Nonlocal quantum resource transformations & unitary bidirectional quantum channels

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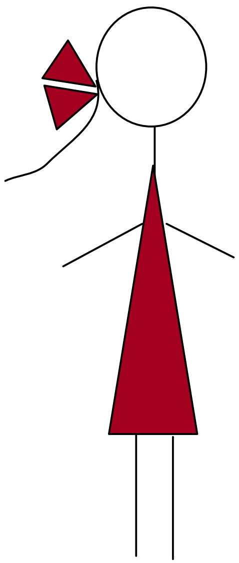
Nonlocal resources for (2-party) distributed info processing:

101011001011 ... 1. Shared randomness 101011001011 ...



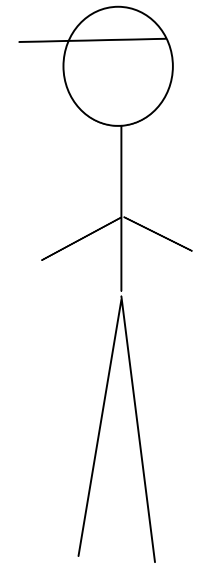
Alice

2. Entanglement $|00\rangle + |11\rangle$

3. Classical communication

4. Quantum communication

5. Nonlocal Interaction
(Bipartite quantum op)



Bob

Outline:

- Nonlocal resource transformations - qualitative equivalences
- Capacities of a given unitary interaction to create other resources
- Example
- General bounds on capacities
- Entanglement capacity
- Entanglement-assisted 1-way classical capacity
  of general arbitrary unitary interaction

Resource transformation:

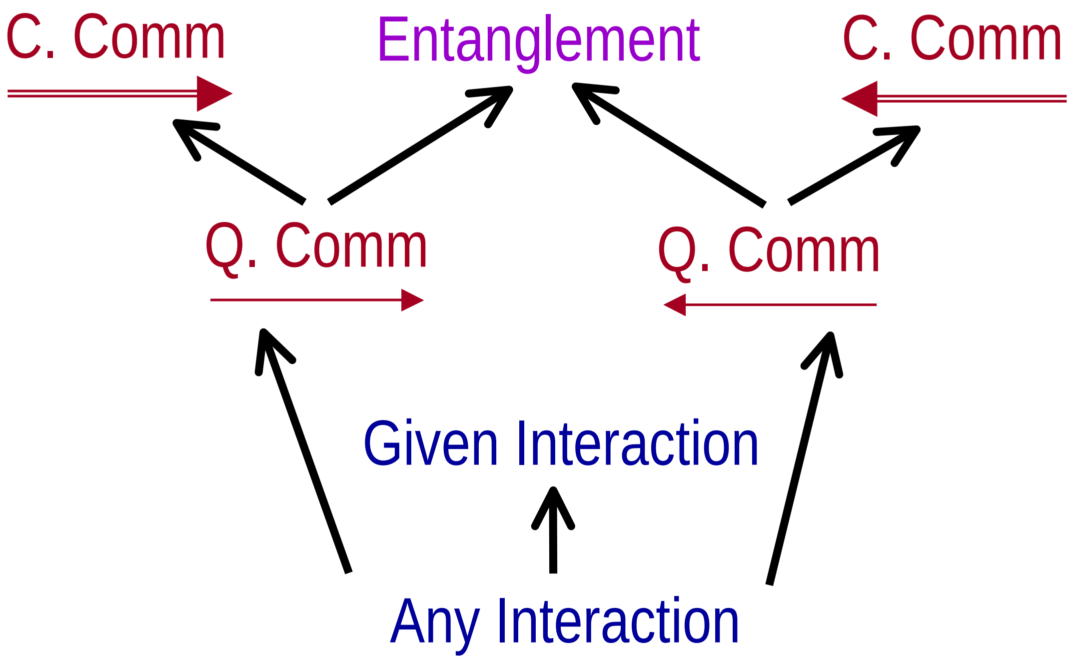
Local resources are free.

Incomparable resources

C. Comm $\not\rightarrow$ / $\not\leftarrow$ Entanglement $\not\rightarrow$ / $\not\leftarrow$ C. Comm

Resource transformation:

Local resources are free.

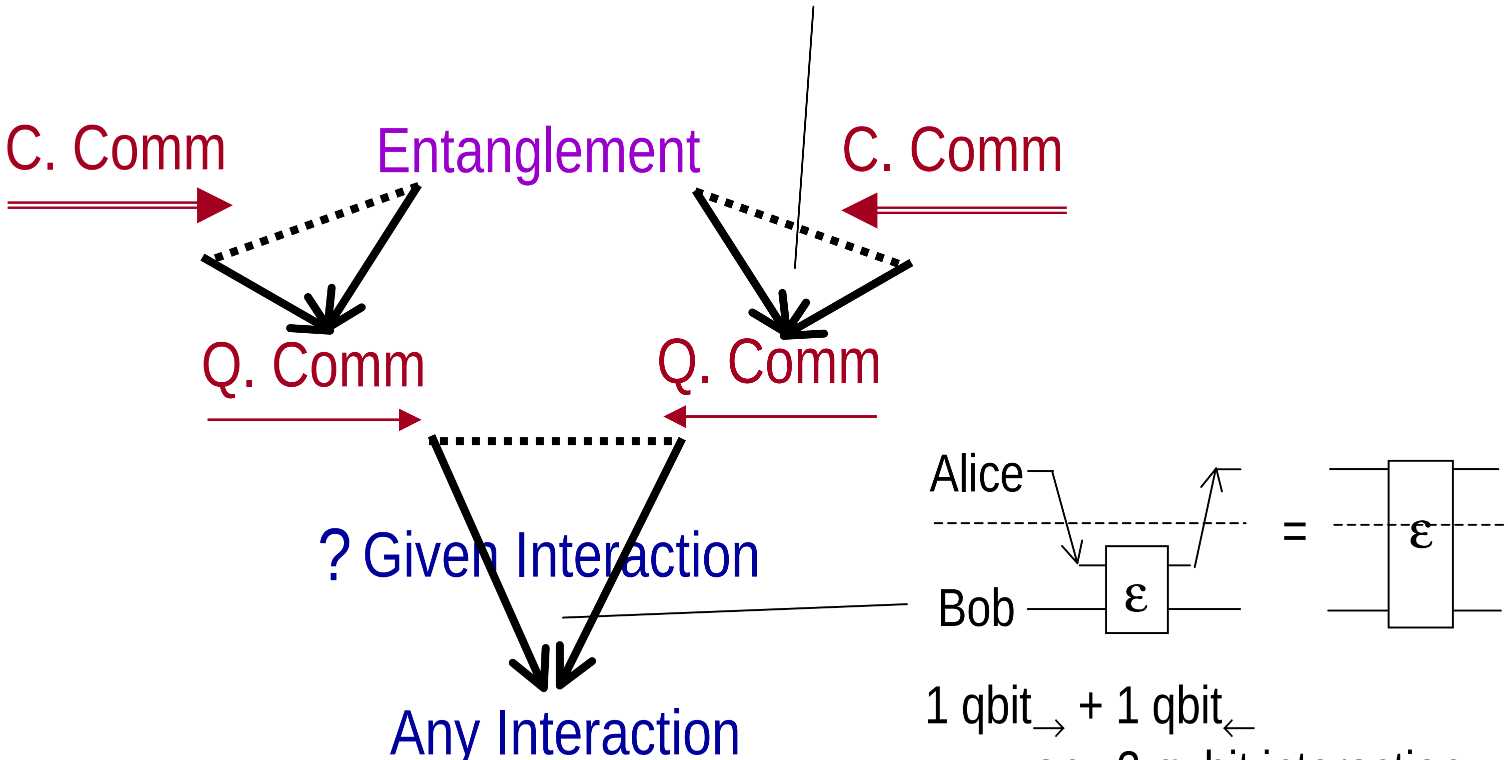


Resource transformation:

Local resources are free.

