Entanglement-Breaking Channels

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M.B. Ruskai, "Entanglement Breaking Channels"
quant-ph/0201700 submitted to *Rev. Math. Phys.*

and related work for $d > 2$ by M. Horodecki and P. Shor

M.B. Ruskai, S. Szarek and E. Werner,
"An analysis of completely positive trace- preserving maps on $\mathcal{M}_2$"
Lin. Alg. Appl. **347**, 159–187 (2002). quant-ph/0101003

P. Shor, "Additivity of the Classical Capacity of
Entanglement-Breaking Quantum Channels"
J. Math. Phys. **43,** 4334-4340 (2002). quant-ph/0201149

Let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a linear map on op. alg., e.g. $n \times n$ matrices

- positivity preserving if $P > 0 \quad \Rightarrow \quad \Phi(P) > 0$

- completely positive if pos. pres, on $\mathbf{C}^{n \times n} \otimes \mathcal{A}_1$, i.e.,

  $\Gamma > 0 \quad \Rightarrow \quad (I \otimes \Phi)(\Gamma) > 0 \qquad \Gamma$ in $\mathbf{C}^{n \times n} \otimes \mathcal{A}_1$.

  By Choi suffices to consider Γ max entang.

- entanglement breaking if $(I \otimes \Phi)(\Gamma)$ separable, i.e.,

  $(I \otimes \Phi)(\Gamma) = \sum_k t_k \gamma_k \otimes \rho_k$ for all Γ in $\mathbf{C}^{n \times n} \otimes \mathcal{A}_1$

  Φ maps entangled states to separable ones

Holevo channel $\Omega(P) = \sum_k R_k \,\mathrm{Tr}\,(PX_k)$ where
each R_k a density matrix and
$\{X_k\}$ a POVM (recall $X_k > 0$ & $\sum_k X_k = I$)

Special cases CQ: $X_k = |e_k\rangle\langle e_k|$ QC: $R_k = |e_k\rangle\langle e_k|$

Point: POVM $= \{I\}$ so that $\Omega(P) = R_0 \;\; \forall \; P$ [also CQ]

Thm: (M. Horodecki and Shor) Φ is Ent Break $\Leftrightarrow$ Holevo

Thm: (Shor) If Φ Ent Break and Ω arbitrary
a) minimal entropy of $\Phi \otimes \Omega$ additive, and
b) Holevo capacity of $\Phi \otimes \Omega$ additive

Set of E.B. channels is convex. What are extreme points?
$\{X_k\}$ and extreme POVM and all R_k pure NOT sufficient

Thm: For qubit channels, the following are equivalent

A) Φ has the Holevo form $\Phi(P) = \sum_k R_k \, \text{Tr}\,(PX_k)$.

B) Φ is entanglement breaking.

C) $\Phi \circ T$ is completely positive, where $T(\rho) = \rho^T$ is the transpose.

D) Φ has "sign-change" property: changing any $\lambda_k \to -\lambda_k$
in canon. param. yields another completely positive map.

E) Φ is in the convex hull of CQ maps.

For $d > 2$ have only (E) $\Rightarrow$ (A) $\Leftrightarrow$ (B) $\Rightarrow$ (C) See erratum at end

Can get (C') $\Rightarrow$ (B) if T replaced by a set of entang witnesses

Recall TFAE for $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$

- Completely Positive and Trace-Preserving

- Stinespring: can find reps π_j of algs s.t.

$$\Phi[\pi_1(B)] = V[\pi_2\Phi(B)]V^\dagger, \quad V^\dagger V = I$$

- Kraus (also Choi): can find A_k s.t.

$$\Phi(P) = \sum_k A_k P A_k^\dagger, \qquad \sum_k A_k^\dagger A_k = I$$

unital, i.e. $\Phi(I_1) = I_2 \;\Leftrightarrow \sum_k A_k A_k^\dagger = I$

non-unique, but can define canonical

- Stinespring/Lindblad: can find $\mathcal{H}_B$, Q_B, unitary U s.t.

