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Unknown quantum operations: A de Finetti representation theorem

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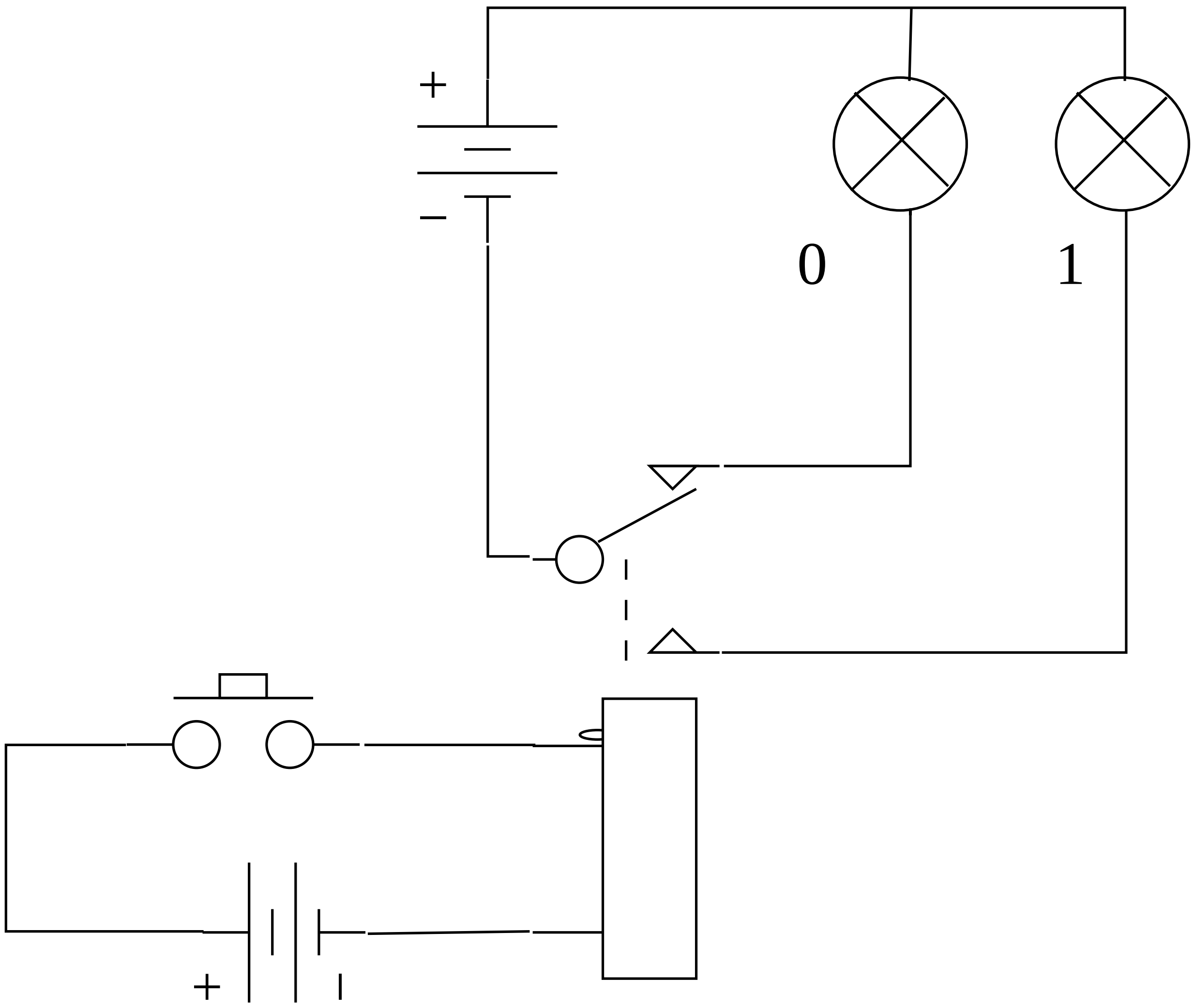
An early project



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Overview

1. What’s the point of this story?
2. Why a representation theorem for quantum operations?
3. The classical de Finetti theorem
4. The quantum de Finetti theorem
5. Bayesian quantum process tomography
6. De Finetti theorem for quantum operations

What's the point of this story?