Teleportation:

2 cbits + 1 ebit $\Rightarrow$ 1 qbit



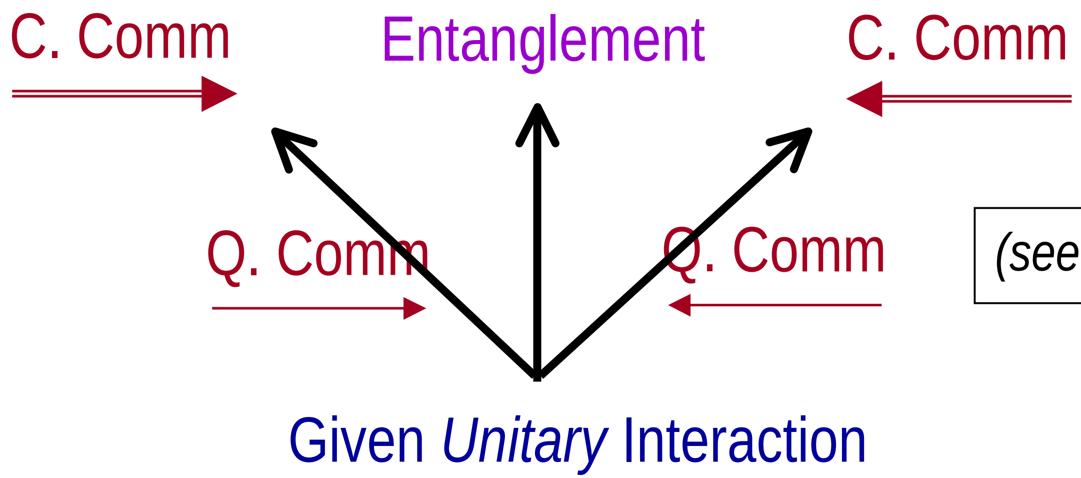
1 qbit$_{\rightarrow}$ + 1 qbit$_{\leftarrow}$

$\Rightarrow$ any 2-qubit interaction

2 cbit$_{\rightarrow}$ + 2 ebit + 2 cbit$_{\leftarrow}$ $\Rightarrow$ any 2-qubit interaction

Resource transformation:

Local resources are free.

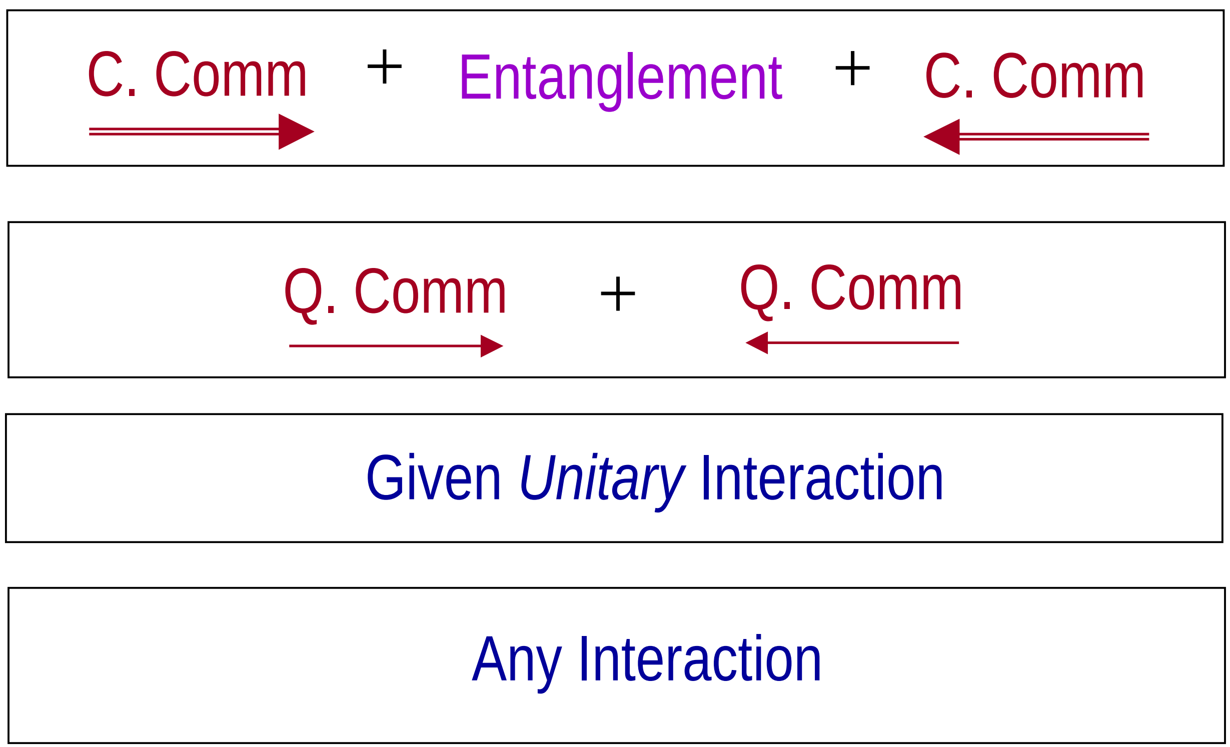


(see 0205057 for proof)

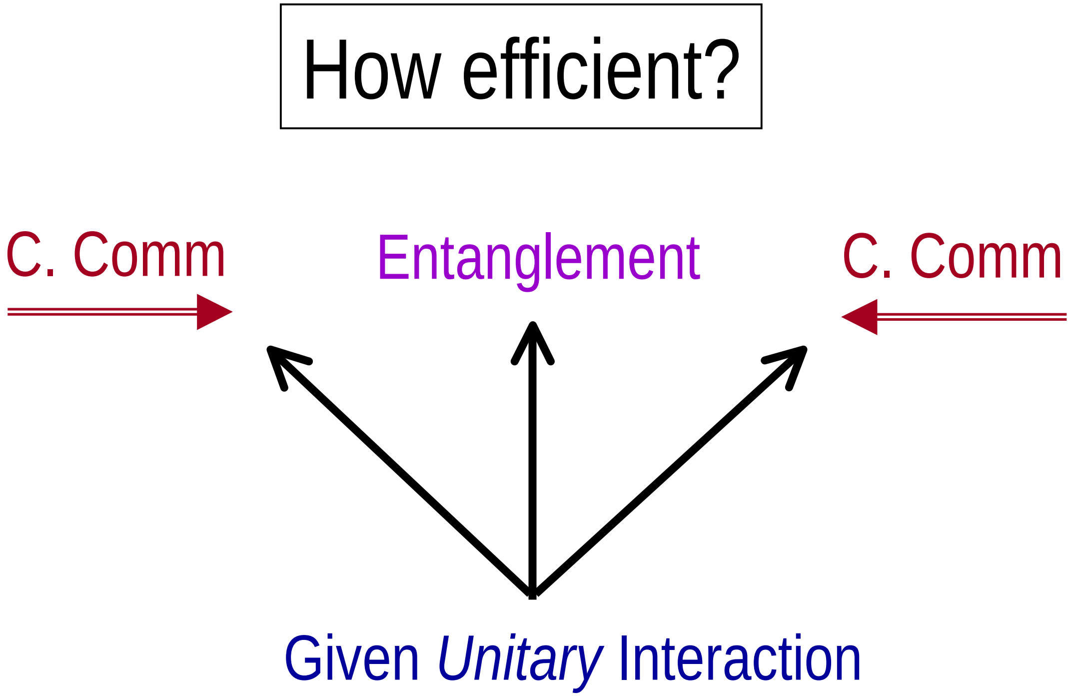
Any Interaction

Qualitatively equivalence resources:

Local resources are free.



