The communication cost of entanglement transformations

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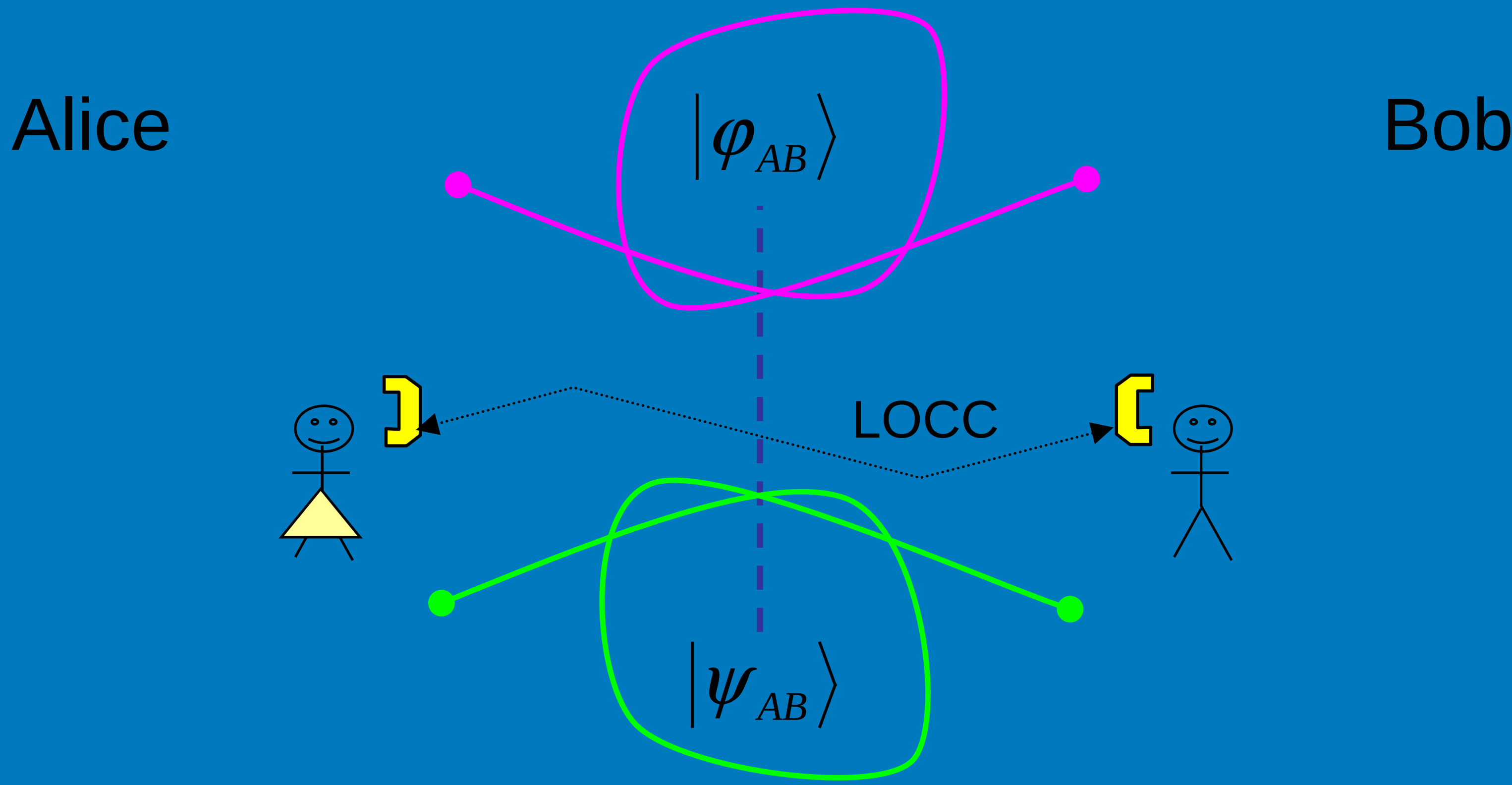
Timely advice: Wim van Dam and Eric Rains

Outline

- Part I: Microscopic view (exact)
  [with Daftuar, work in progress]
- Part II: Asymptotic view (approximate)
  [with Winter, quant-ph/0204092]

A starting point



$\Leftrightarrow \varphi_A \,\square\, \psi_A$ [Nielsen99]

Paying the phone bill

$$|\varphi_{AB}\rangle \xrightarrow{LOCC} |\psi_{AB}\rangle \quad \Leftrightarrow \quad \varphi_A \,\square\, \psi_A$$

log *n* is the number of bits that need to be communicated (almost)
(See Harrow & Lo 2002.)

$$\Leftrightarrow \varphi_A = \sum_{i=1}^{n} \underbrace{p_i U_i \psi_A U_i^*}$$

spectrum: $p_i \lambda(\psi_A)$

Add up n matrices with spectra proportional to $\lambda(\psi_A)$.
What are the possible sums?

HORN'S PROBLEM!

Horn's Problem

Given $\lambda(X), \lambda(Y), \lambda(Z)$ do there exist Hermitian matrices X, Y, Z such that $X + Y + Z = 0$?