No randomness in —
no randomness out!

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Bayesian probability theory: p is a degree of belief, not part of physical reality.

Does quantum mechanics help?

It certainly looks like it:

A measurement in the basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ yields the result $+$. The post-measurement state is

$$\mathcal{F}(\rho) = \frac{|+\rangle\langle+|\rho|+\rangle\langle+|}{\langle+|\rho|+\rangle} = |+\rangle\langle+| \, .$$

A subsequent measurement in the 0-1 basis gives $p = \mathrm{Pr}(0) = \frac{1}{2}$.

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Is p part of physical reality?

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$\mathcal{F}$ is part of physical reality $\Longrightarrow$ $|+\rangle$ is part of physical reality $\Longrightarrow$ p is part of physical reality.

Two kinds of probability?

Consider a mixture

$$\rho = p|+\rangle\langle+| + (1-p)|-\rangle\langle-| \ ,$$

where p is a Bayesian degree of belief, and $|\pm\rangle\langle\pm|$ are elements of physical reality.

But ρ can be rewritten

$$\rho = q|\psi\rangle\langle\psi| + (1-q)|\phi\rangle\langle\phi| \ .$$

What kind of probability is q?

Preferred decomposition of density operators?

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The argument:

$\mathcal{F}$ is part of physical reality $\Longrightarrow$ $|+\rangle$ is part of physical reality $\Longrightarrow$ p is part of physical reality.

Equivalently,

p is not part of physical reality $\Longrightarrow$ $|+\rangle$ is not part of physical reality $\Longrightarrow$ $\mathcal{F}$ is not part of physical reality.

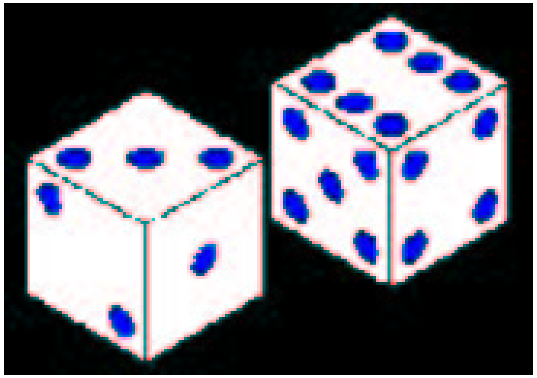
A Bayesian version of quantum process tomography?

Quantum process tomography: Determine a quantum operation by making measurements on the output for a set of well-chosen inputs.

What does it mean to determine an unknown quantum operation, $\mathcal{F}$, if $\mathcal{F}$ is not part of physical reality?

Needed: A Bayesian formulation of quantum process tomography.

Classical tomography



Result in 10 throws:

1 times $k = 1$,
4 times $k = 2$,
2 times $k = 3$,
2 times $k = 4$,
1 times $k = 5$,
0 times $k = 6$.

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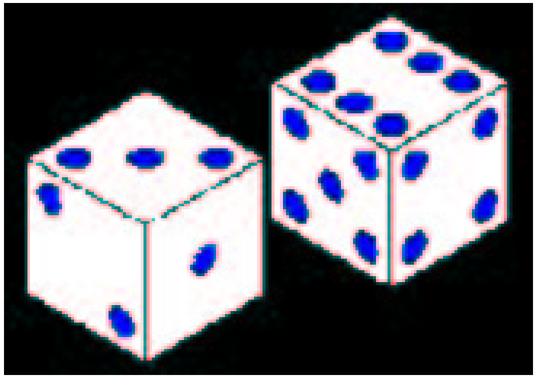

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Question: What probability p do you assign to a 6 in the next throw?

Answer: It depends on your prior.