Capacities of unitary bidirectional channels:



t-shot capacity for resource X =

(amount of X obtained by best strategy

with t uses of U) ÷ t

(always allow local resources in strategy)

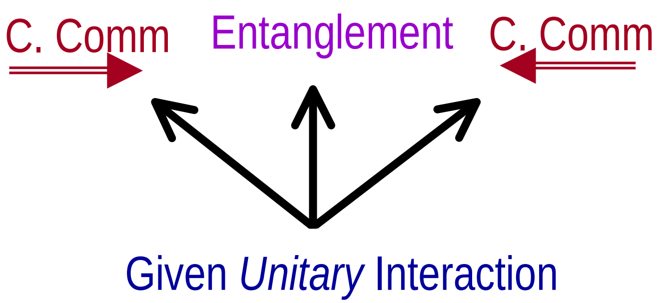
e.g. 1-shot capacity

Asymptotic capacity = t-shot capacity in large-t limit

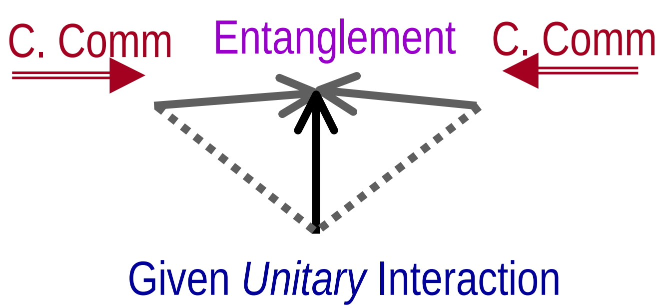
Can also allow specific auxiliary resources

Auxiliary resources:

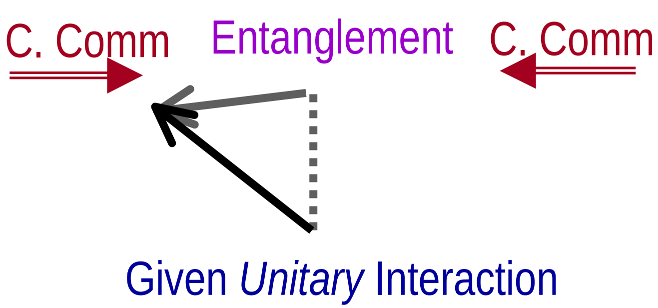
Unassisted capacities:



Entanglement Capacity assisted by CC:



Entanglement-assisted 1-way classical capacity:



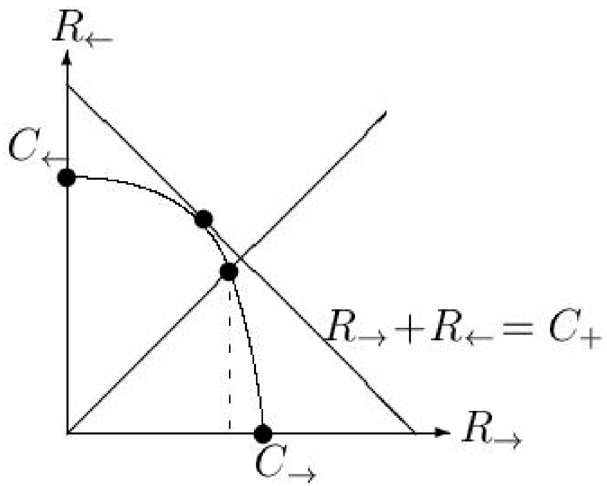
Classical capacities of bidirectional channels:

Simultaneous forward & backward communication – pair of resources.

The rate pair $(R_{\rightarrow}, R_{\leftarrow})$ is "achievable"
if $R_{\rightarrow}$ & $R_{\leftarrow}$ cbits can be sent forward and backward per use of U

Unassisted capacities:

Forward: $C_{\rightarrow} = \sup R_{\rightarrow}$
Backward: $C_{\leftarrow} = \sup R_{\leftarrow}$
Total: $C_{+} = \sup (R_{\rightarrow} + R_{\leftarrow})$



Entanglement-assisted:

$C^E_{\rightarrow}$, $C^E_{\leftarrow}$, C^E_{+}

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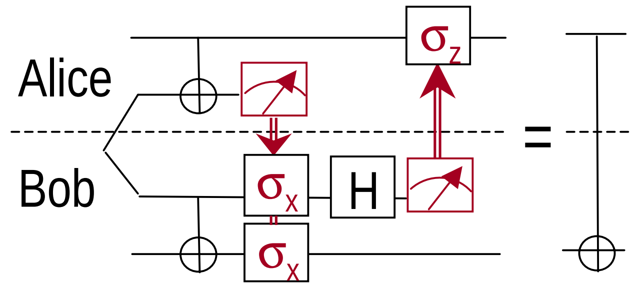
Example U = CNOT_{ab}:

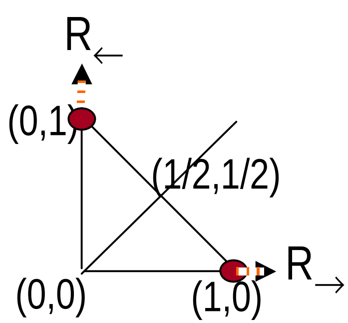
- 1 ebit + 1 $\text{cbit}_{\rightarrow}$ + 1 $\text{cbit}_{\leftarrow}$ $\Rightarrow$ 1 CNOT
  t ebit + t $\text{cbit}_{\rightarrow}$ + t $\text{cbit}_{\leftarrow}$
  $\Rightarrow$ t CNOT $\Rightarrow$ t-use protocol
  $\therefore E, C^E_{\rightarrow}, C^E_{\leftarrow} \leq 1$
- $(H \otimes H)\ \text{CNOT}_{ab}\ (H \otimes H) = \text{CNOT}_{ba}$
  $\therefore C_{\rightarrow} = C_{\leftarrow}\ ,\ C^E_{\rightarrow} = C^E_{\leftarrow}$

Gottesman 98 :

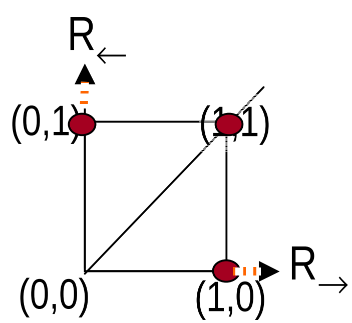


$E = 1,\ C^E_{\rightarrow} = C_{\rightarrow} (= C_{\leftarrow} = C^E_{\leftarrow}) = 1$

$(|0\rangle+|1\rangle)\,|0\rangle \rightarrow |00\rangle+|11\rangle,\quad |i\rangle|0\rangle \rightarrow |ii\rangle$



Unassisted



Assisted

$C^E_{+} \geq 2$

i, j	$(\sigma_x^{\,i} \otimes \sigma_z^{\,j})(\|00\rangle+\|11\rangle)$		
0,0	$\|00\rangle+\|11\rangle$		$(\|0\rangle+\|1\rangle)\|0\rangle$
0,1	$\|00\rangle-\|11\rangle$	$\rightarrow$	$(\|0\rangle-\|1\rangle)\|0\rangle$
1,0	$\|10\rangle+\|01\rangle$		$(\|0\rangle+\|1\rangle)\|1\rangle$
1,1	$\|10\rangle-\|01\rangle$		$(\|0\rangle-\|1\rangle)\|1\rangle$

NB Asymptotic capacity = 1-shot

Example $U = CNOT_{ab}$:

- 1 ebit $+ 1$ cbit$_{\rightarrow}$ $+ 1$ cbit$_{\leftarrow}$ $\Rightarrow 1$ CNOT

  t ebit $+ t$ cbit$_{\rightarrow}$ $+ t$ cbit$_{\leftarrow}$

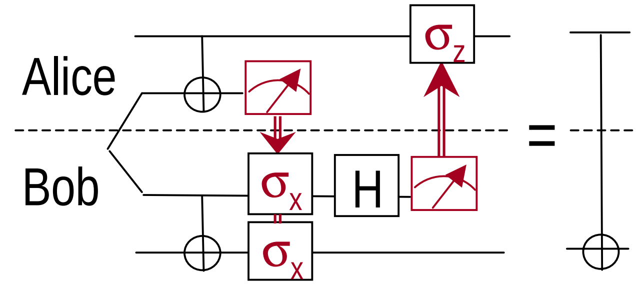
  $\Rightarrow t$ CNOT $\Rightarrow t$-use protocol

  $\therefore E, C^E_{\rightarrow}, C^E_{\leftarrow} \leq 1$

- $(H \otimes H)\, CNOT_{ab}\, (H \otimes H) = CNOT_{ba}$

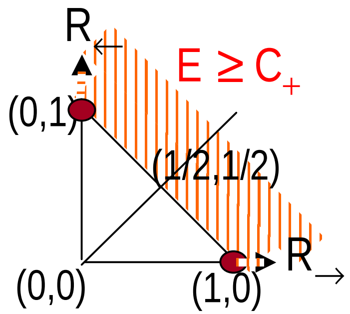
  $\therefore C_{\rightarrow} = C_{\leftarrow}$, $C^E_{\rightarrow} = C^E_{\leftarrow}$