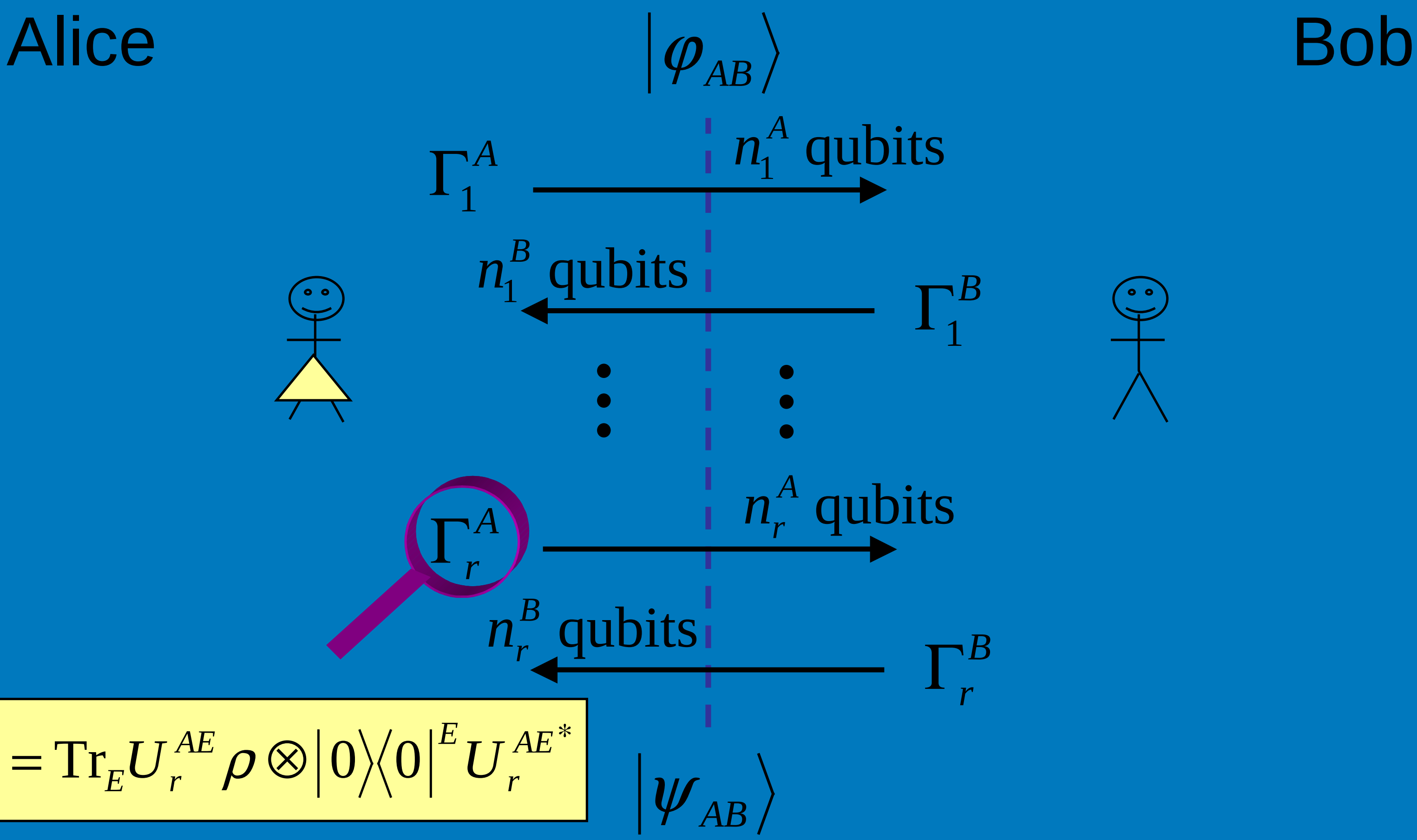
Problem solved by:
Klyachko, Helmke, Rosenthal, Totaro, Knutson, Tao…

One consequence in this context:
All p_i may be taken to equal to *1/n*.

$$\varphi_A = \sum_{i=1}^{n} p_i U_i \psi_A U_i^* = \tfrac{1}{n} \sum_{i=1}^{n} V_i \psi_A V_i^*$$

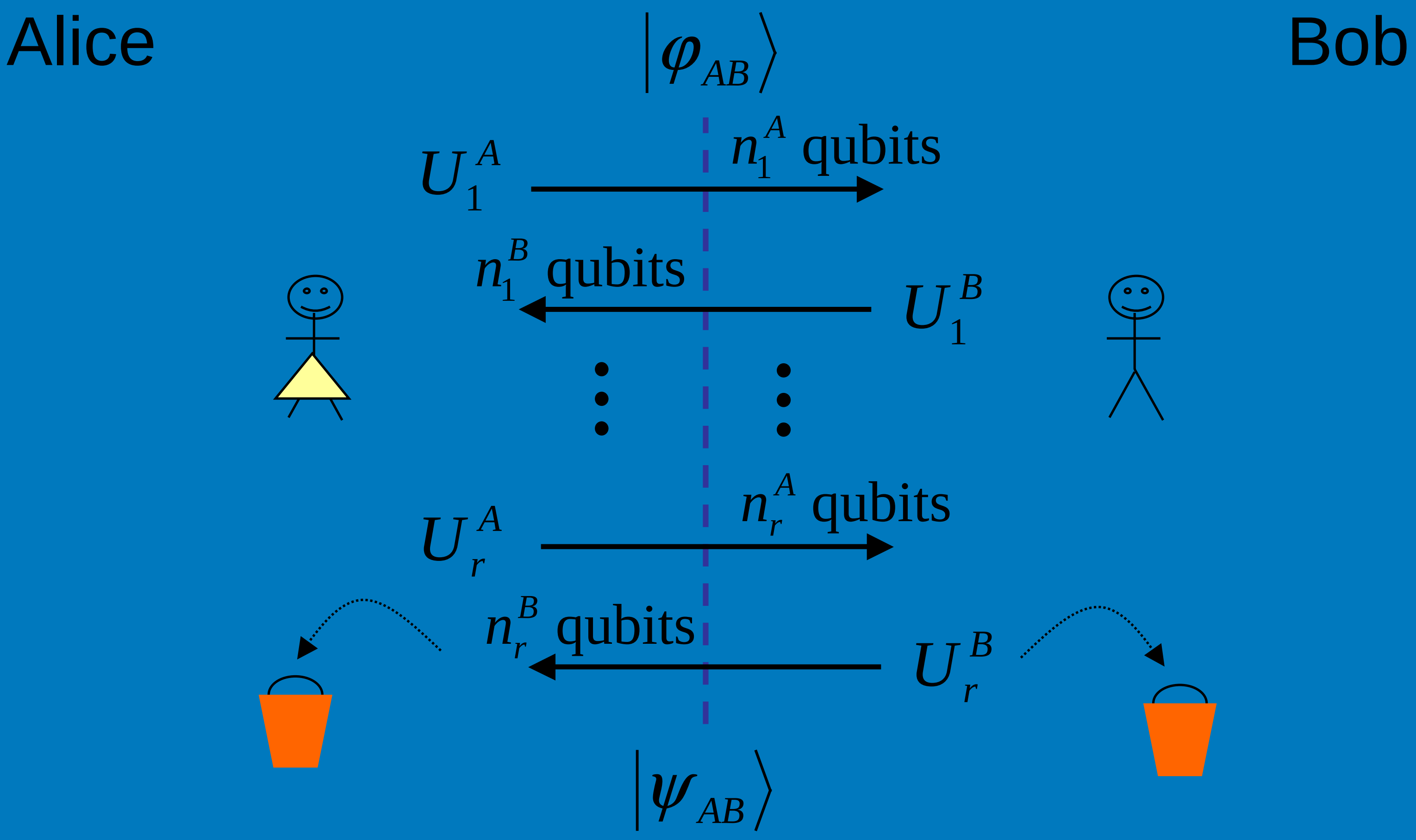
In physical terms, all measurement outcomes equiprobable in optimal (minimal communication) protocol.

