Local Hidden Variable Theories for Quantum States

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Motivation

Relationship between entanglement and non-locality is poorly understood. (Do any bound entangled states violate a Bell inequality?)

Is there some general structure to local hidden variable theories?

Are there algorithms that can construct local hidden variable theories for quantum states?

Bell Experiment

Alice and Bob make measurements in distant laboratories.

Shared randomness.

S_a measurements
O_a outcomes

S_b measurements
O_b outcomes

The outcomes of Alice's measurements are unaffected by Bob's choice of setting.

Bell Experiment: Deterministic

s_a measurements
o_a outcomes

s_b measurements
o_b outcomes

Each of Alice's measurements has some outcome with probability one, independent of the measurement made by Bob.

Alice's outcomes. $\mathbf{m} = (m_1, \ldots, m_{s_a}) \quad m_i = 1, \ldots, o_a$

Bob's outcomes. $\mathbf{n} = (n_1, \ldots, n_{s_b}) \quad n_k = 1, \ldots, o_b$

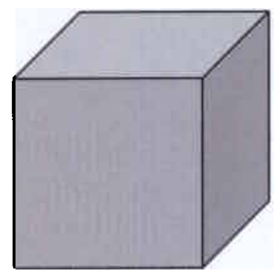
$$P_{ij,kl} = B^{\mathbf{m},\mathbf{n}}_{ij,kl} = \delta_{m_i j} \delta_{n_k l}$$

Bell Experiment II

Shared randomness.

Each of the deterministic outcomes occurs with some probability.

$$P_{ij,kl} = \sum_{\mathbf{m},\mathbf{n}} p_{\mathbf{m},\mathbf{n}} B^{\mathbf{m},\mathbf{n}}_{ij,kl}$$



Bell Experiment: Quantum

Quantum State.

ρ

s_a measurements
o_a outcomes

s_b measurements
o_b outcomes

Each measurement is described by a POVM.

$$P_{ij,kl}(\rho) = \mathrm{Tr}\, E_{ij}^{A} \otimes E_{kl}^{B} \rho$$

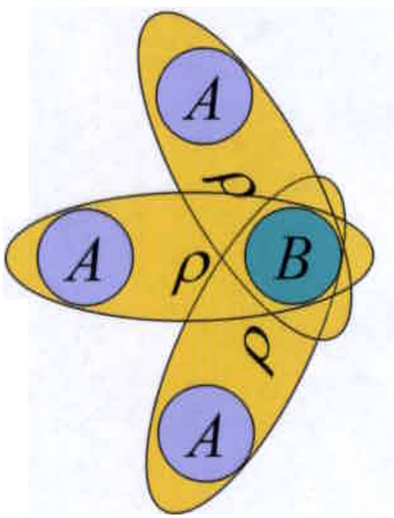
$$\left\{ E_{ij}^{A} \geq 0 : \sum_j E_{ij}^{A} = I_{d_A} \right\} \quad \left\{ E_{ij}^{B} \geq 0 : \sum_j E_{ij}^{B} = I_{d_B} \right\}$$

``Strategy''

No Bell inequality violations if different POVMs *commute* or if Bob only has *one setting*.

Try to replace ρ by another state for which LHV is obvious and

$$P_{ij,kl}(\rho) = P_{ij,kl}(\tilde{\rho})$$



Extensions of ρ fit the bill if they are suitably symmetric.

$$\tilde{\rho} \geq 0, \ \mathrm{Tr}_{A^{\otimes s_a - 1}}[\tilde{\rho}] = \rho$$ Extension for ρ

$$\frac{1}{s_a!}\sum_{\pi_A} \pi\tilde{\rho}\pi = \tilde{\rho}$$ Whichever spaces are traced over.

Local Hidden Variable Theory

Reproduces the measurement probabilities of a quantum mechanical Bell experiment as a convex combination of the deterministic classical outcomes.

$$P_{ij,kl}(\rho) = \mathrm{Tr}\, E_{ij}^{A} \otimes E_{kl}^{B} \rho$$
$$= \sum_{\mathbf{m},\mathbf{n}} p_{\mathbf{m},\mathbf{n}}\left(\left\{E_{ij}^{A}, E_{kl}^{B}\right\}, \rho\right) B_{ij,kl}^{\mathbf{m},\mathbf{n}}$$

If this is not possible the state violates some Bell inequality for this number of settings. Entanglement is necessary for such a violation.

Extensions and LHVs

If ρ has a (s,1)-symmetric extension then it does not violate a Bell inequality for s settings for Alice and any number of settings for Bob.