$$\Phi(P) = \text{Tr}_B[UP \otimes Q_B U^\dagger]$$

- Choi $(I \otimes \Phi)(\Gamma_C) > 0$ where Γ_C max entang

Closer look at Choi's Thm. and consequences

Horodecki reform $\phi = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |jj\rangle$ max entang

$$\Gamma = |\phi\rangle\langle\phi| \;=\; \frac{1}{\sqrt{d}} \sum_{jk} |j\rangle\langle k| \otimes |j\rangle\langle k|$$

$$(I \otimes \Phi)(|\phi\rangle\langle\phi|) \;=\; \frac{1}{\sqrt{d}} \sum_{jk} |j\rangle\langle k| \otimes \Phi(|j\rangle\langle k|) > 0$$

Choi matrix Γ_C is $d^2 \otimes d^2$ with $d \times d$ blocks E_{jk}
 where $E_{jk} = |j\rangle\langle k|$ has 1 in j-k spot, 0's elsewhere

Get 1-1 corr. between linear op on $\mathbf{C}^{d \times d}$ and states on $\mathbf{C}^d \otimes \mathbf{C}^d$
 both $d^2 \times d^2$ $\quad\quad \Phi \;\leftrightarrow\; (I \otimes \Phi)(\Gamma_C)$

Aside: Eigenvecs of $(I \otimes \Phi)(\Gamma_C)$ are $d^2 \times 1$: corr. to $d \times d$
 yields Kraus ops (see Leung quant-ph/0201119 JMP)

Let $G_1 \ldots G_{d^2}$ be O.N. basis for $\mathbf{C}^{d \times d}$

$\langle G_m, G_n \rangle = \text{Tr } G_m^\dagger G_n = \delta_{mn}$

Can rep Φ by matrix $g_{mn} = \text{Tr } G_m^\dagger \Phi(G_n)$

If $G_0 = \frac{1}{\sqrt{d}} I$, then $\text{Tr}\, G_m = 0$ for $m = 1, 2 \ldots d^2 - 1$ so that trace-preserving, implies first row has form $(\,1\ 0\ 0 \ldots\ 0\,)$

$d = 2$: Pauli matrices natural choice $\{I, \sigma_x, \sigma_y, \sigma_z\}$

One extension: $d > 2$, $G_0 = \frac{1}{\sqrt{d}} I$, $G_1 \ldots G_{d-1}$ diagonal

rest: $G_d \ldots G_{d^2-1}$ have form $\frac{1}{\sqrt{2}}(E_{jk} + E_{kj})$ self-adjoint

Downside: Density matrix $\rho = G_0 + \sum_{k=1}^{d^2-1} u_k G_k$

no simple cond on u_k guarantee ρ pos semi-def

unrelated to mult props of Pauli which can extend for $d = 2^n$

Qubit channels: Rep. Φ in basis $\{I, \sigma_x, \sigma_y, \sigma_z\}$ for $\mathbf{C}^{2\times 2}$

Density matrix $\rho = \frac{1}{2}[I + \mathbf{w} \cdot \sigma]$ where $\mathbf{w}$ in $\mathbf{R}^3$

ρ a one-dim proj (pure state) $\Leftrightarrow$ $|\mathbf{w}| = 1.$

After rotation and diag can assume wlog ϕ is rep. by

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \Lambda \end{pmatrix}$$

or, equivalently,

$$\Phi : \ \tfrac{1}{2}\left[I + \mathbf{w}\cdot\sigma\right] \ \mapsto \ \tfrac{1}{2}\left[I + \sum_k (t_k + \lambda_k w_k)\sigma_k\right]$$

Image $\Phi(\rho)$ is translated ellipsoid

$$\left(\frac{x_1 - t_1}{\lambda_1}\right)^2 + \left(\frac{x_2 - t_2}{\lambda_2}\right)^2 + \left(\frac{x_3 - t_3}{\lambda_3}\right)^2 = 1$$

But NOT all ellipsoids come from CPT map — need conds on t_k, λ_k

Rep. Φ in form $\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \Lambda \end{pmatrix}$ in fixed basis

$\Phi(\rho) \to \Phi(\sigma_j^\dagger \rho \sigma_j)$ takes $\lambda_k \to -\lambda_k$ for $k \neq j$: change two signs

$\Phi(\rho) \to \Phi(\rho^T)$ takes $\lambda_2 \to -\lambda_2$: change one sign — not C.P.