A new source of revenue for WorldCom?

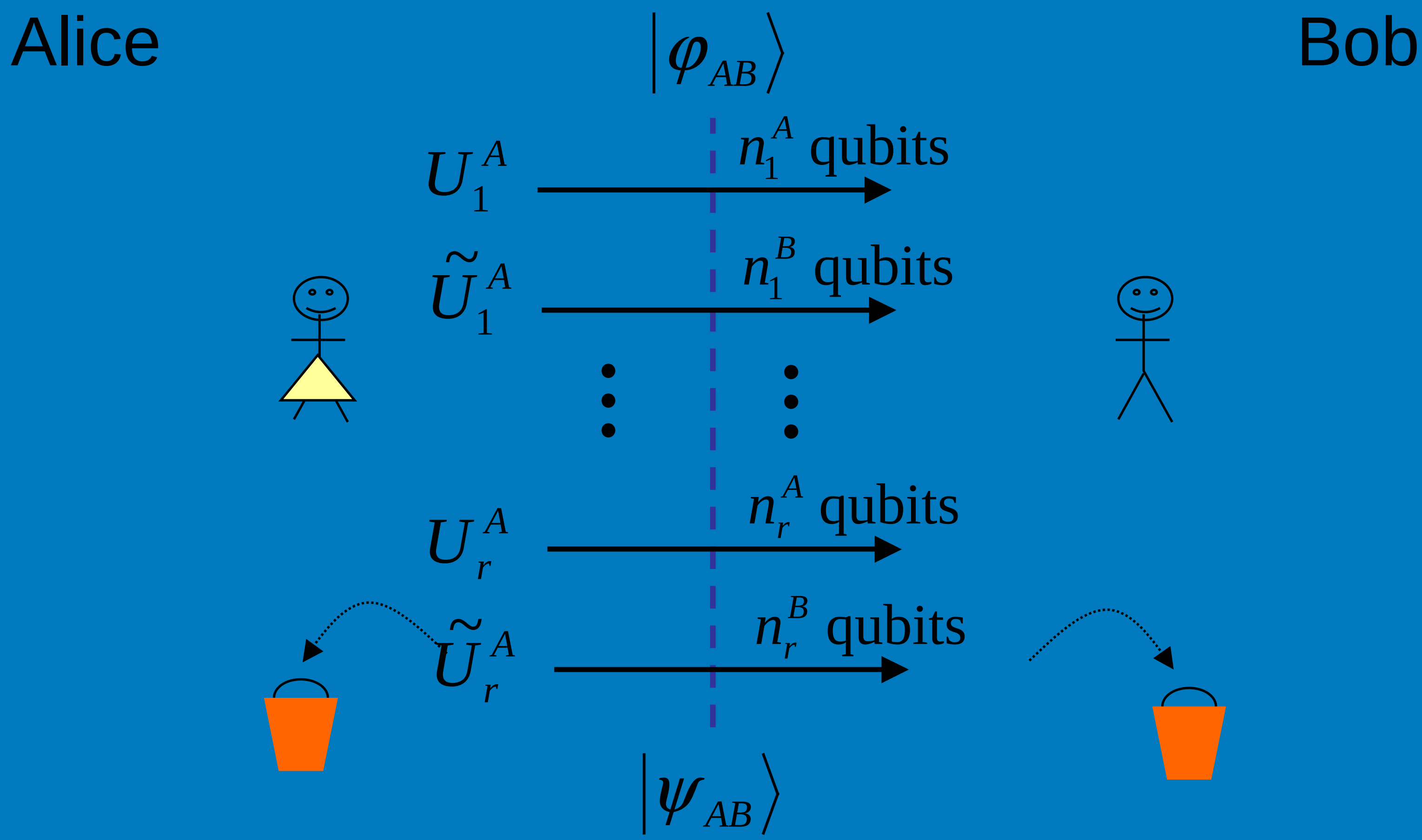


$$\Gamma_r^A(\rho) = \mathrm{Tr}_E U_r^{AE} \rho \otimes |0\rangle\langle 0|^E U_r^{AE*}$$