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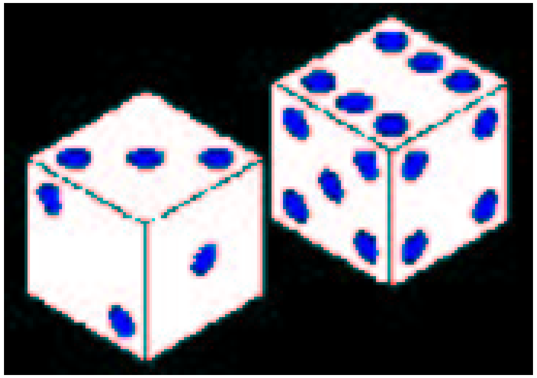

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Question: What probability p do you assign to a 6 in the next throw?

Answer I: $p = 1/6$ if you believe that the die is fair.

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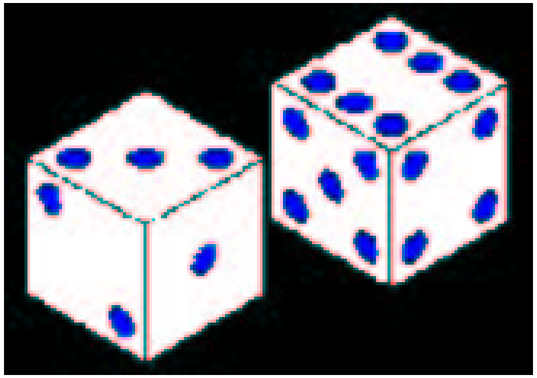

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Question: What probability p do you assign to a 6 in the next throw?

Answer II: $p = 1/12$ given a totally uninformative prior (Laplace's rule of succession).

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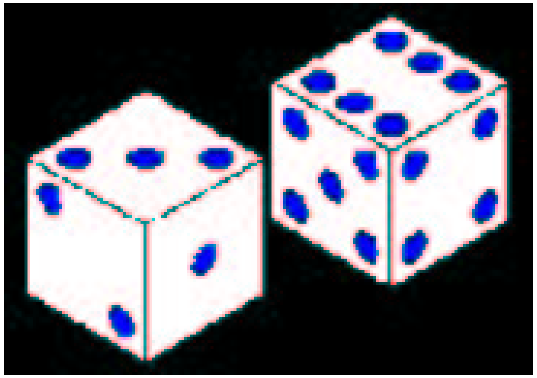

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Question: What probability p do you assign to a 6 in the next throw?

Answer III: $p = 0$ if you know the die came from a box that contains only trick dice of two types: type A never comes up 1, type B never comes up 6.

Exchangeability (binary case)

Let $x_k \in \{0,1\}$ be binary random variables.

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For given N, we say that $p^{(N)}(x_1,\ldots,x_N)$ is exchangeable if it is part of an exchangeable sequence.

De Finetti's representation theorem (binary case)

$p^{(N)}(x_1, \ldots, x_N)$ is exchangeable

if and only if

$$p^{(N)}(x_1, \ldots, x_N) = \int_0^1 P(p)\, p^k (1-p)^{N-k} dp$$

where $P(p)$ is unique and k is the number of zeroes in $(x_1, \ldots, x_N)$.

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Bayesian (classical) tomography

(i) Start from an exchangeable prior for $N + M$ trials.

(ii) Collect data for N trials.

(iii) Use Bayes' rule to obtain (posterior) probabilities for the remaining M trials.

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Two agents starting from different priors will always converge to a joint posterior in the limit of large N (under mild assumptions about the priors).

Key assumption: Exchangeability.

Exchangeability for quantum systems

A state $\rho^{(N)}$ of N systems is exchangeable if it is a member of an exchangeable sequence $\rho^{(n)}$, i.e.,

(i) (symmetry) each $\rho^{(n)}$ is invariant under permutations of the n systems on which it is defined; and

(ii) (extendibility) $\rho^{(n)} = \mathrm{tr}_{n+1}\rho^{(n+1)}$ for all n, where tr_{n+1} denotes the partial trace over the $(n+1)$th system.

