varphi_1\rangle$	$\|\varphi_0\rangle\langle\varphi_0\| - \|\varphi_1\rangle\langle\varphi_1\|$
$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix} = \sigma_1$
$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \sigma_2$
$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \sigma_3$

complementary orthonormal bases

$$\left|\langle\psi_a^i|\psi_b^j\rangle\right|^2 = \frac{1}{2}$$

all $i \neq j \in \{1,2,3\}$

all $a,b \in \{0,1\}$

$\underline{\underline{d=3}}$: find 4 orthonormal bases

$\{ |\varphi_a^j\rangle \}$ $\qquad j = 0, 1, 2, 3$

$\qquad a = 0, 1, 2$

such that

$$|\langle \varphi_a^i | \varphi_b^j \rangle|^2 = \frac{1}{3} \qquad \text{all } i \neq j$$

$$\text{all } a, b$$

$$\boxed{\omega = e^{\frac{2\pi i}{3}}}$$

$j = 0$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \omega \\ \bar{\omega} \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \bar{\omega} \\ \omega \end{pmatrix}$
$j = 1$	$\frac{1}{\sqrt{3}}\begin{pmatrix} \bar{\omega} \\ 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \bar{\omega} \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ \bar{\omega} \end{pmatrix}$
$j = 2$	$\frac{1}{\sqrt{3}}\begin{pmatrix} \omega \\ 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \omega \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ \omega \end{pmatrix}$
$j = 3$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Recall: Pauli $|\varphi_0\rangle\langle\varphi_0| - |\varphi_1\rangle\langle\varphi_1|$

Now $|\varphi_0\rangle\langle\varphi_0| + \omega|\varphi_1\rangle\langle\varphi_1| + \bar{\omega}|\varphi_2\rangle\langle\varphi_2| = P_j$

and $|\varphi_0\rangle\langle\varphi_0| + \bar{\omega}|\varphi_1\rangle\langle\varphi_1| + \omega|\varphi_2\rangle\langle\varphi_2| = P_j^*$

$j=0$	$P_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$P_0^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$j=1$	$P_1 = \begin{pmatrix} 0 & \bar{\omega} & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}$	$P_1^* = \begin{pmatrix} 0 & 0 & \bar{\omega} \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$j=2$	$P_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ \bar{\omega} & 0 & 0 \end{pmatrix}$	$P_2^* = \begin{pmatrix} 0 & 0 & \omega \\ \bar{\omega} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$j=3$	$P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}$	$P_3^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix}$

Properties.

$$P_a P_a^* = I, \qquad P_a^3 = I, \quad P_a^* = P_a^2$$

$$\mathrm{Tr}\,(P_a P_b) = \mathrm{Tr}\,(P_a P_b^*) = 0 \qquad \text{for } a \neq b.$$

9 matrices

$\{I, P_i, P_i^*\}$ closed under multiplication.

e.g.

$$P_0 P_1 = \bar{\omega} P_2^*$$
$$P_1 P_2^* = \omega P_3^*$$
$$P_2^* P_3^* = \bar{\omega} P_0^*$$
$$P_3^* P_0^* = \omega P_1^*$$
$$P_0^* P_1^* = \bar{\omega} P_2$$
$$P_1^* P_2 = \omega P_3$$
$$P_2 P_3 = \bar{\omega} P_0$$
$$P_3 P_0 = \omega P_1$$

c.f.

$$\sigma_1 \sigma_2 = i \sigma_3$$
$$\sigma_2 \sigma_3 = i \sigma_1$$
$$\sigma_3 \sigma_1 = i \sigma_2$$

State ρ on $\mathbb{C}^3$ has unique representation

$$\rho = \frac{1}{3}\left(I + \sum_{i=0}^{3} a_i P_i + \sum_{i=0}^{3} \bar{a}_i P_i^* \right)$$

$a_i \in \mathbb{C}$.

Conditions on $\{a_i\}$ for $\rho \geq 0$?

Complicated ... not needed here.