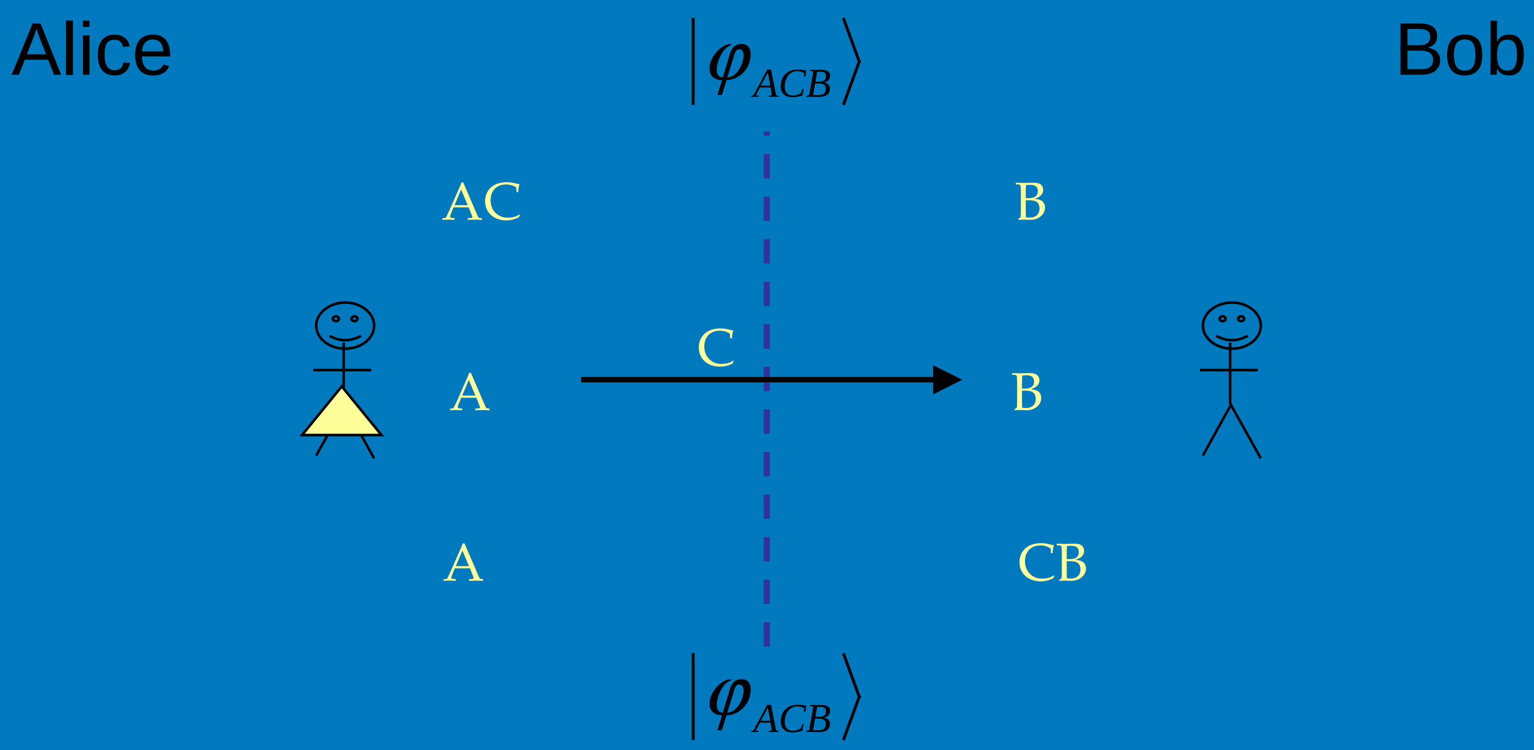
Simplified protocol



Simplified protocol



Essential question



How is $\lambda(\varphi_A)$ related to $\lambda(\varphi_{AC})$?

Inequalities

How is $\lambda(\varphi_A)$ related to $\lambda(\varphi_{AB})$?

The prototype:

$$\sum_{i=1}^{k}\lambda_i(\varphi_A) = \max_{V\in \mathrm{Gr}_k(A)} \mathrm{Tr}(\varphi_A \mathrm{P}_V)$$

$$= \max_{V\in \mathrm{Gr}_k(A)} \mathrm{Tr}(\varphi_{AB} \mathrm{P}_{V\otimes B})$$

$$\leq \max_{V\in \mathrm{Gr}_{kd_B}(A\otimes B)} \mathrm{Tr}(\varphi_{AB} \mathrm{P}_V)$$

$$= \sum_{i=1}^{kd_B}\lambda_i(\varphi_{AB})$$

Key steps: 1) Variational principle for eigenvalues
2) Non-empty intersection

Variational principle…

Let F_i be the i-dimensional subspace of A corresponding to the i largest eigenvalues of φ_A.

Π = 0010111001 a binary string of length dim(A)

Introduce the *Schubert cycle*

$$W_\pi(F) = \{V \subseteq A : \dim(V \cap F_i) - \dim(V \cap F_{i-1}) \geq \pi(i)\}$$

Then [HZ]

$$\sum_i \pi(i)\lambda_i(\varphi_A) = \min_{V \in W_\pi(\varphi_A)} \mathrm{Tr}(\varphi_A P_V)$$

…plus intersections give inequalities

$$\sum_{i=1}^{d_A} \pi(i)\lambda_i(\varphi_A) + \sum_{i=1}^{d_A d_B} \mu(i)\lambda_i(-\varphi_{AB})$$

$$= \min_{V\in W_\pi(\varphi_A)} \mathrm{Tr}\left(\varphi_A P_V\right) + \min_{V\in W_\mu(\varphi_{AB})} \mathrm{Tr}\left(-\varphi_{AB} P_V\right)$$

$$= \min_{V\in W_\pi(\varphi_A)} \mathrm{Tr}\left(\varphi_{AB} P_{V\otimes B}\right) + \min_{V\in W_\mu(\varphi_{AB})} \mathrm{Tr}\left(-\varphi_{AB} P_V\right)$$

If common choice possible

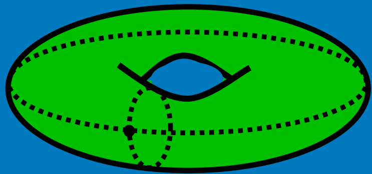
$$\leq \mathrm{Tr}\left((\varphi_{AB} - \varphi_{AB})P_{V_0}\right) = 0$$

IF there exists $V_0 \in W_\pi(\varphi_A) \otimes B$ ✋ $W_\mu(\varphi_{AB})$

Intersections can be studied using Schubert calculus

Finding intersections

$$W_\pi(F) = \{V \subseteq A : \dim(V \cap F_i) - \dim(V \cap F_{i-1}) \geq \pi(i)\}$$



$$H_*(T^2) = \langle [pt], [l_1], [l_2], [T^2] \rangle$$

$$H_*(Gr_k(A)) = \langle [W_\pi(F)] \rangle$$

Ring structure in (co)homology: intersection pairing/cup product

$\iota_B(W_\pi(\varphi_A))$ 🖐 $W_\mu(\varphi_{AB}) \neq \emptyset$ if $\iota_*([W_\pi]) \cdot [W_\mu] \neq 0$

Tricky to evaluate…

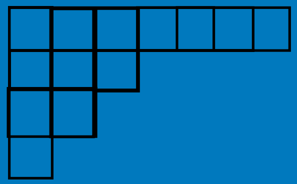
One version of the solution (Or another of the problem?)

Irreducible representations of S_n: V_μ

Another representation of S_n: $B^{\otimes n}$

Inequalities come from decomposition into irreps of: $V_\mu \otimes B^{\otimes n}$

01010100001 $\longleftrightarrow$



Some mathematical context

Set of Hermitian matrices with fixed spectrum λ is a symplectic manifold O_{λ}.