Non-diag form requires 12 parameters
 Two rotations use 6 param — leaves 6 in above canon form

Use convex subsets: All stochastic (CPT) Φ for qubits 12 param
 $\supset$ all CPT Φ with T in canon form in fixed basis 6 param
 $\supset$ all CPT Φ with basis and (t_1, t_2, t_3) fixed 3 param
  special case: $\mathbf{t} = (0, 0, 0)$ gives unital $\Phi(I) = I$
 $\supset$ all CPT Φ with basis, (t_1, t_2, t_3) and λ_3 fixed 2 param
  can plot in 2-dim λ_1, λ_2 use $\lambda_\pm = \lambda_1 \pm \lambda_2$

In each case, Ent.-Break Φ also convex subset
 in fact intersection with one (or three) sign changes for λ_k.

Choi's C.P. cond for $d = 2$:

$$(I \otimes \Phi)(\Gamma_C) > 0 \;\Leftrightarrow\; R_\Phi^\dagger R_\Phi \leq I$$

$$\begin{aligned}(I \otimes \Phi)(\Gamma_C) &= \begin{pmatrix} \Phi(E_{11}) & \Phi(E_{12}) \\ \Phi(E_{21}) & \Phi(E_{22}) \end{pmatrix} \\ &= \begin{pmatrix} \Phi(E_{11}) & \sqrt{\Phi(E_{11})}R_\Phi\sqrt{\Phi(E_{22})} \\ \sqrt{\Phi(E_{22})}R_\Phi^\dagger\sqrt{\Phi(E_{11})} & \Phi(E_{22}) \end{pmatrix}\end{aligned}$$

actually apply to adjoint $\widehat{\Phi}$

$$R_\Phi = \begin{pmatrix} \frac{t_1 + it_2}{(1+t_3+\lambda_3)^{1/2}(1-t_3-\lambda_3)^{1/2}} & \frac{\lambda_1+\lambda_2}{(1+t_3+\lambda_3)^{1/2}(1-t_3+\lambda_3)^{1/2}} \\ \frac{\lambda_1-\lambda_2}{(1+t_3-\lambda_3)^{1/2}(1-t_3-\lambda_3)^{1/2}} & \frac{t_1+it_2}{(1+t_3-\lambda_3)^{1/2}(1-t_3+\lambda_3)^{1/2}} \end{pmatrix}$$

Can rewrite $I - R_\Phi^\dagger R_\Phi = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} > 0$ as ineq for λ_k, t_k

Have reduced pos semi-def conds from 4×4 to 2×2

$$I - R_\Phi^\dagger R_\Phi = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} > 0$$

Diag conditions using $\lambda_\pm = \lambda_1 \pm \lambda_2$

$$\begin{aligned} m_{11} \geq 0 &\Leftrightarrow |\lambda_+|^2 \leq |1 + \lambda_3|^2 - |\mathbf{t}|^2 \pm \ldots \\ m_{22} \geq 0 &\Leftrightarrow |\lambda_-|^2 \leq |1 - \lambda_3|^2 - |\mathbf{t}|^2 \pm \ldots \end{aligned}$$

and $\quad \det(I - R_\Phi^\dagger R_\Phi) = m_{11}m_{22} - |m_{12}|^2 \geq 0$

Extreme points need equality in $m_{kk} \geq 0$ and det is redundant.

In most other situations, $\det M > 0$ is stronger. (Fig. 1)

Can plot allowed $\lambda_\pm = \lambda_1 \pm \lambda_2$ for Ent-Break maps with fixed (t_1, t_2, t_3) and λ_3 using det cond with and without sign change