Imaginary Bell experiment: Alice and Bob share $\widetilde{\rho}$ and Alice performs each of her POVMs on a different one of her copies of system A. (Essentially only a single POVM)

$$p_{\mathbf{m},\mathbf{n}}\left(\left\{E_{ij}^{A}, E_{kl}^{B}\right\}, \rho\right) = \frac{\prod_{i'=1}^{s_b}\left(\mathrm{Tr}\,\mathbf{E}_{\mathbf{m}}^{A} \otimes E_{i'm_{i'}}^{B}\,\widetilde{\rho}\right)}{\left(\mathrm{Tr}\,\mathbf{E}_{\mathbf{m}}^{A} \otimes I_{B}\,\widetilde{\rho}\right)^{s_b-1}}$$

$$\mathbf{E}_{\mathbf{m}}^{A} = E_{1m_1}^{A} \otimes E_{2m_2}^{A} \otimes \ldots \otimes E_{sm_s}^{A}$$

Quasi-Extensions and LHVs

Checking the LHV just requires verifying the equality

$$\boxed{\operatorname{Tr} E_{ij}^{A} \otimes E_{kl}^{B} \rho = \sum_{\mathbf{m},\mathbf{n}} p_{\mathbf{m},\mathbf{n}}\left(\left\{E_{ij}^{A}, E_{kl}^{B}\right\}, \rho\right) B_{ij,kl}^{\mathbf{m},\mathbf{n}}}$$

This holds as a result of the partial trace and symmetry properties of $\widetilde{\rho}$ and the normalization of the POVMs.

Positivity of $\widetilde{\rho}$ only serves to guarantee positivity of the probabilities. So in fact we only need

$$\boxed{\left(\operatorname{Tr} \mathbf{E}_{\mathbf{m}}^{A} \otimes E_{i'm_{i'}}^{B} \widetilde{\rho}\right) \geq 0}$$

$\widetilde{\rho}$ only needs to be positive on product states: *quasi-extension* H_{ρ}

Comments

Result works for all POVMs (not just projective measurements) and any number of measurement outcomes for each setting.

Since all separable states have symmetric extensions there is an LHV of this kind for *every* Bell experiment on *every* separable state.

Many entangled states (PPT and NPT) have (1,s)-symmetric quasi-extensions for small s (only). (e.g. real UPB states, Werner states.)

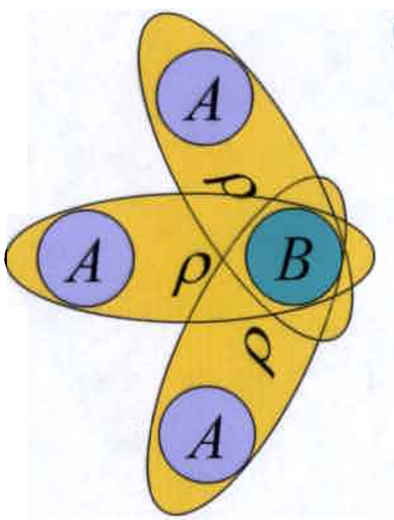
There must be other kinds of LHV theories that are relevant for entangled states. (Result of Werner.)

Some Physical Insight

If two systems are highly entangled they are less able to be entangled with other systems. (Monogamy of entanglement)

On the other hand separable states can be shared out as much as you like.

ρ is separable $\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$



Consider the state

$$\tilde{\rho} = \sum p_i \left(|\psi_i\rangle\langle\psi_i|\right)^{\otimes s} \otimes \left(|\phi_i\rangle\langle\phi_i|\right)$$

State of any copy of A and B is ρ

Existence of symmetric extension is a weak kind of separability criterion.

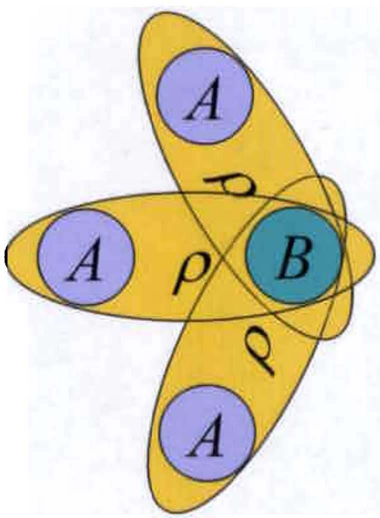
Constructing Symmetric Extensions

This problem can be cast as a *semidefinite program.*

Partial trace condition $\quad \mathrm{Tr}\,\mathrm{Sym}(X \otimes \mathbf{I})\tilde{\rho} = \mathrm{Tr}\, X\rho \quad \forall X$

Expand in basis for matrices $\quad X = \sum_{i=0}^{d_A d_B - 1} x_i \sigma_i \quad$ $\boxed{\mathrm{Tr}\,\sigma_i \sigma_j = \delta_{ij}}$