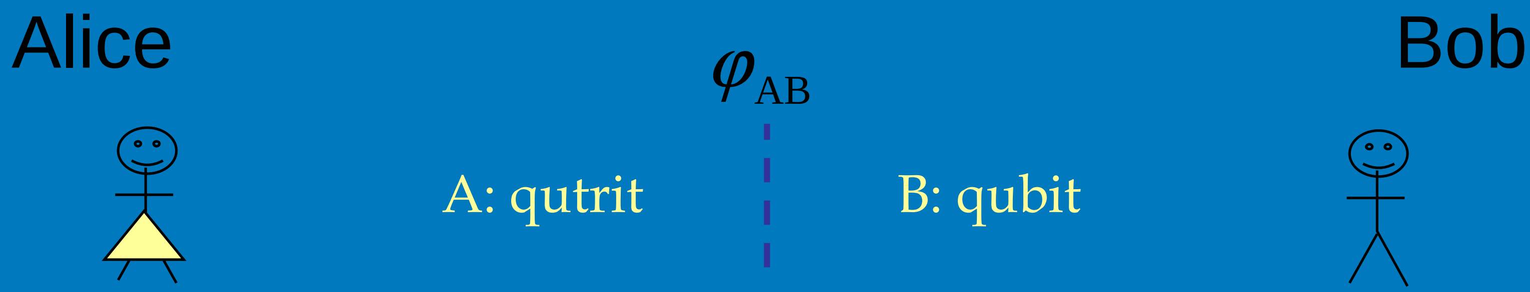
$U(A)$ acts on O_{λ}^{AB} by conjugation : φ_{AB} ♋ $(U \otimes I)\varphi_{\mathrm{AB}}(U^{*} \otimes I)$

This is an example of a *Hamiltonian group action.*
The partial trace over B is a *moment map* for this action.
Thus, our problem is to describe the image of the symplectic manifold O_{λ}^{AB} under the moment map Tr_{B}.

Machinery exists: See, for example, "Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion" By Berenstein and Sjamaar, math.sg/9810125.

Upshot: Inequalities derived by method of previous slides are sufficient.

A small example



$$\overbrace{\tilde{\lambda}_1}^{\varphi_A} \leq \overbrace{\lambda_1 + \lambda_2}^{\varphi_{AB}}$$

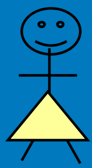
$$\tilde{\lambda}_3 \leq \lambda_2 + \lambda_3$$

$$\tilde{\lambda}_1 \geq \lambda_4 + \lambda_5$$

$$\tilde{\lambda}_3 \geq \lambda_5 + \lambda_6$$

More simple cases

Alice

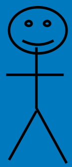


φ_{AB}

A: qubit
or qutrit

B: qubit or larger/
qutrit or larger

Bob



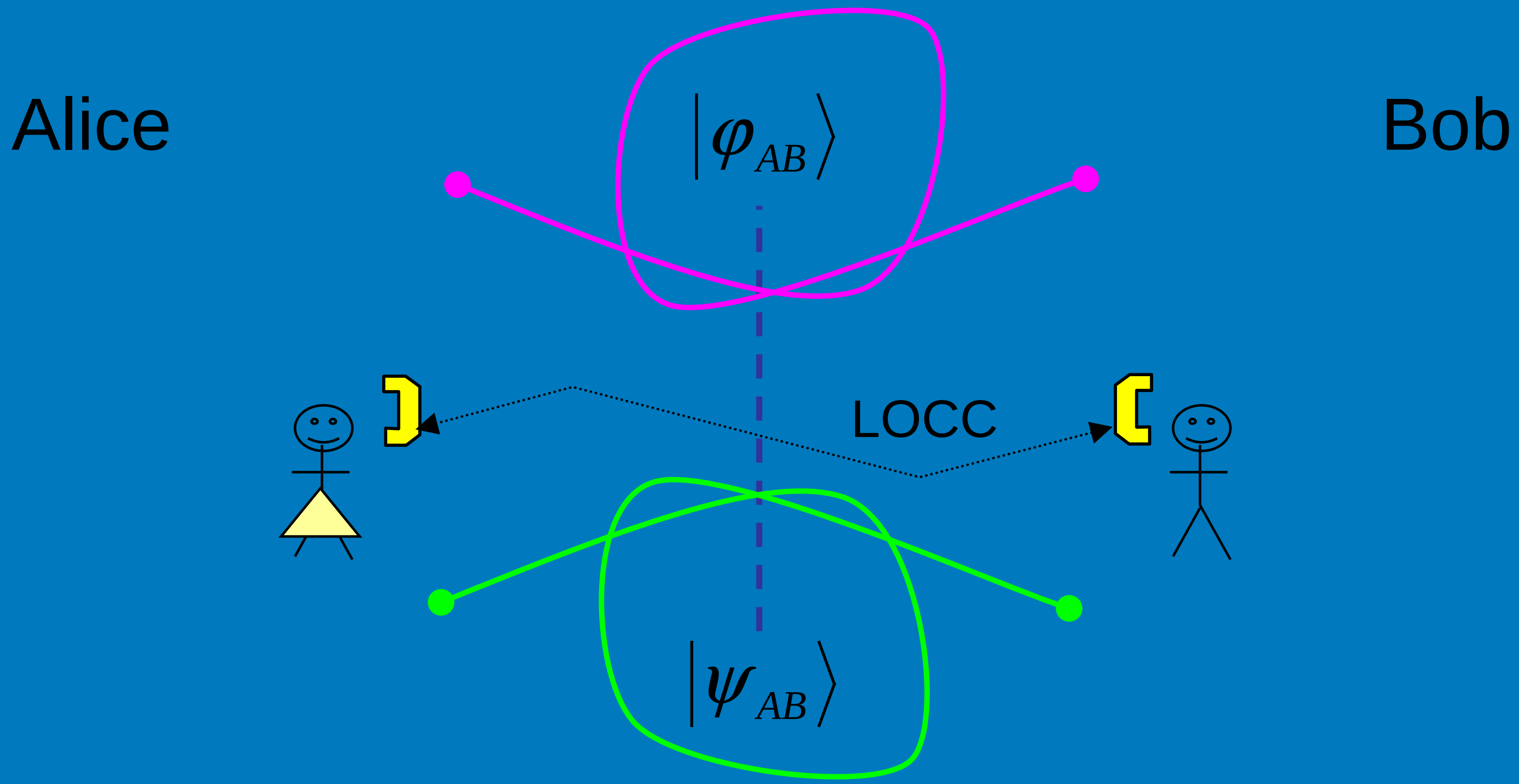
In these situations, the original prototype inequalities are *necessary* and *sufficient*:

$$\sum_{i=1}^{k} \lambda_i(\varphi_A) \leq \sum_{i=1}^{kd_B} \lambda_i(\varphi_{AB})$$