Gottesman 98 :

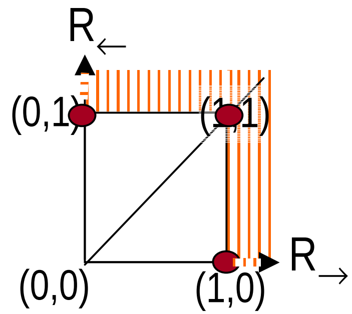


$E = 1,\ C^E_{\rightarrow} = C_{\rightarrow} (= C_{\leftarrow} = C^E_{\leftarrow}) = 1$

$(|0\rangle+|1\rangle)\,|0\rangle \rightarrow |00\rangle+|11\rangle$, $|i\rangle|0\rangle \rightarrow |ii\rangle$



Unassisted



Assisted

$C^E_{+} = 2$

i, j	$(\sigma_x^i \otimes \sigma_z^j)(\|00\rangle+\|11\rangle)$		
0,0	$\|00\rangle+\|11\rangle$		$(\|0\rangle+\|1\rangle)\|0\rangle$
0,1	$\|00\rangle-\|11\rangle$	$\rightarrow$	$(\|0\rangle-\|1\rangle)\|0\rangle$
1,0	$\|10\rangle+\|01\rangle$		$(\|0\rangle+\|1\rangle)\|1\rangle$
1,1	$\|10\rangle-\|01\rangle$		$(\|0\rangle-\|1\rangle)\|1\rangle$

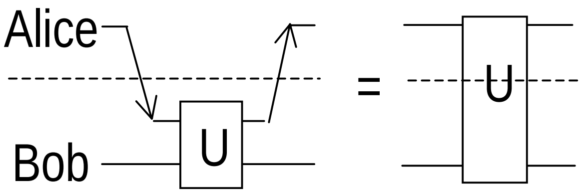
NB Asymptotic capacity = 1-shot

General bound I $\forall U$:

Bound 1: E , $C^E_{\rightarrow}$, $C^E_{\leftarrow} \leq 2 \log d_a$

Proof:

$2 \log d_a$ cbit$_{\rightarrow}$+ $2 \log d_a$ cbit$_{\leftarrow}$+ $2 \log d_a$ ebit $\Rightarrow$ any U on d_a-dim and d_b-dim systems



General bound II ∀U:

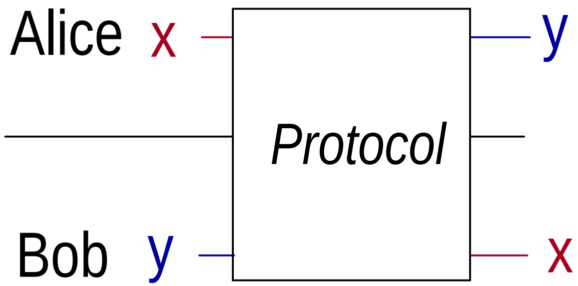
Bound 2 : $E \geq C_+$

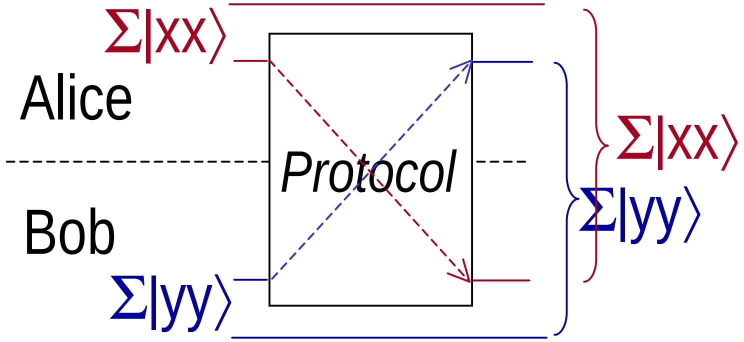
Idea : ∃ t & t-use protocol that sends $x \in \{1,\ldots,N_a\}$, $y \in \{1,\ldots,N_b\}$ with

$$(\log N_a + \log N_b) \div t \geq C_+ - \varepsilon$$

The same protocol can create a maximally entangled state over

two N_aN_b-dim systems. $\therefore E \geq (\log N_a + \log N_b) \div t \geq C_+ - \varepsilon$





Capacities generally difficult to find. Need to optimize over

1. arbitrary input (with ancillas, pure or mixed, possibly entangled over AB and different uses of U)
2. adaptive, interactive strategies
3. consider various auxiliary resources

- Approach: find upper & lower bounds, hope they coincide

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Entanglement capacity of U:

Max entanglement created per use of U

Need to specify :

1. # of uses
2. Auxiliary resources (e.g. free classical communication)
3. Input, output entanglement measures E_{in} , E_{out}

Result: Allowing any input state & ancilla (pure/mixed, possibly entangled), for a variety of reasonable entanglement measures:

- 1-shot capacity $E^{(1)}$ — optimal protocol: "apply U", optimal input is pure
- t-shot capacity $E^{(t)}$ = 1-shot capacity
- asymptotic capacity $E^{(\infty)}$ = 1-shot capacity, can "start-with-nothing"

Free CC, finite catalyst (or other sublinear resources) do not help.

Precise proof : 0205057

Entanglement capacity of U *(simplified)*:

Pure but arbitrary input states

Max entanglement created per use of U

Leifer, Henderson, Linden 0205055

Need to specify :

1. # of uses = t
2. Auxiliary resources (LOCC, arbitrary pure input with ancilla)
3. Unique asymptotic measure E_e = entropy of reduced density matrix

   E_e additive & nonincreasing under LOCC

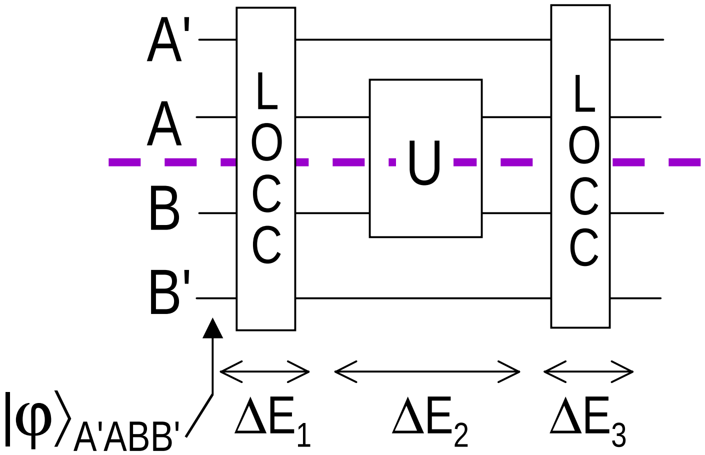
Then: t-shot capacity $E^{(t)} \equiv \sup_{\text{t-use protocol}, \rho_{in}} [E_e(\rho_{out}) - E_e(\rho_{in})] \div t$

any entangled input with any ancilla & catalyst, but subtract input entanglement

• 1-shot capacity $E^{(1)}$ — optimal protocol: "apply U", optimal input is pure

Claim: $E^{(1)} = \Delta E \equiv \sup_{|\varphi\rangle_{A'ABB'}} E_e(U_{AB}|\varphi\rangle_{A'ABB'}) - E_e(|\varphi\rangle_{A'ABB'})$.

Pf: *Any 1-use protocol:*



Entanglement increases

by $\Delta E_1 + \Delta E_2 + \Delta E_3 \leq \Delta E$.