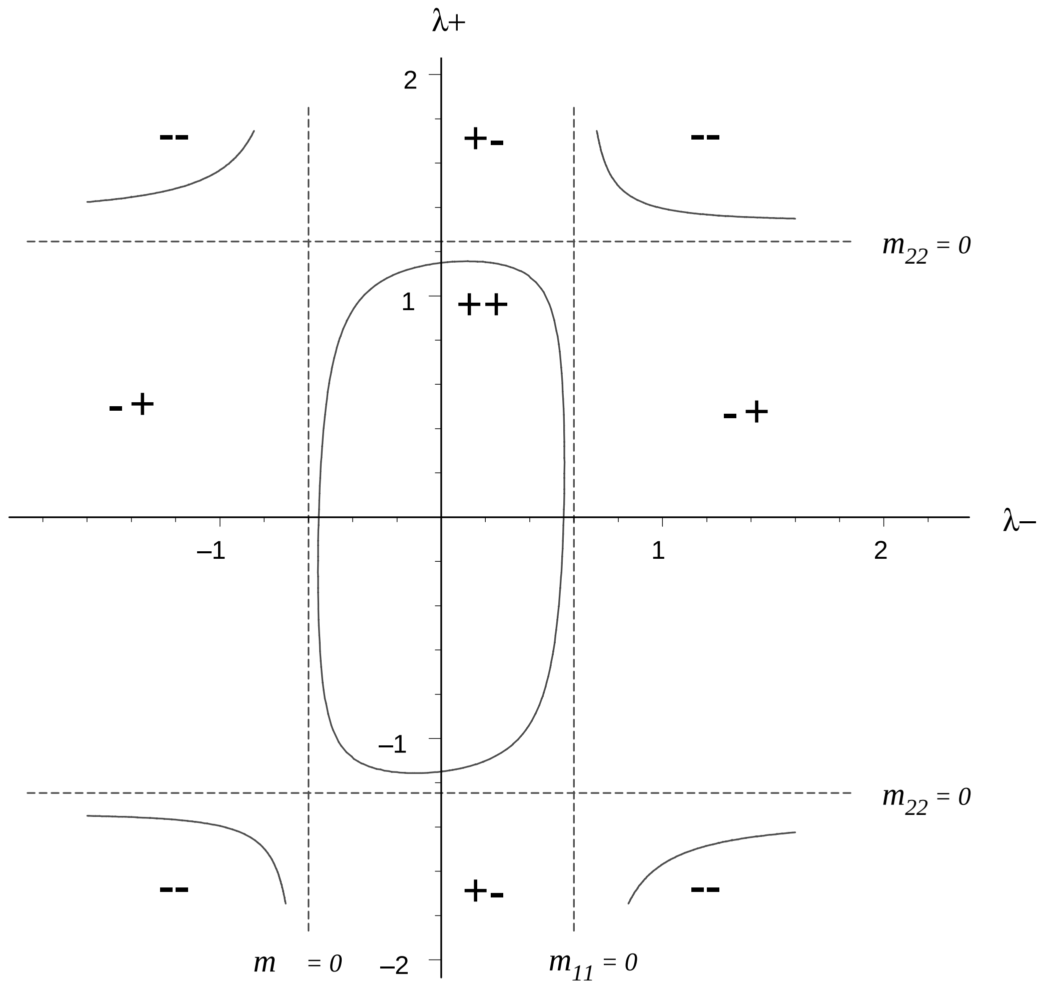

Figure 1: The λ_+-λ_- plane showing the regions described by the diagonal conditions (dotted lines) and the curves corresponding to $\det(I - R_\Phi^\dagger R_\Phi) = 0$ for $\mathbf{t} = (0.2, 0.3, 0)$ and $\lambda_3 = 0.35$. The closed curve and its interior describes the parameters for which the corresponding map is completely positive.