Quantum de Finetti Theorem

$\rho^{(N)}$ is exchangeable

if and only if

$$\rho^{(N)} = \int d\rho \, p(\rho) \, \rho^{\otimes N} = \int d\rho \, p(\rho) \, \rho \otimes \cdots \otimes \rho,$$

where $p(\rho) \geq 0$ is unique.

(Hudson, Moody 1976; Caves, Fuchs, RS 2002)

Bayesian quantum tomography

$$\rho^{(N+M)} = \int d\rho \; p(\rho) \; \rho^{\otimes (N+M)}$$

measure N subsystems

get outcome $\vec{\alpha}$

$$\rho^{(M)} = \int d\rho \; p(\rho|\vec{\alpha}) \; \rho^{\otimes M}$$

$p(\rho|\vec{\alpha})$ given by the quantum Bayes rule.

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Quantum process tomography: limited resources

Any finite version of quantum process tomography depends ineluctably on a prior.

Wanted: A representation theorem for priors in the space of quantum operations on N copies of a system.

Bayesian quantum process tomography

We describe N uses of a quantum channel by a quantum operation (trace-preserving cpm)

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5. We deduce $\Phi^{(M)}(\rho^{(M)})$.

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2.) (Extendibility) $\Phi^{(n)}(\rho^{(n)}) = \mathrm{tr}_{n+1}\Big(\Phi^{(n+1)}(\rho^{(n+1)})\Big)$ for any states $\rho^{(n)}$, $\rho^{(n+1)}$ such that $\rho^{(n)} = \mathrm{tr}_{n+1}\rho^{(n+1)}$.

De Finetti representation for quantum operations

$\Phi^{(N)}$ is exchangeable (i.e., part of an exchangeable sequence)

if and only if

$$\Phi^{(N)} = \int d\Phi \; p(\Phi) \; \Phi^{\otimes N}$$

where the integral ranges over all single-system quantum operations $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$, and where $p(\Phi) \geq 0$ is unique.

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De Finetti for quantum operations proof sketch (1)

Φ is a quantum operation on $\mathcal{H}_d$ if it is a trace-preserving completely positive map (cpm), i.e.,

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1.) $\Phi(\rho)$ is a density operator for any density operator ρ on $\mathcal{H}_d$;

2.) $(I \otimes \Phi)(\rho^{(2)})$ is a density operator for any density operator $\rho^{(2)}$ on $\mathcal{H}_{d'} \otimes \mathcal{H}_d$ (d' arbitrary).

De Finetti for quantum operations
proof sketch (2)

Let $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |e_j\rangle \otimes |e_j\rangle \quad \in \mathcal{H}_d \otimes \mathcal{H}_d$

be a maximally entangled state.

Theorem (Jamiołkowski):

A linear map Φ on $\mathcal{H}_d$ is a cpm

if and only if

$(I \otimes \Phi)(|\psi\rangle\langle\psi|)$ is a positive operator.

De Finetti for quantum operations proof sketch (3)

Denote by $\rho^{(N)}$ the Jamiołkowski density operator corresponding to the quantum operation $\Phi^{(N)}$.

$\Phi^{(N)}$ exchangeable

$\Longrightarrow$ $\rho^{(N)}$ exchangeable

$\Longrightarrow$ unique de Finetti representation for $\rho^{(N)}$

$\Longrightarrow$ unique de Finetti representation for $\Phi^{(N)}$.

De Finetti for quantum operations proof sketch (4)

There exist non-trace-preserving cpm's Φ such that $(I \otimes \Phi)(|\psi\rangle\langle\psi|)$ is a density operator.

How does one see that the domain of the integral

$$\Phi^{(N)} = \int d\Phi \; p(\Phi) \; \Phi^{\otimes N}$$

includes only trace-preserving cpm's?

De Finetti for quantum operations proof sketch (5)

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Assume $\mathrm{tr}[\Phi_1(\rho)] \neq 1$ for some ρ.

Then there is k such that $\mathrm{tr}[\Phi_k(\rho)] > 1$.