Diagonal 3D channels.

$$\rho = \frac{1}{3}\left(I + \sum a_i P_i + \sum \bar{a}_i P_i^* \right)$$

$$\mapsto \frac{1}{3}\left(I + \sum_{i=0}^{3} z_i a_i P_i + \sum_{i=0}^{3} \bar{z}_i \bar{a}_i P_i^* \right)$$

Represent by $[z_0, z_1, z_2, z_3]$.

Conditions for CP?

Recall 2D :

$$1 + \alpha\lambda_1 + \beta(\lambda_2 + \alpha\lambda_3) \geq 0$$

$$\alpha = \pm 1$$

$$\beta = \pm 1$$

3D:

$$1 + \alpha z_0 + \bar\alpha z_0 + \beta\left(z_1 + \alpha z_2 + \bar\alpha z_3\right)$$

$$+ \bar\beta\left(\bar z_1 + \bar\alpha \bar z_2 + \alpha \bar z_3\right) \geq 0$$

$$\alpha, \beta \in \{1, \omega, \bar\omega\}$$

9 hyperplanes

9 extreme points:

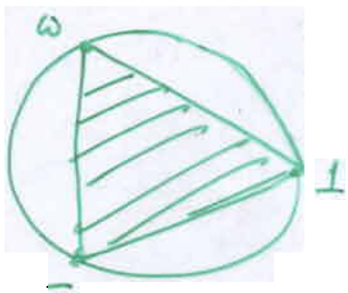
$[1, 1, 1, 1]$ $\qquad \rho \mapsto \rho$

$[1, \omega, \omega, \omega]$ $\qquad \rho \longmapsto P_0 \rho P_0^*$

$[1, \bar\omega, \bar\omega, \bar\omega]$ $\qquad \rho \longmapsto P_0^* \rho P_0$

etc. $\qquad \rho \longmapsto P_j \rho P_j^*$

Hence $\quad z_i = c_0 + c_1\omega + c_2\bar\omega \quad ; \quad c_0 + c_1 + c_2 = 1$



$[z_0, z_1, z_2, z_3]$

$= \cancel{\sum c_i \rho + P_i \rho P_i^*}$

$$[z_0, z_1, z_2, z_3] = c_0 \rho + \sum_{i=0}^{3} \left(d_i P_i \rho P_i^* + f_i P_i^* \rho P_i \right)$$

<u>Goal:</u> compute information-carrying capacity of these diagonal channels.

So far ... only done for depolarizing channel.

<u>Depolarizing channel</u>. $(d=3)$

$$\Delta_\lambda(\rho) = \lambda \rho + \frac{1-\lambda}{3} I \qquad \left(-\frac{1}{8} \le \lambda \le 1\right).$$

$$\Delta_\lambda = [\lambda, \lambda, \lambda, \lambda]$$

<u>Result:</u> capacity is same as classical channel

$$P_{ij} = \lambda \delta_{ij} + \frac{1-\lambda}{3}$$

Idea of proof: uses non-commutative p-norms

$$\|A\|_p = \left(\mathrm{Tr}\,|A|^p\right)^{\frac{1}{p}} \qquad (p \geq 1)$$

Channel Φ: $\nu_p(\Phi) = \sup_{\rho} \|\Phi(\rho)\|_p$ [Amosov, Holevo, Werner]

Show

$$\boxed{\nu_p(\Delta_\lambda \otimes \Omega) = \nu_p(\Delta_\lambda) \cdot \nu_p(\Omega)}$$

for all channels Ω, all $p \geq 1$.

Hence deduce additivity of Holevo capacity, hence compute capacity of Δ_λ.