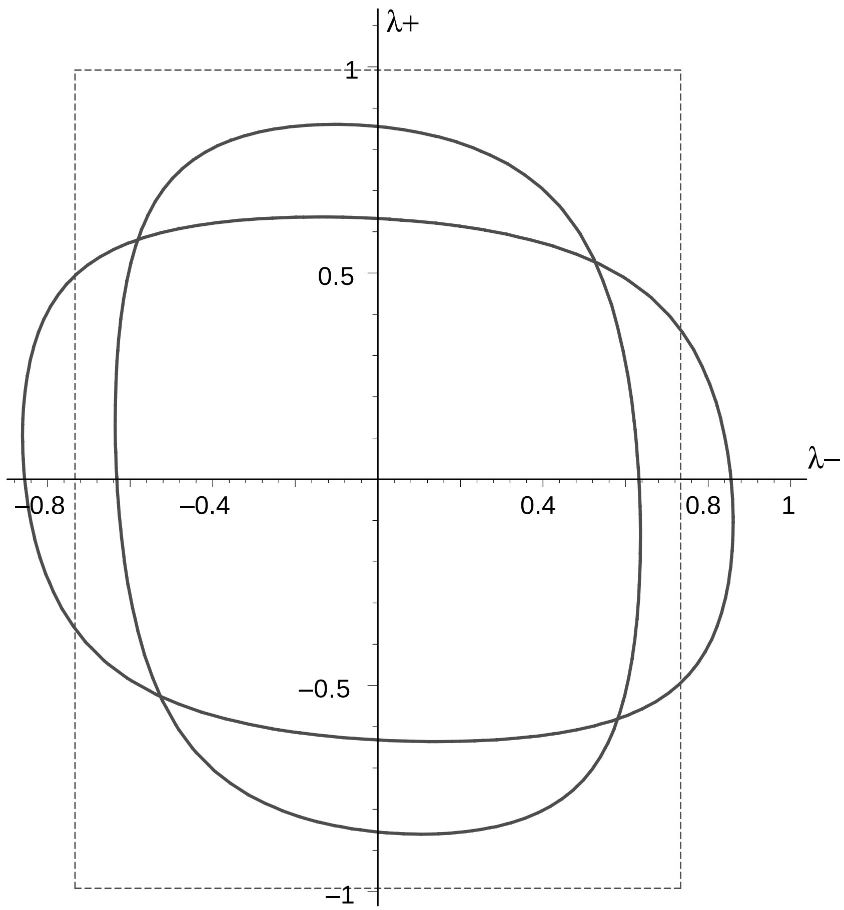

Figure 2: The λ_+-λ_- plane showing the region determined by determinant condition when $\mathbf{t} = (0.4, 0.3, 0.0)$ and $\lambda_3 = 0.15$ and the corresponding region with λ_+ andλ_- interchanged. The intersection corresponds to the entanglement breaking maps with the indicated parameters.

qubit QC and CQ channels

recall param and qubit extreme points

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos u & 0 & 0 \\ 0 & 0 & \cos v & 0 \\ \sin u \sin v & 0 & 0 & \cos u \cos v \end{pmatrix}$$

QC and CQ both have two $\lambda_k = 0$: Bloch sphere maps to line

QC: $\Phi : \ \frac{1}{2}[I + \mathbf{w}\cdot\sigma] \ \mapsto \ \frac{1}{2}[I + (t_3 + \lambda_3 w_3)\sigma_3]$

CQ: $\Phi : \ \frac{1}{2}[I + \mathbf{w}\cdot\sigma] \ \mapsto \ \frac{1}{2}[I + t_1\sigma_1 + \lambda_3 w_3 \, \sigma_3]$ shift orthog to line

extreme $\mapsto \ \frac{1}{2}[I + \cos\theta \, \sigma_1 + \sin\theta \, w_3 \, \sigma_3]$

Qubit map is both extreme and Ent. Break $\Leftrightarrow$ CQ

turns out (non-trival) all extreme pts of Qubit E.B. are CQ

Thm: Φ Ent-Break $\Rightarrow \;\sum_k |\lambda_k| \leq 1$

Interp: E.B. maps are "noisy" (extends to $d > 2$)

BUT some noisy Φ are not E.B.

extreme quibt Φ no E.B. unless image is point $\Leftrightarrow \lambda_k = 0$

(cond for extreme to be CQ)

extreme (amp. damp) can have $\sum_k |\lambda_k|$ very small not be E.B.

Thm: Unital qubit Φ Ent-Break $\Leftrightarrow \sum_k |\lambda_k| \leq 1$

Fig 3: Octahedron of unital qubit E.B.

Fig 4: Rounding of tetrahedron for fixed $\mathbf{t} = (0, 0, t_3)$ with $t_3 \neq 0$.