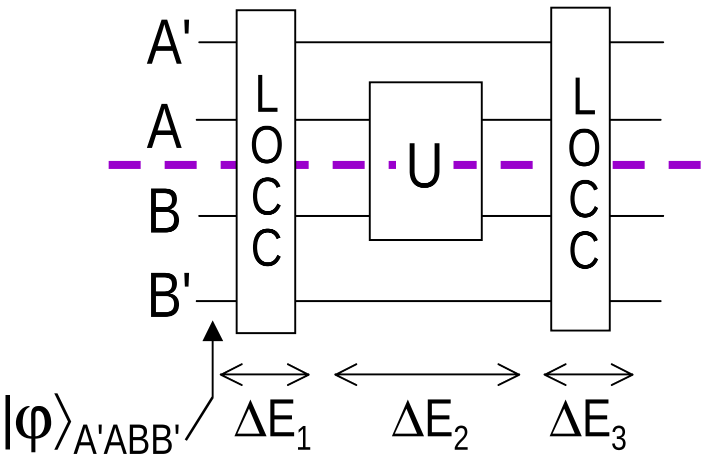
$\wedge|$ $\wedge|$ $\wedge|$

0 ΔE 0

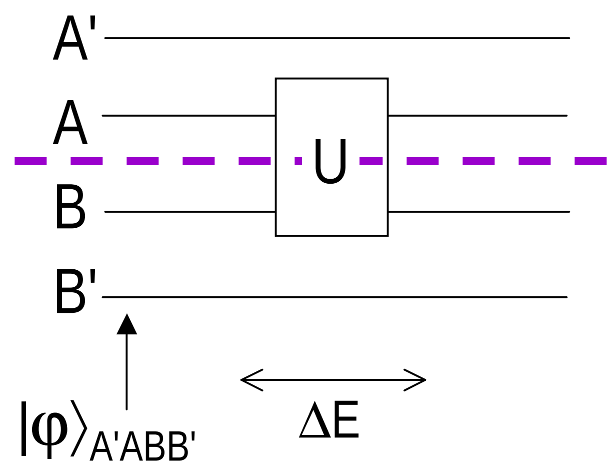
• 1-shot capacity $E^{(1)}$ — optimal protocol: “apply U”, optimal input is pure

Claim: $E^{(1)} = \Delta E \equiv \sup_{|\varphi\rangle_{A'ABB'}} E_e(U_{AB}|\varphi\rangle_{A'ABB'}) - E_e(|\varphi\rangle_{A'ABB'})$

Pf: *Any 1-use protocol:*



Optimal 1-use protocol:



Entanglement increases

by $\Delta E_1 + \Delta E_2 + \Delta E_3 \leq \Delta E$.

$$\begin{array}{ccc} \Delta E_1 & \Delta E_2 & \Delta E_3 \\ \wedge\!| & \wedge\!| & \wedge\!| \\ 0 & \Delta E & 0 \end{array}$$

NB. Classical communication does not help.

Simply explicit expression for $E^{(1)}$ and simple optimal protocol.

Open question: how large A' B' need to be? ☹

- Unclear if infinite dim is strictly better.
- Limit to unbounded finite dims:

Let $e_n = \sup_{|\varphi\rangle_{A'ABB'}\,:\,A', B'\text{ n-dim}} E(U|\varphi\rangle_{A'ABB'}) - E(|\varphi\rangle_{A'ABB'})$.

$\{e_n\}$ is increasing & upper bounded, thus converges.

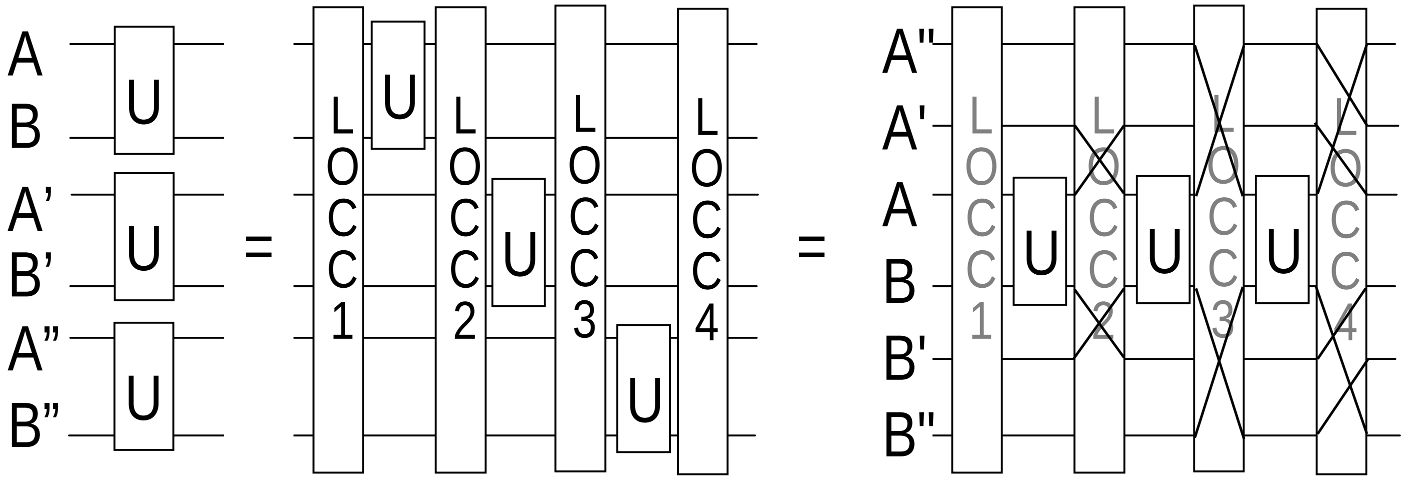
$\forall \varepsilon > 0$ $E^{(1)} - \varepsilon$ can be attained by large enough dim of A' B'.

Call $|\varphi\rangle_{A'ABB'}$ optimal input (attaining $E^{(1)} - \varepsilon$).

• t-shot capacity $E^{(t)}$ = 1-shot capacity

Claim: $E^{(1)} = E^{(t)}$

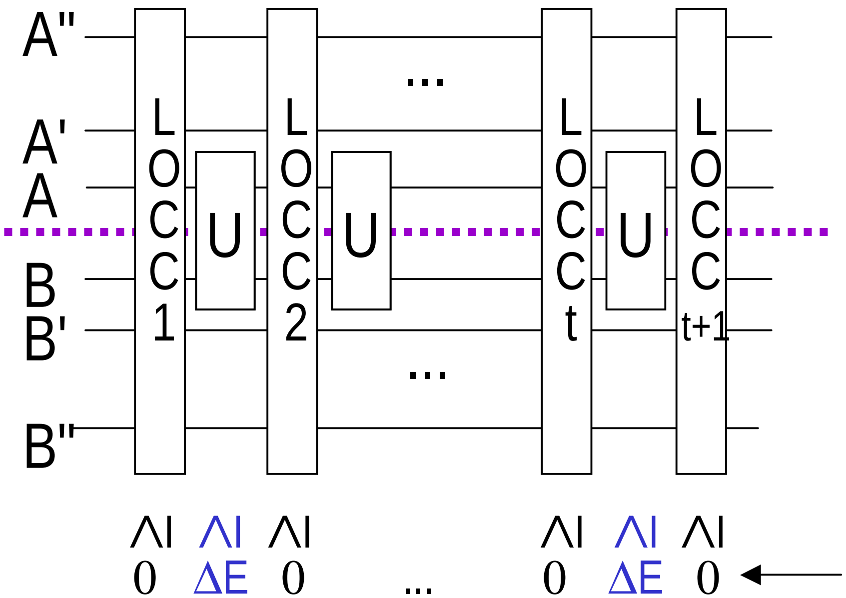
Observation: parallel use of U is special case of sequential use.



- t-shot capacity $E^{(t)}$ = 1-shot capacity

Claim: $E^{(1)} = E^{(t)}$

Most general t-use protocol

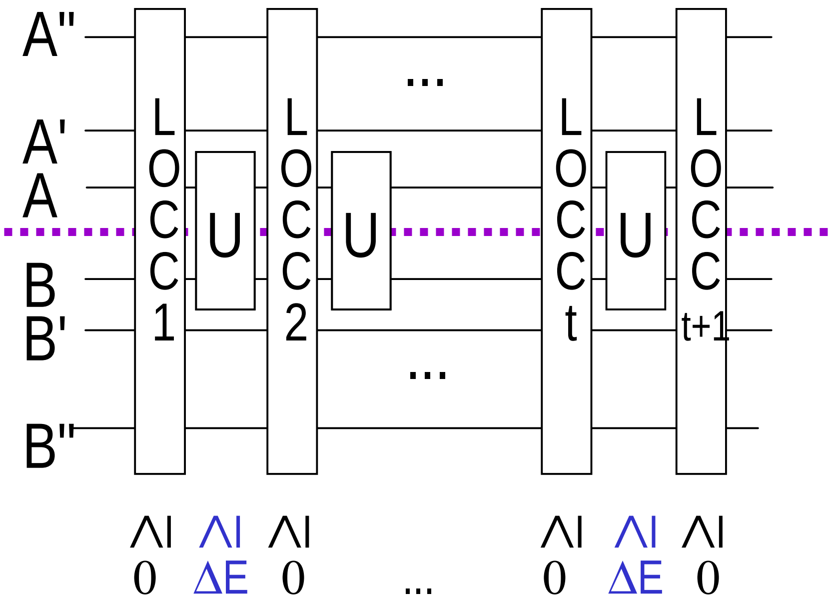


total increase $\leq t\Delta E$

• t-shot capacity $E^{(t)}$ = 1-shot capacity

Claim: $E^{(1)} = E^{(t)}$

Most general t-use protocol



- *Entanglement increases by no more than* t $E^{(1)}$

  $\therefore E^{(t)} \leq E^{(1)}$

- $E^{(t)} \geq E^{(1)}$ by repeating optimal 1-shot protocol t times

Corollary: asymptotic capacity $E^{(\infty)} = E^{(1)}$ *.*

NB: CC & collective strategy do not help.