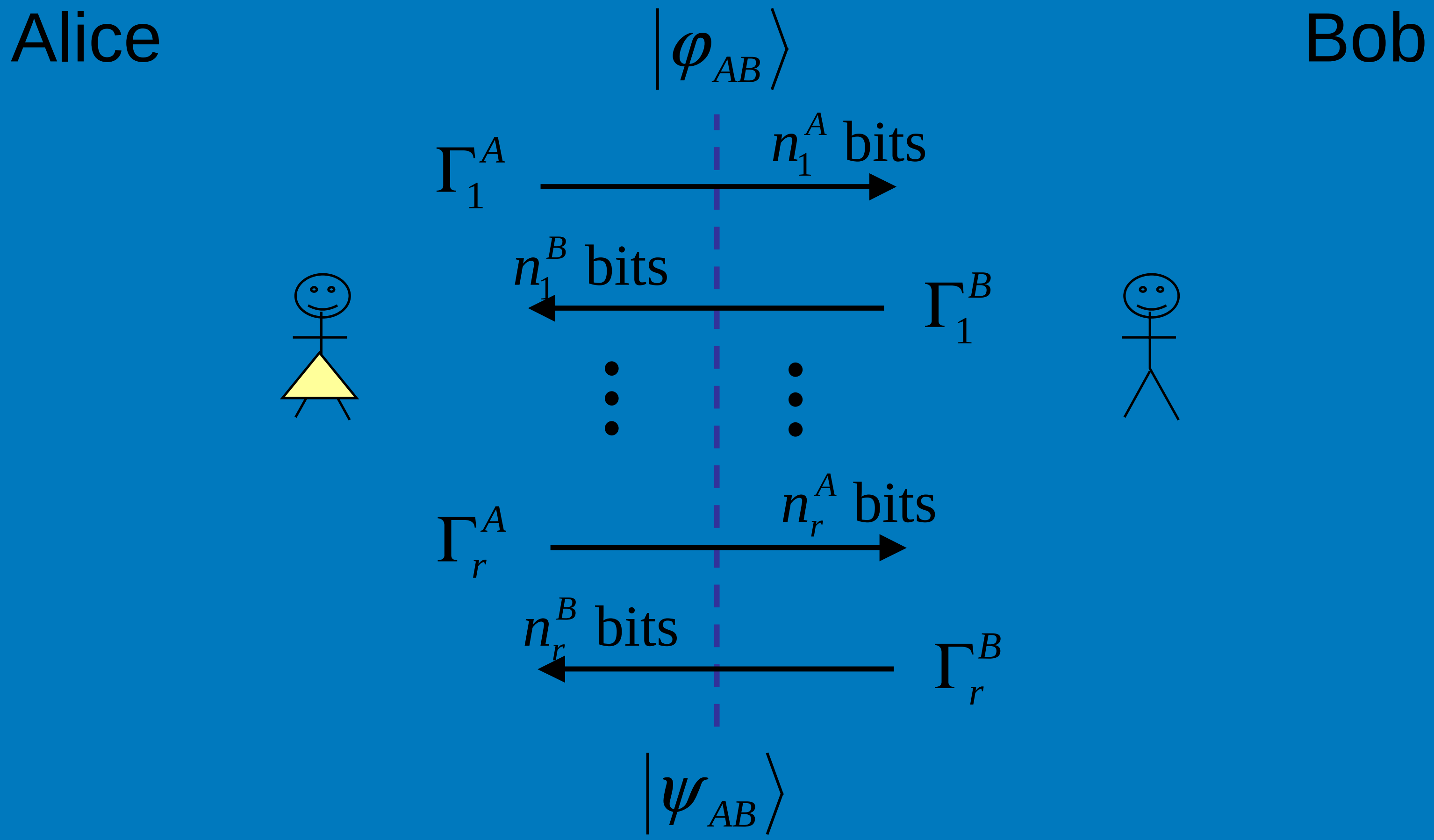
Conclusions: Part I

- Detailed description of those state transformations possible with limited communication is mathematically tractable
- Provides a link between quantum information theory and an area of active research in mathematics
- Other problems in QIT can likely be analyzed using similar tools