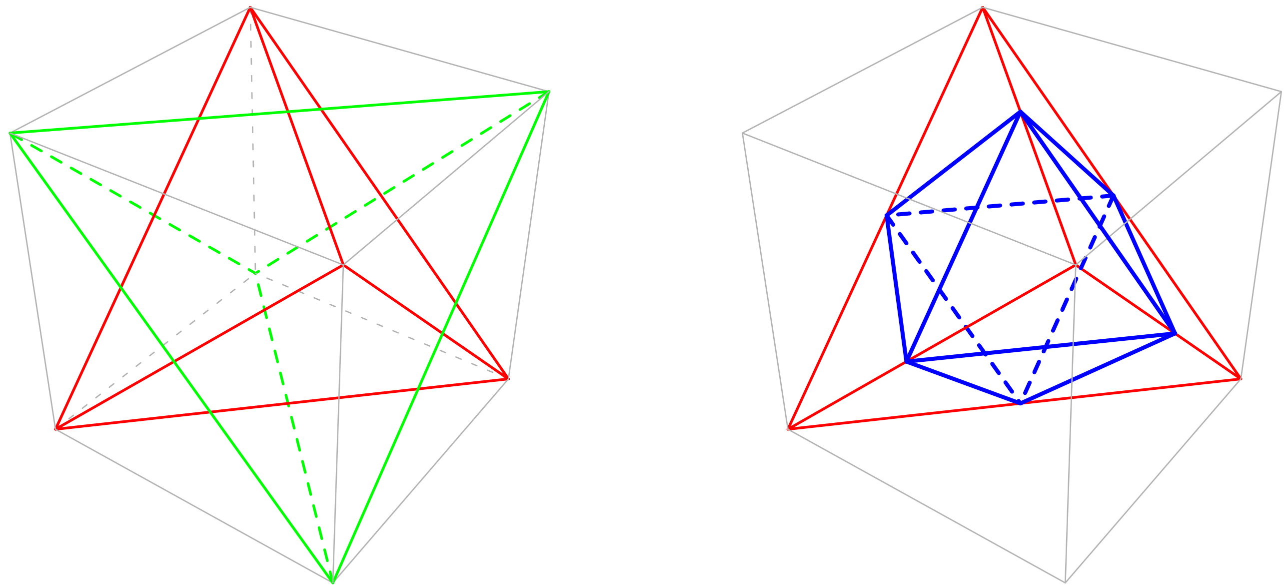

Figure 3: The tetrahedron of bistochastic maps and its inversion through the origin (left); their intersection gives the octahedron of unital entanglement breaking maps (right).

(Figures by K. Durstberger appeared in R.A. Bertlmann, H. Narnhofer and W. Thirring "A Geometric Picture of Entanglement and Bell Inequalities" quant-ph/0111116.)