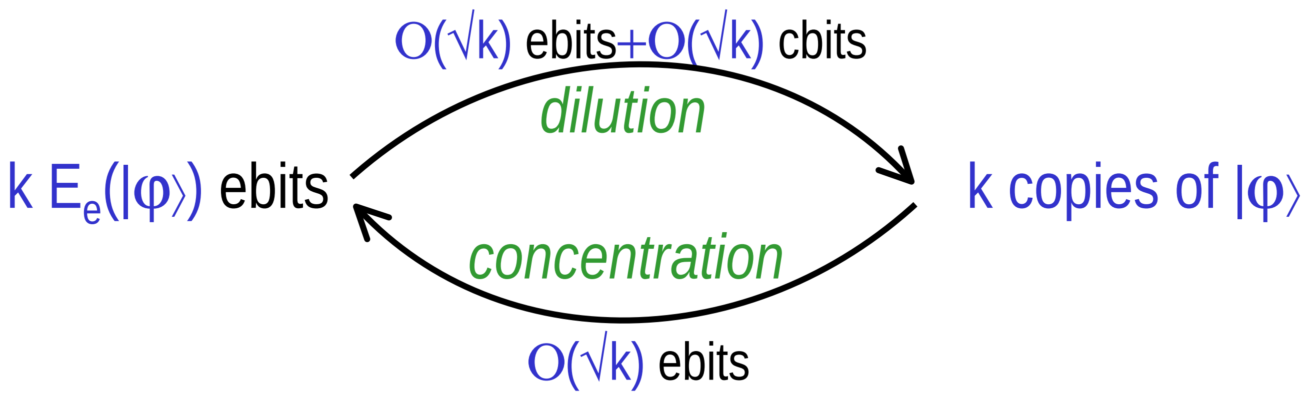
• Asymptotic capacity $E^{(\infty)}$ = 1-shot capacity, can “start-with-nothing”

i.e. Asymptotically, allowing arbitrary input states does not help.

Idea :

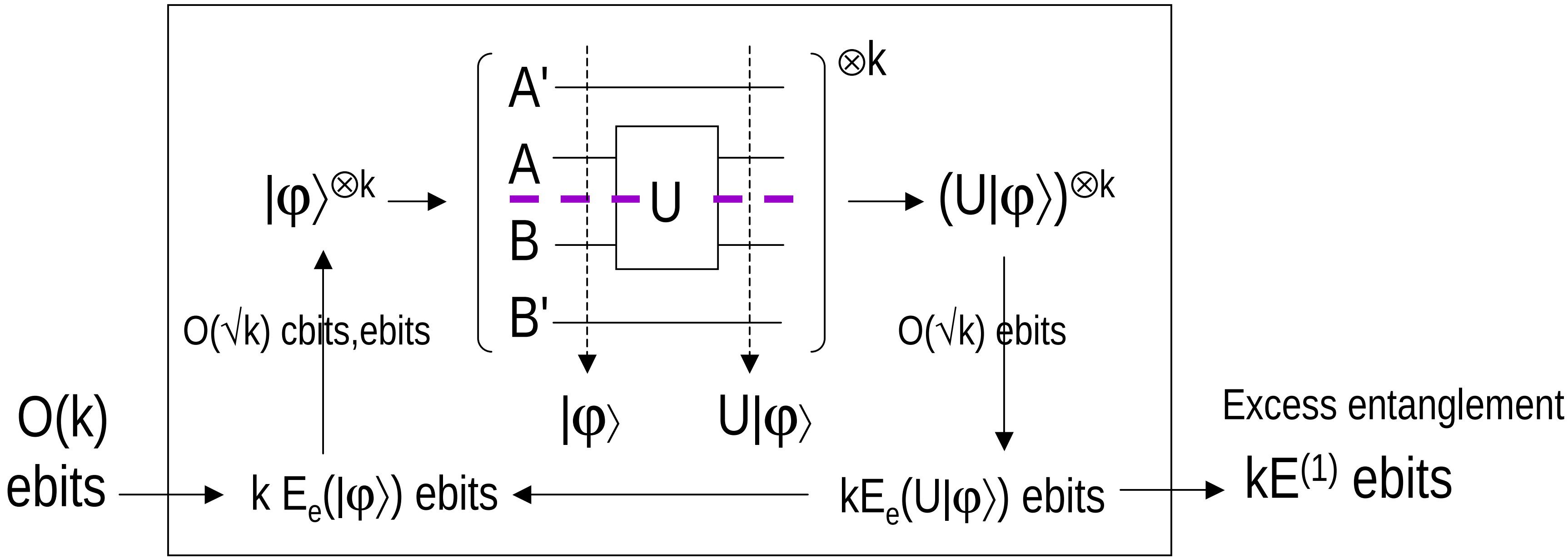
Let $|\varphi\rangle$ = optimal input for 1-shot capacity

Create $|\varphi\rangle$ using ebits by *dilution & convert* $U|\varphi\rangle$ to ebits by *concentration*.



Protocol: First create $|\varphi\rangle^{\otimes k}$, then repeat the cycle:

- apply 1-use protocol to each
- concentration $(U|\varphi\rangle)^{\otimes k}$ to $k\,E_e(U|\varphi\rangle)$ ebits
- dilution $k\,E_e(|\varphi\rangle)$ ebits to $|\varphi\rangle^{\otimes k}$



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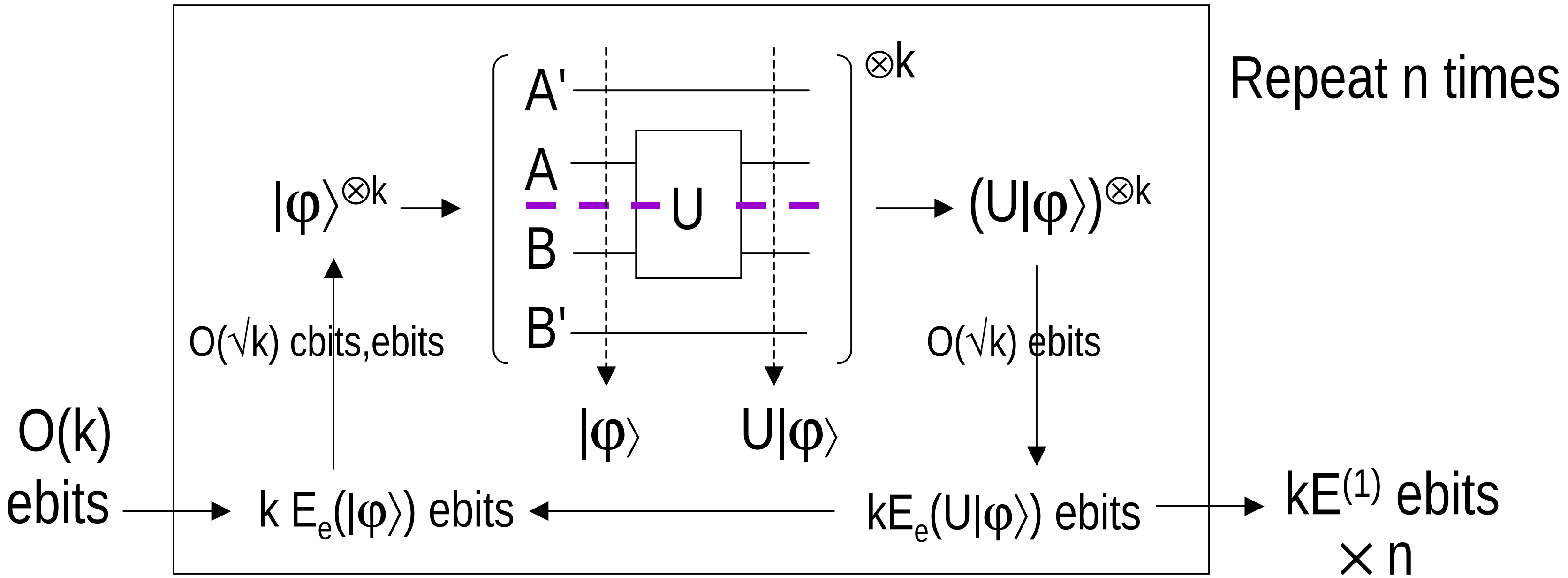

$$\frac{n\,k\,E^{(1)}}{n\,k + n\,O(k^{1/2}) + O(k)} \rightarrow E^{(1)} = E^{(\infty)}$$