Then $1 \geq p_k \big(\mathrm{tr}[\Phi_k(\rho)]\big)^N \to \infty$

Bayesian quantum process tomography

We describe N uses of a quantum channel by a quantum operation (trace-preserving cpm)

$$\Phi^{(N)} : \mathcal{L}(\mathcal{H}^{\otimes N}) \longrightarrow \mathcal{L}(\mathcal{H}^{\otimes N})$$

1. We start from an exchangeable prior $\Phi^{(N+M)}$.
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States that are not exchangeable

1.) $\rho = \rho_1 \otimes \rho_2$ is not exchangeable for $\rho_1 \neq \rho_2$.

2.) $\rho_{\mathrm{GHZ}} = \frac{1}{2}(|000\rangle + |111\rangle)(\langle 000| + \langle 111|)$ is not exchangeable:

there exists no permutation-invariant $\rho^{(4)}$ such that $\rho_{\mathrm{GHZ}} = \mathrm{tr}_4(\rho^{(4)})$.

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Quantum Bayes rule: detail

$$p(\rho|\vec{\alpha}) = \frac{p(\rho)p(\vec{\alpha}|\rho)}{p_\alpha} ,$$

where $p(\vec{\alpha}|\rho) = \mathrm{tr}\big(\rho^{\otimes N} E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N} \big)$,

$\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$,

$\{E_k\}$ is a POVM,

and $p_\alpha = \int d\rho \, p(\rho) \, p(\vec{\alpha}|\rho)$.

Jamiołkowski operator

Matrix elements of Φ:

$$\Phi(|e_j\rangle\langle e_k|) = \sum_{l,m} S_{lj,mk}|e_l\rangle\langle e_m|$$

Matrix elements of Jamiołkowski's ρ:

$$\begin{aligned}\rho &= (I \otimes \Phi)(|\psi\rangle\langle\psi|) \\ &= \frac{1}{d}\sum_{l,j,m,k} S_{lj,mk}(|e_j\rangle \otimes |e_l\rangle)(\langle e_k| \otimes \langle e_m|)\end{aligned}$$

Quantum de Finetti: proof sketch

Let $\{E_1, \ldots, E_{d^2}\}$ be a minimal informationally complete POVM on d-dim. Hilbert space $\mathcal{H}_d$.

Then there is a one-to-one correspondence between states ρ and probabilities $(p_1, \ldots, p_{d^2})$:

$$\mathrm{tr}(\rho E_1) = p_1$$

$$\mathrm{tr}(\rho E_2) = p_2$$

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$\{E_1, \ldots, E_{d^2}\}$ form a basis of the d^2-dimensional vector space $\mathcal{L}(\mathcal{H}_d)$.

Measurements on N systems

Results are random variables $\alpha_k \in \{1, \ldots, d^2\}$ with distribution

$$p^{(N)}(\alpha_1, \ldots, \alpha_N) = \mathrm{tr}\big(\, \rho^{(N)}\, E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N} \big)$$

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$$p^{(N)}(\alpha_1, \ldots, \alpha_N) = \mathrm{tr}\big(\, \rho^{(N)}\, E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N} \big)$$

$\rho^{(N)}$ exchangeable

$\Longrightarrow$ $p^{(N)}$ exchangeable

$\Longrightarrow$ classical de Finetti representation for $p^{(N)}$

$\Longrightarrow$ quantum de Finetti representation for $\rho^{(N)}$.

Construction of a minimal ICPOVM (1)

1.) Let $\{|e_j\rangle\}$ be an orthonormal basis of $\mathcal{H}_d$. A basis of $\mathcal{L}(\mathcal{H}_d)$ is then given by the d^2 projectors Π_α of the form

$$\Pi_\alpha = |e_j\rangle\langle e_j|$$

or $\Pi_\alpha = \frac{1}{2}\Big(|e_j\rangle + |e_k\rangle\Big)\Big(\langle e_j| + \langle e_k|\Big)$

or $\Pi_\alpha = \frac{1}{2}\Big(|e_j\rangle + i|e_k\rangle\Big)\Big(\langle e_j| - i\langle e_k|\Big)$

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5.) $E_\alpha = G^{-1/2} \Pi_\alpha G^{-1/2}$ form a minimal ICPOVM.