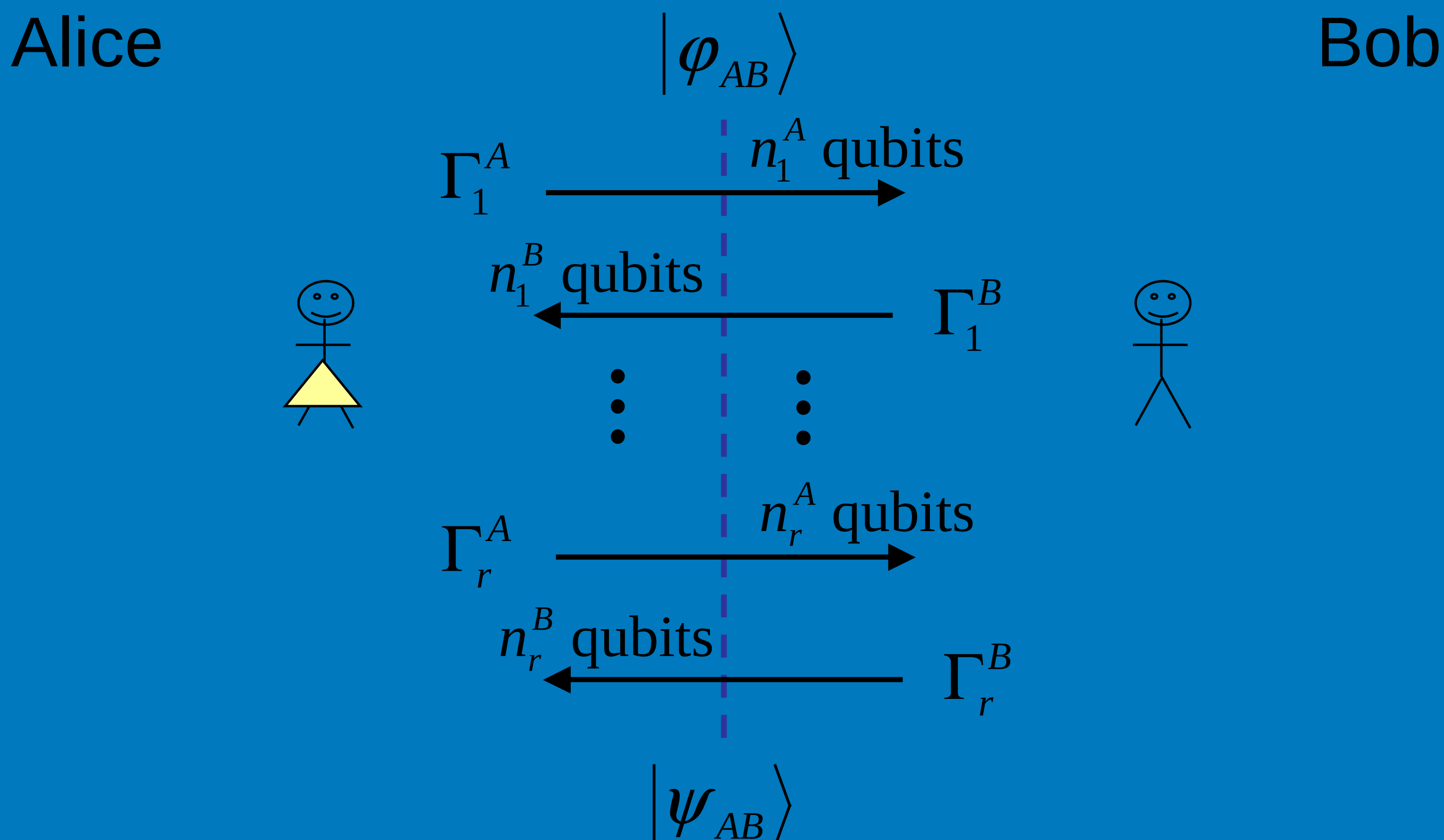
Back to the original problem



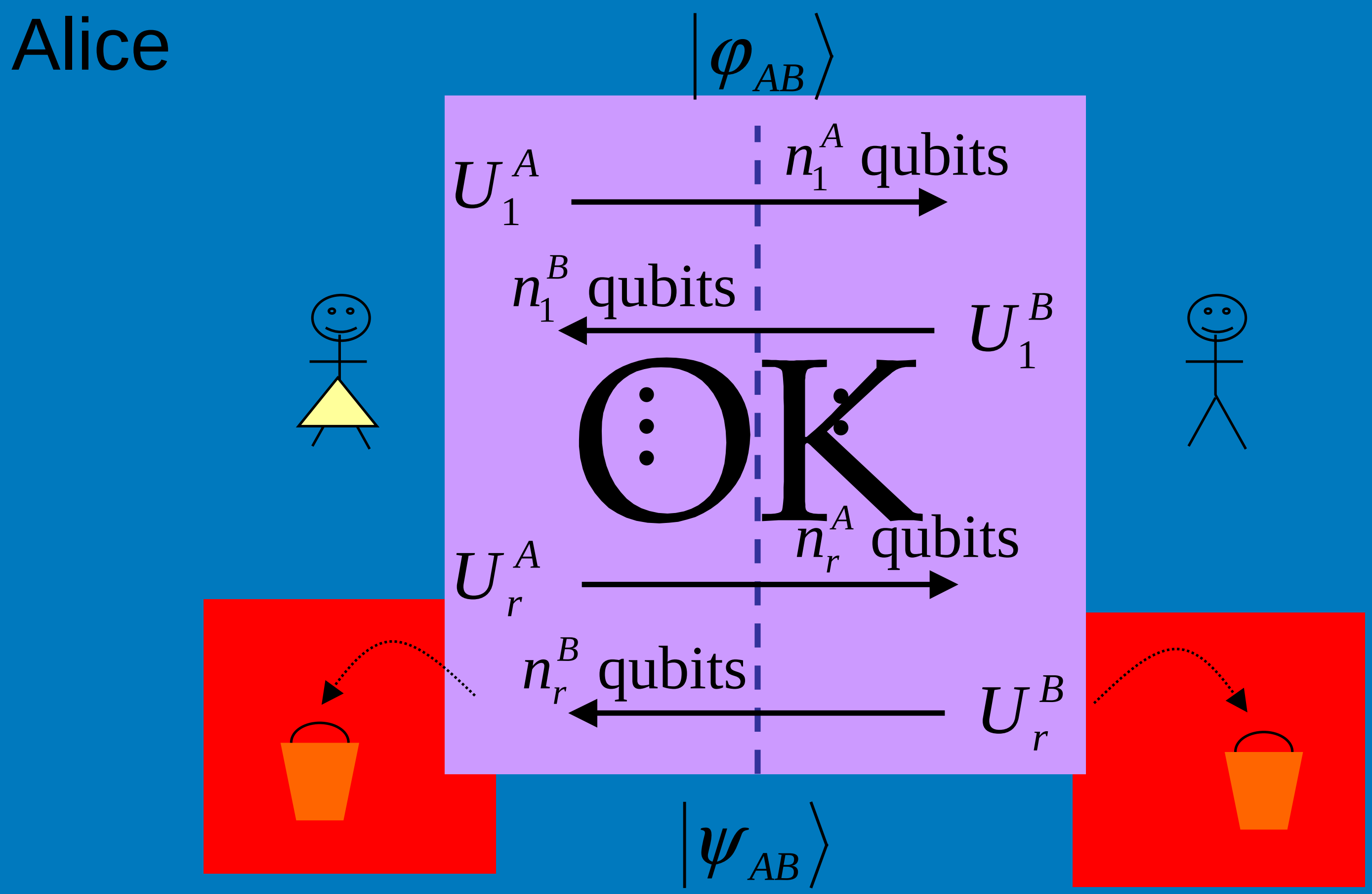
Protocol anatomy:



Qubits are better than bits



Simplified protocol



How to handle the final discard step?