NB. Protocol has optimal rate but uses no initial/auxiliary resources. Collective only in dilution & conc.

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- Entanglement-assisted 1-way classical capacity

Asymptotic capacity is a 1-shot expression with protocol acting on product states (additive).

Entanglement-assisted 1-way classical capacity $C^E_{\rightarrow}$:

Asymptotic $C^E_{\rightarrow} = \Delta\chi^{(1)} \equiv \sup_{\mathcal{E}=\{p_i,|\varphi_i\rangle_{A'ABB'}\}} [\, \chi\, (tr_{AA'}\, U_{AB}\, \mathcal{E}\,) - \chi\, (tr_{AA'}\, \mathcal{E}\,)\,]$

where for $\mathcal{E} = \{p_i, |\varphi_i\rangle_{A'ABB'}\}$

$U_{AB}\, \mathcal{E} = \{p_i, U_{AB}\, |\varphi_i\rangle_{A'ABB'}\}$

$tr_{AA'}\, \mathcal{E} = \{p_i, tr_{AA'}\, |\varphi_i\rangle\langle\varphi_i|_{A'ABB'}\}$ "reduced ensemble"

$tr_{AA'}\, U_{AB}\, \mathcal{E} = \{p_i, tr_{AA'}\, U_{AB}\, |\varphi_i\rangle\langle\varphi_i|_{A'ABB'}\, U^{\dagger}_{AB}\}$

and $\chi(\{p_i, \rho_i\}) = S(\Sigma_i\, p_i\, \rho_i\,) - \Sigma_i\, p_i\, S(\rho_i\,)$ = Holevo information

$\Delta\chi^{(1)}$ = optimal increase in Holevo information of Bob's reduced ensemble.

Preliminaries:

Let $\mathcal{E} = \{p_i, \rho_i\}$ be an ensemble.

Holevo: If Alice prepares ρ_i w.p. p_i and Bob measures,

Bob's information on i $\leq \chi(\mathcal{E}) = S(\Sigma_i\, p_i\, \rho_i) - \Sigma_i\, p_i\, S(\rho_i)$.

HSW: Bob's information on $(i_1, i_2, \ldots, i_n)$ approaches $n\chi(\mathcal{E})$ if Bob measures $\rho_{i1} \otimes \rho_{i2} \otimes \ldots \otimes \rho_{in}$ for large n .

RSP: Let $\mathcal{E} = \{p_i, |\varphi_i\rangle\}$ be a bipartite ensemble (Alice and Bob). Alice can choose $|\varphi_{i1}\rangle \otimes |\varphi_{i2}\rangle \otimes \ldots \otimes |\varphi_{in}\rangle$ and prepare for Bob $n\chi(\mathrm{tr}_A\mathcal{E}) + o(n)$ forward cbits + free entanglement .

Thm 2 (upper half) : $C^E_{\rightarrow} \leq \Delta\chi^{(1)}$

Pf: Given initial ensemble $\mathcal{E}_0 = \{p_i, |\varphi_i\rangle_{AB}\}$,
at each stage, reduced ensemble $tr_{AA'}(\mathcal{E}_i)$
and $\chi(tr_{AA'}(\mathcal{E}_i))$ well defined.

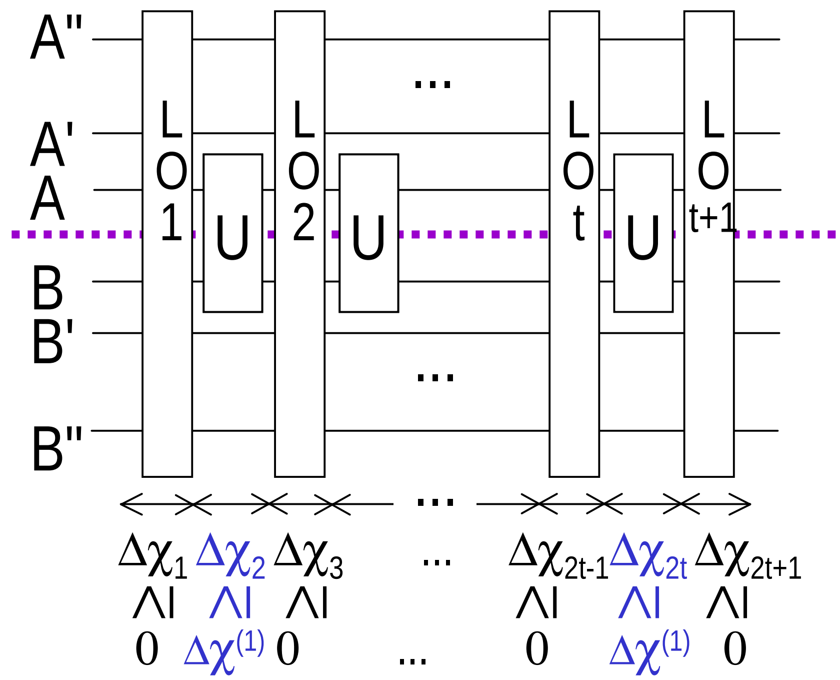
$\chi(\text{final}) - \chi(\text{initial}) \leq \Sigma\Delta\chi_i \leq t\, \Delta\chi^{(1)}$

$\vee$I	I$\wedge$	
Acc Info (comm achieved)	Initial info (comm needed)	by the protocol

$\forall t \;\; R_{\rightarrow} \leq \Delta\chi^{(1)}$

$\therefore\; C^E_{\rightarrow} \leq \Delta\chi^{(1)}$

Most general t-use protocol :