Main idea:

reduce to phase-damping channel

$$\Phi_\lambda : \rho \longmapsto \lambda\rho + (1-\lambda)\,\rho_{diag}$$

$$[\lambda, \lambda, \lambda, 1]$$

Acting on bi-partite state on $\mathbb{C}^3 \otimes \mathbb{C}^K$:

$$\begin{pmatrix} A_1 & C_{12} & C_{13} \\ C_{12}^* & A_2 & C_{23} \\ C_{13}^* & C_{23}^* & A_3 \end{pmatrix}$$

$$\Phi_\lambda \otimes I : \longmapsto \begin{pmatrix} A_1 & \lambda C_{12} & \lambda C_{13} \\ \lambda C_{12}^* & A_2 & \lambda C_{23} \\ \lambda C_{13}^* & \lambda C_{23}^* & A_3 \end{pmatrix}$$

Generalize the Lieb – Ruskai bound: $\forall \ p \geq 1$,

$$\operatorname{Tr} \begin{pmatrix} A_1 & \lambda C_{12} & \lambda C_{13} \\ \lambda C_{12}^* & A_2 & \lambda C_{23} \\ \lambda C_{13}^* & \lambda C_{23}^* & A_3 \end{pmatrix}^p \leq \frac{1}{3} \left(\operatorname{Tr} A_1^p + \operatorname{Tr} A_2^p + \operatorname{Tr} A_3^p \right) \cdot \operatorname{Tr} \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix}^p$$

say

$$\rho_{12} = \begin{pmatrix} \alpha_1 & * & * \\ * & \alpha_2 & * \\ * & * & \alpha_3 \end{pmatrix} \quad \text{state on } \mathbb{C}^3 \otimes \mathbb{C}^K$$

Φ_λ = phase-damping channel on $\mathbb{C}^{3\times 3}$

Ω = C.P. channel on $\mathbb{C}^{K\times K}$

Suppose also that

$$\operatorname{Tr} \alpha_1 = \operatorname{Tr} \alpha_2 = \operatorname{Tr} \alpha_3 = \frac{1}{3}$$

Consider

$$(\Phi_\lambda \otimes \Omega)(\rho_{12}) = \begin{pmatrix} A_1 & \lambda * & \lambda * \\ \lambda * & A_2 & \lambda * \\ \lambda * & \lambda * & A_3 \end{pmatrix}$$

with $\quad A_i = \Omega(\alpha_i) \Rightarrow \operatorname{Tr} A_i = \frac{1}{3}$

L–R bound:

$$\operatorname{Tr} \left|(\Phi_\lambda \otimes \Omega)(\rho_{12})\right|^p$$

$$\leq \frac{1}{3}\left(\operatorname{Tr} A_1^p + \operatorname{Tr} A_2^p + \operatorname{Tr} A_3^p\right) \cdot \operatorname{Tr} \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix}^p$$

$$\leq \frac{1}{3}\left[\nu_p(\Omega)\right]^p \cdot \left[\left(\tfrac{1}{3}\right)^p + \left(\tfrac{1}{3}\right)^p + \left(\tfrac{1}{3}\right)^p\right] \cdot \operatorname{Tr} \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix}^p$$

$$\left(\operatorname{Tr}\left[\Omega(\alpha)\right]^p\right)^{\frac{1}{p}} \leq \operatorname{Tr}(\alpha) \cdot \nu_p(\Omega)$$

Also, let

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{on} \quad \mathbb{C}^3$$

$$\Rightarrow \quad |\psi\rangle\langle\psi| = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{3}\left(I + P_0 + P_0^{*} \right)$$

$$\Delta_\lambda(|\psi\rangle\langle\psi|) = \frac{1}{3} \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix}$$

$$\Rightarrow \quad \operatorname{Tr} \left| (\Phi_\lambda \otimes \Omega)(\rho_{12}) \right|^p$$

$$\leq [\nu_p(\Omega)]^p \quad \operatorname{Tr} \quad \Delta_\lambda(|\psi\rangle\langle\psi|)^p$$

$$\leq [\nu_p(\Omega)]^p \; [\nu_p(\Delta_\lambda)]^p$$

//

How to reduce from depolarizing to phase-damping? Write Δ_λ as a convex combination — easy using the diagonal representation:

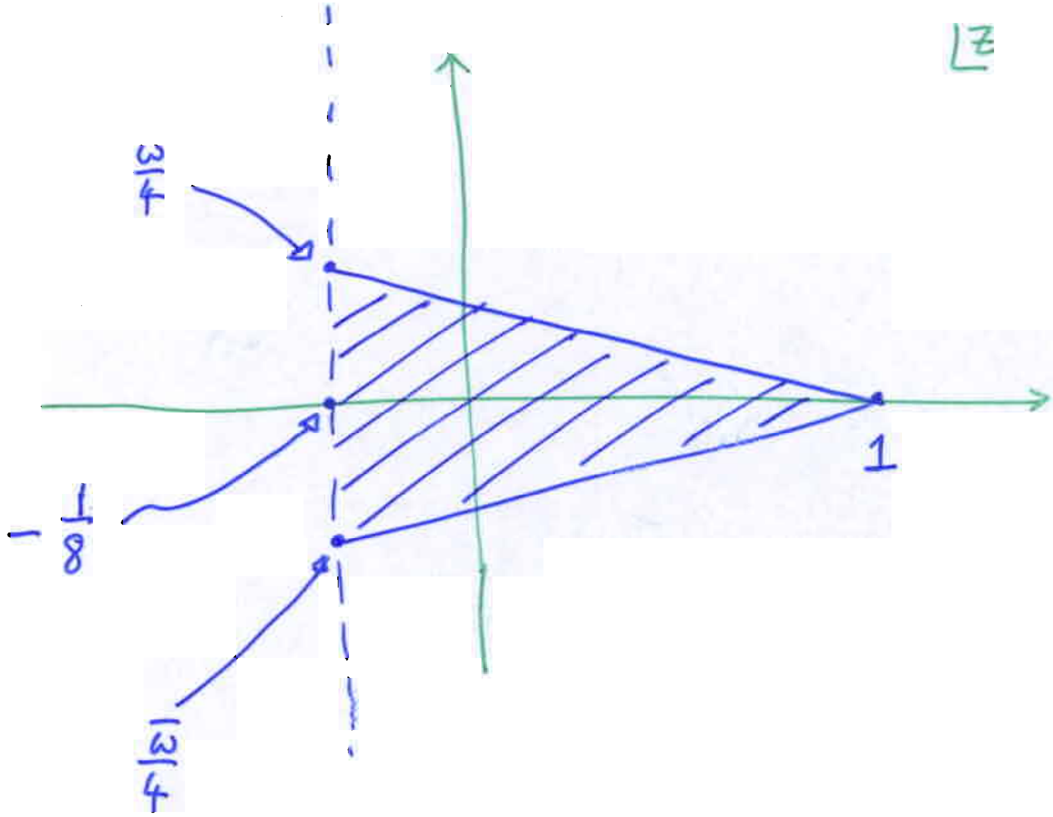
$$\Delta_\lambda = \frac{1+8\lambda}{9(1+2\lambda)}\Big\{[1,\lambda,\lambda,\lambda] + [\lambda,1,\lambda,\lambda] + [\lambda,\lambda,1,\lambda]\Big\}$$

$$+ \frac{1-\lambda}{9(1+2\lambda)}\Big\{[\omega,\omega\lambda,\omega\lambda,\lambda] + [\omega\lambda,\omega,\omega\lambda,\lambda] + [\omega\lambda,\omega\lambda,\omega,\lambda]\Big\}$$

$$+ \frac{1-\lambda}{9(1+2\lambda)}\Big\{[\bar\omega,\bar\omega\lambda,\bar\omega\lambda,\lambda] + [\bar\omega\lambda,\bar\omega,\bar\omega\lambda,\lambda] + [\bar\omega\lambda,\bar\omega\lambda,\bar\omega,\lambda]\Big\}$$

Some special diagonal channels.

$z \in \mathbb{C}$: $[z, z, z, z]$



"generalized depolarizing channels"

Partial symmetries: [c.f. 2-Pauli channel in $d=2$]

$[\lambda, \lambda, \lambda, x]$ $\qquad x = \dfrac{8\lambda^2+\lambda-1}{3+5\lambda}$

$[\lambda, \lambda, z, \bar{z}]$ $\qquad z = \dfrac{1}{2+\lambda}\left[3\lambda^2 + (1-\lambda)(\lambda\omega+\bar{\omega})\right]$

Try to imitate $d=2$ result:

use partial symmetries to bootstrap up ....

Work in progress!!