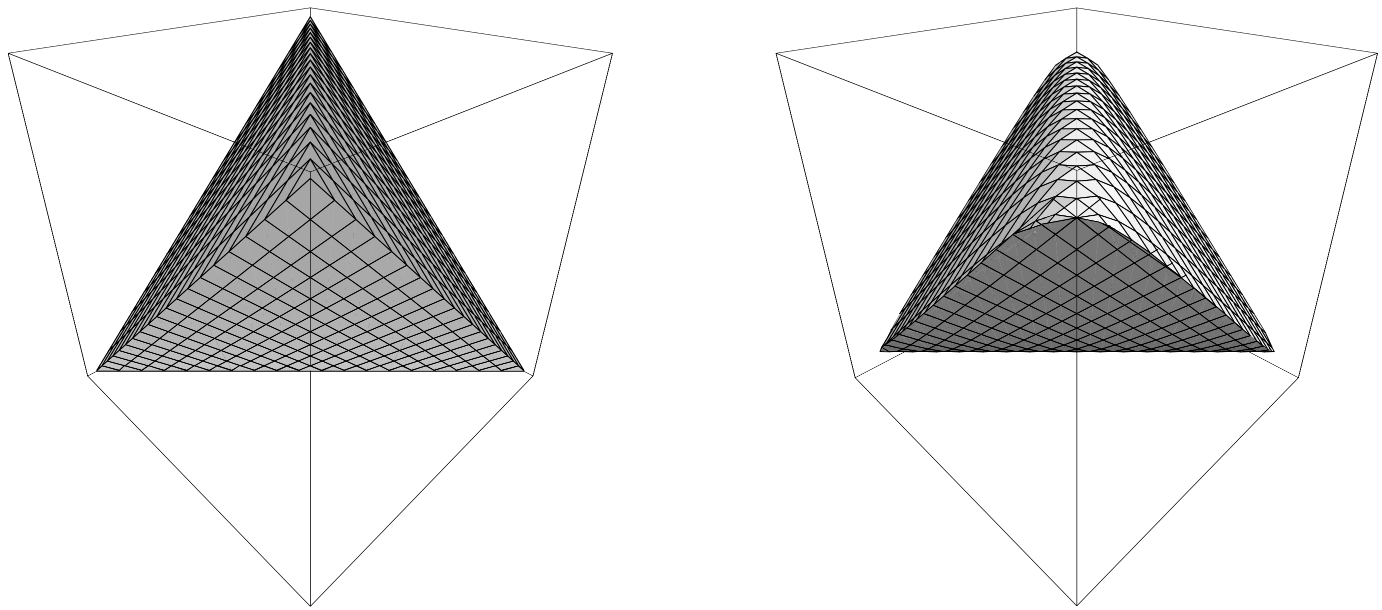

Figure 4: Tetrahedron of unital maps (left); and rounding of tetrahedron (right) which occurs for $\mathbf{t} = (0, 0, t_3)$ with $t_3 \neq 0$ in 3 param space of λ_k
(From Ruskai, Szarek and Werner)

In 12 parameter space of all qubit CPT maps, NO straight edges. .

Why are CQ only extreme points of qubit E.B. maps ?

Thm: For qubit CPT Φ either

(I) Φ generalized extreme point (R_Φ unitary) OR

(II) Φ in interior of plane in convex set of all CPT.

$\Rightarrow$ No edges except for unital tetrahedron

Can show that if $R_\Phi = V \begin{pmatrix} \cos\theta_1 & 0 \\ 0 & \cos\theta_2 \end{pmatrix} W^\dagger$ is not unitary, then it can be written in two distinct ways as conv comb of unitary

Note: Any contraction R (e.g., unitary U) defines a CPT Φ via $\Phi(E_{12}) = [\Phi(E_{11})]^{-1/2} R [\Phi(E_{22})]^{-1/2}$

BUT (subtle point) not all U yield Φ in canonical form even when convex comb is — need full 12 param space

For qubit channels, the following are equivalent

A) Φ has the Holevo form $\Phi(P) = \sum_k R_k \, \text{Tr}\,(PX_k)$.

B) Φ is entanglement breaking.

C) $\Phi \circ T$ is completely positive, where $T(\rho) = \rho^T$ is the transpose.

D) Φ has "sign-change" property: changing any $\lambda_k \to -\lambda_k$ in canon. param. yields another completely positive map.

E) Φ is in the convex hull of CQ maps.

For $d > 2$ have only (E) $\Rightarrow$ (A) $\Leftrightarrow$ (B) $\Rightarrow$ (D) $\Rightarrow$ S_{jk} $\Rightarrow$ (C)

Erratum: Need to Replace (D) and S_{jk} conditions – See last slide

For $d > 2$ (E) $\Rightarrow$ (A) $\Leftrightarrow$ (B) $\Rightarrow$ (D) $\Rightarrow$ S_{jk} $\Rightarrow$ (C)

For $d = 3$ Shor found extreme E.B. channel which is not CQ
extreme point of E.B. which not extreme in all CPT

Do NOT expect $\Phi \circ T$ C.P. $\Rightarrow$ E.B. for $d > 2$ because there are
channels which break PPT entang., but preserve other types
indpe by (a) Horodecki's and (b) Shor + subset of IBM group

Below is Not True !!

BUT Φ E.B. $\Rightarrow$ stronger cond than $\Phi \circ T$ is Comp Pos.

Let S_{jk} denote "selective transpose", i.e.,
$S_{jk}(A)$ swaps only particular $a_{jk} \leftrightarrow a_{jk}$

Then Φ E.B. $\Rightarrow$ $\Phi \circ S_{jk}$ also C.P. for every fixed $\{j, k\}$

BUT S_{jk} not even pos. preserving — can't be entang. witness

$\Phi \circ S_{jk}$ also C.P. is a very strong condition

The claim that Φ Ent Break implies a "sign-change" conditons is false for $d > 2$.

The point is that "sign-change" or "selective transpose" preserves only the $\sum_k E_k = I$ property, but not the $E_k > 0$ property needed for a POVM

Instead we have only the much weaker statement that a C.P. map Φ is Ent. Break $\Leftrightarrow$ $\Phi \circ \Upsilon$ is also C.P. $\Leftrightarrow$ $\Upsilon \circ \Phi$ is also C.P. for any positivity preserving map Υ.

If we know a set of entanglement witnesses for the space on which Φ acts, then it suffices to check the above for Υ in this set.