Just consider basis elements $\quad \mathrm{Tr}\,\mathrm{Sym}(\sigma_i \otimes \mathbf{I})\tilde{\rho} = \mathrm{Tr}\,\sigma_i \rho \quad \forall i$



minimize	$\mathrm{Tr}\,\tilde{\rho}$
subject to	$\mathrm{Tr}\,\mathrm{Sym}(\sigma_i \otimes \mathbf{I})\tilde{\rho} = \mathrm{Tr}\,\sigma_i \rho \quad \forall i$
	$\tilde{\rho} \geq 0$

Extension exists if optimum less than one.

$$\mathrm{Sym}\, Z = \frac{1}{s_a!} \sum_{\pi_A} \pi \rho Z \pi$$

Semidefinite Programs

Primal

minimize $c^{\mathrm{T}} w$

subject to $F_0 + \sum w_i F_i \geq 0$

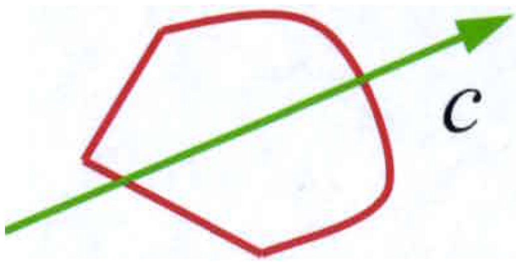
Dual

maximize $-\mathrm{Tr}[F_0 W]$

subject to $\mathrm{Tr}[F_i W] = c_i$

$W \geq 0$

If conditions can be satisfied problem is *feasible*



Minimize linear function over intersection of affine subspace with cone of positive matrices.

Large class of convex optimizations.

e.g. minimize sum of largest r eigenvalues

(separability, distillation Rains, CPM optimization Audenaert De Moor)

Excellent numerical methods available.

Semidefinite Programming Duality II

A nice sub-class of problems are termed strictly feasible.

> If there is a feasible point w, W such that
>
> $$F(w) > 0, W > 0$$
>
> then the primal and dual optimal values are the same and there is a point w^*, W^* attaining these values.

Numerical methods tend to attempt to minimize the difference between the two optimal values (duality gap).

For the strictly feasible case we are guaranteed 'certificates' of optimality as well as points attaining the optimum.

Semidefinite Programming Duality

Consider a feasible point w, W

$$c^T w + \mathrm{Tr}[F_0 W] = \sum \mathrm{Tr}[F_i W] w_i + \mathrm{Tr}[F_0 W]$$
$$= \mathrm{Tr}[F(w) W] \geq 0$$

So for feasible points

$$c^T w \geq -\mathrm{Tr}[F_0 W]$$

Constraints guarantee that feasible primal values bound dual optimum and vice versa.

Dual Problem

The optimization dual to the one that constructs extensions searches for an entanglement witness.

maximize	$1-\mathrm{Tr}\, Z\rho$
subject to	$\mathrm{Sym}(Z\otimes\mathbf{I})\geq 0$

If optimum is greater than one (no extension exists) Z is an entanglement witness.

As a test for entanglement this is neither weaker nor stronger than partial transpose criterion.

Fewer variables means easier to deal with analytically.

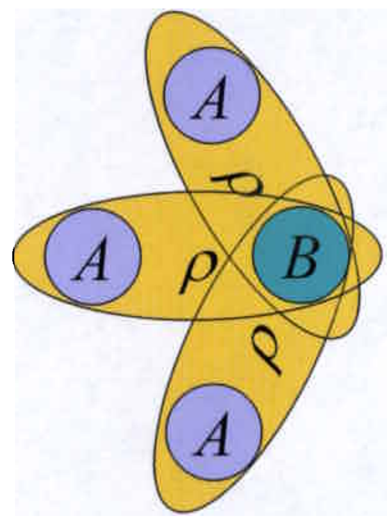
Results

Tested on many examples of bound entangled states in $\mathcal{H}_3 \otimes \mathcal{H}_3$

Computation scales at worst like $\left(d_A^{2.5s+4}\right)\left(d_B^4\right)$

Bound entangled states taken from:

Horodecki 1999, Bennett *et al.* 1999
Horodecki, Lewenstein 2000,
Bruss, Peres 1999



Analytically extensions exist for real UPB states and s=2, and Werner states for s<d.

Summary and Outlook

Described a method of constructing local hidden variable theories for quantum states that works for all separable states and some entangled ones.

Construction works for a fixed number of settings for Alice (say) but for any POVMs with any number of outcomes.

Numerically and analytically tractable since a semidefinite program.

Other kinds of local hidden variable exist (Werner). Perhaps it would be fruitful to find a more complete characterization of LHV theories.