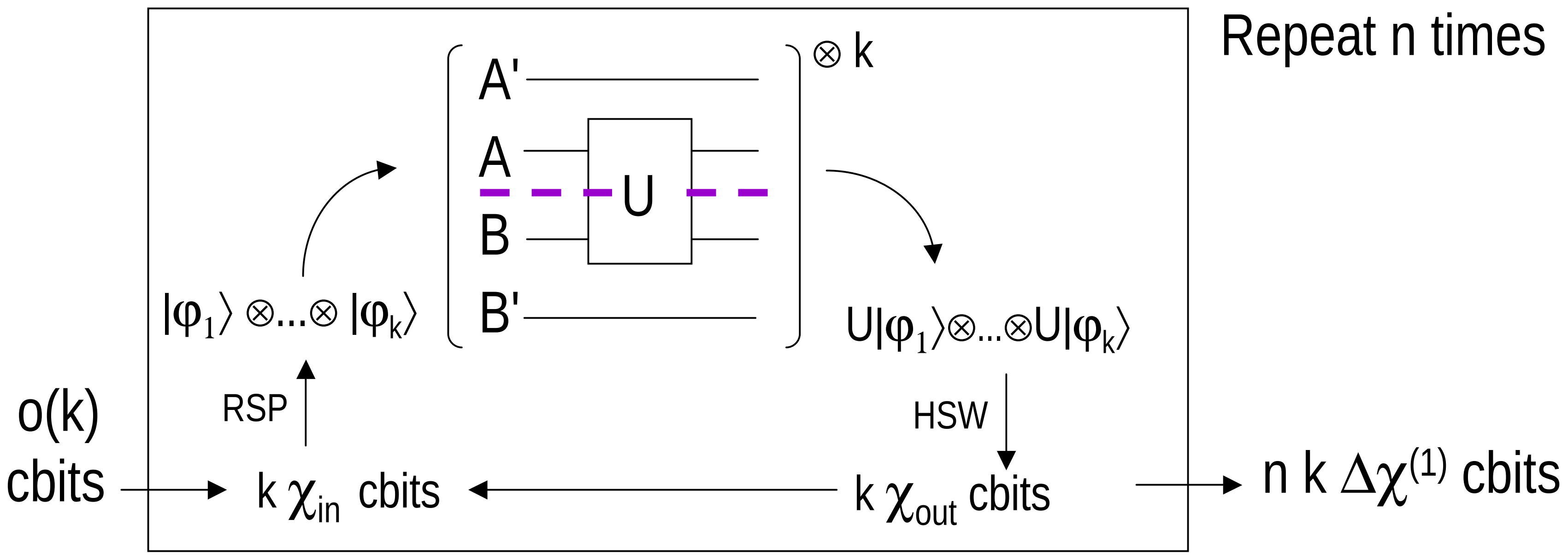


Thm 2 (lower half) : $C^E_{\rightarrow} \geq \Delta\chi^{(1)}$

Pf: Let $\mathcal{E} = \{p_i, |\varphi_i\rangle_{A'ABB'}\}$ be optimal ensemble for $\Delta\chi^{(1)}$

Let $\chi_{in} = \chi(\mathrm{tr}_{AA'} \mathcal{E})$, $\chi_{out} = \chi(\mathrm{tr}_{AA'} U_{AB} \mathcal{E})$

Repeat the entanglement generation method ??



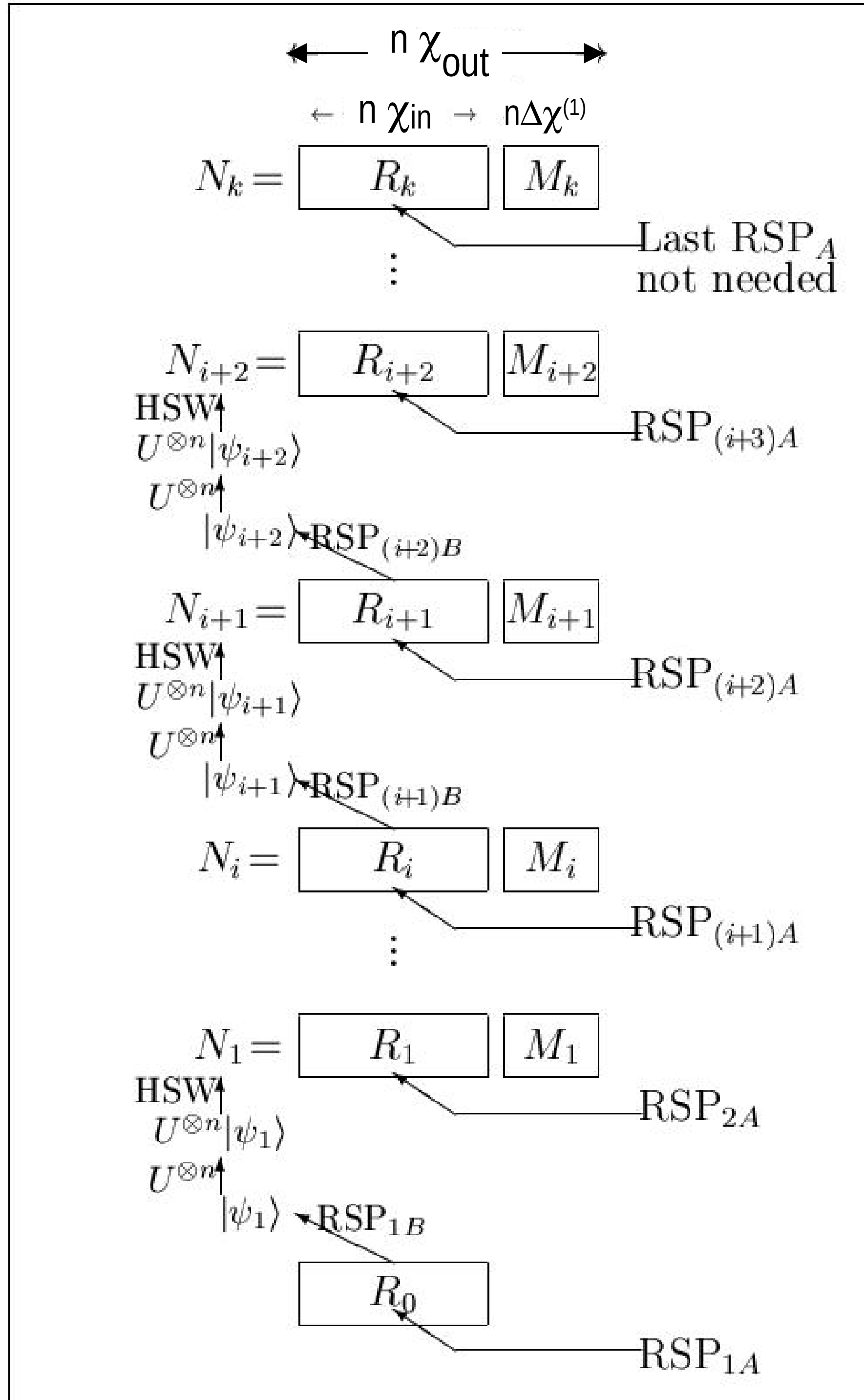
To send k messages $M_1, M_2, \ldots, M_k$ each of length $n(\chi_{out} - \chi_{in}) = n\,\Delta\chi^{(1)}$

Alice calculates k $(n\chi_{out})$-bit messages

- $N_k = (R_k, M_k)$ encoded in $U|\varphi_{k1}\rangle \otimes U|\varphi_{k2}\rangle \ldots \otimes U|\varphi_{kn}\rangle$
- $N_{k-1} = (R_{k-1}, M_{k-1})$ encoded in $U|\varphi_{k-1\,1}\rangle \otimes U|\varphi_{k-1\,2}\rangle \ldots \otimes U|\varphi_{k-1\,n}\rangle$ where R_{k-1} helps Bob create $|\varphi_{k1}\rangle \otimes |\varphi_{k2}\rangle \ldots \otimes |\varphi_{kn}\rangle$ with RSP

...

- $N_1 = (R_1, M_1)$ encoded in $U|\varphi_{1\,1}\rangle \otimes U|\varphi_{1\,2}\rangle \ldots \otimes U|\varphi_{1\,n}\rangle$ R_1 helps Bob create $|\varphi_{21}\rangle \otimes |\varphi_{22}\rangle \ldots \otimes |\varphi_{2n}\rangle$ by RSP.
- Create $|\varphi_{11}\rangle \otimes |\varphi_{12}\rangle \ldots \otimes |\varphi_{1n}\rangle$ with O(n) uses of the gate



Discussions:

1. Hamiltonian capacities $\equiv$ amount of X produced per time use of H are infinitesimal gate capacities:

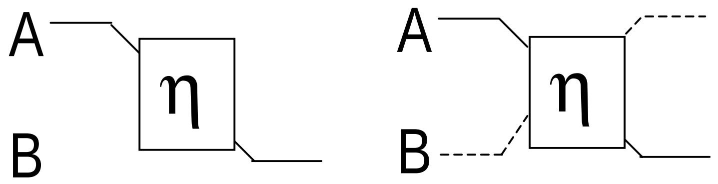
$$X_H = \lim_{s\to 0} \ 1/s \ \ X_{U=e^{-iHs}}$$

0207052: 2-qubit H

For $H = \alpha\, \sigma_x \otimes \sigma_x + \beta\, \sigma_y \otimes \sigma_y$

optimal input has no ancillas & $E_H \approx 1.9\ (\alpha+\beta)$ (0207052)

2. Interaction as a generalization of usual quantum channel



Nonunitary case: work in progress & various complications.

3. Details: 0205057, inspirations: 0006034, very related: 0205055, 0207052, remotely related: 0011050, 0205181, 0207065, 0208077.