Renyi entropy

Definition: $$S_\alpha(\varphi) = \frac{1}{1-\alpha} \log \mathrm{Tr}\left(\varphi^\alpha\right)$$

Properties:

(1) $$S_\alpha\left(|\varphi\rangle\langle\varphi|\right) = 0 \quad S_\alpha\left(\tfrac{1}{d} I\right) = \log d$$

(2) $$S_\alpha(\varphi \otimes \rho) = S_\alpha(\varphi) + S_\alpha(\rho)$$

(3) $$S_\alpha(\varphi_A) - \log \dim B \leq S_\alpha(\varphi_{AB}) \leq S_\alpha(\varphi_A) + \log \dim B$$

(4) $$\alpha \leq \beta \Rightarrow S_\alpha(\varphi) \geq S_\beta(\varphi)$$ [0204093]

Keep track of: $$\Delta(\varphi) = S_0(\varphi) - S_\infty(\varphi)$$

Renyi entropy and spectral fluctuations

Definition: $S_\alpha(\varphi) = \frac{1}{1-\alpha} \log \mathrm{Tr}(\varphi^\alpha)$

Properties: (1) $S_\alpha(|\varphi\rangle\langle\varphi|) = 0 \quad S_\alpha(\frac{1}{d} I) = \log d$

(2) $S_\alpha(\varphi \otimes \rho) = S_\alpha(\varphi) + S_\alpha(\rho)$

(3) $S_\alpha(\varphi_A) - \log \dim B \leq S_\alpha(\varphi_{AB}) \leq S_\alpha(\varphi_A) + \log \dim B$

(4) $\alpha \leq \beta \Rightarrow S_\alpha(\varphi) \geq S_\beta(\varphi)$

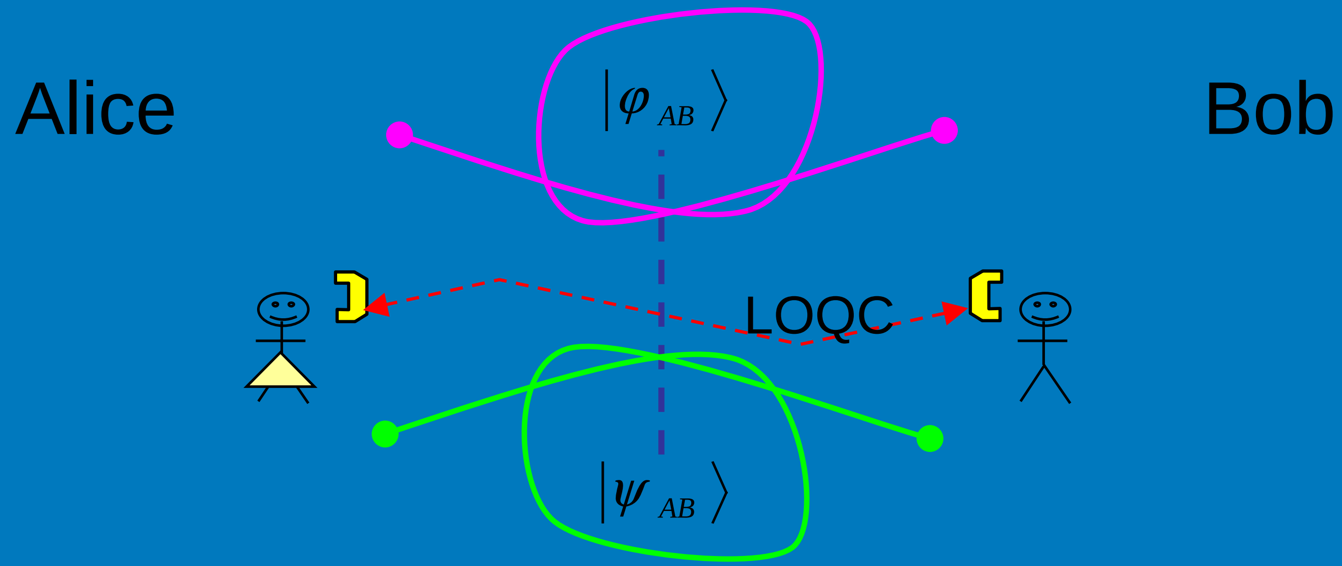
Keep track of: $\Delta(\varphi) = S_0(\varphi) - S_\infty(\varphi) \geq 0$

$$\Delta(\varphi_A) - 2\log \dim B \leq \Delta(\varphi_{AB}) \leq \Delta(\varphi_A) + 2\log \dim B$$

$$\Delta(\varphi \otimes \rho) = \Delta(\varphi) + \Delta(\rho) \geq \Delta(\varphi)$$

$$\Delta(|\varphi\rangle\langle\varphi|) = \Delta(\tfrac{1}{d} I) = 0$$

Theorem:



Starting from the state $|\varphi_{AB}\rangle$, if Alice and Bob perform local operations and exchange at most n qubits (or bits) of communication to create $|\psi_{AB}\rangle$, then $\Delta(\psi_A) - \Delta(\varphi_A) \leq 2n$.

What's this good for?

Entanglement concentration: [BBPS]

$$\left|\varphi_{AB}\right\rangle^{\otimes n} \xrightarrow{LOCC} \approx \left|\Phi_{+}\right\rangle^{\otimes n(S(\varphi_A)-\varepsilon)}$$

Entanglement dilution:

$$\left|\Phi_{+}\right\rangle^{\otimes n(S(\varphi_A)+\varepsilon)} \xrightarrow{LOCC} \approx \left|\varphi_{AB}\right\rangle^{\otimes n}$$

$$\Delta\left(\Phi_{+}^{A}\right) = 0 \qquad \tilde{\Delta}\left(\varphi_A^{\otimes n}\right) \sim \sqrt{n}$$

Best known protocol consumes $O(n^{1/2})$ bits of communication [LP]

Theorem: Any protocol for producing a high-fidelity copy of $|\varphi_{AB}\rangle^{\otimes n}$ from EPR pairs requires $\Omega(n^{1/2})$ bits (or even qubits) of communication. [Hayden-Winter, Harrow-Lo]

Conclusions

- Asymptotic, pure state LOCC entanglement transformations require $\Omega(n^{1/2})$ bits of communication
- Fundamental asymmetry between concentration and dilution due to fluctuations
- General open problem: Bridge the gap between exact and asymptotic techniques!