Universal Source Coding, Soft Tomography, and Universal concentration

Keiji Matsumoto
ERATO QCI Project,Japan Science and Technology
keiji@qci.jst.go.jp

collaboration with M.HAYASHI

I. Universal Source Coding & Tender Measurement

1. Definitions
2. Tender Measurement & State Estimation
3. Application to Universal Source Code
4. Optimal Protocol & Group Representation

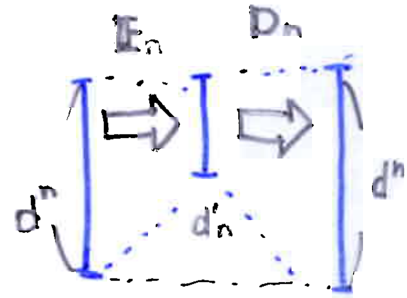
II Universal Entanglement Concentration without distortion

1. Definition & Protocol
2. Optimality

Quantum Source Coding

$$\rho_x \sim p_x, \qquad \rho_x \in \mathcal{S}(\mathbf{C}^d)$$

$$\begin{array}{ccc} \rho_{x_1} \otimes \rho_{x_2} \cdots \otimes \rho_{x_n} & \sim & p_{x_1} p_{x_2} \cdots p_{x_n} \\ (:= \rho_{\boldsymbol{x}_n}) & & (:= p_{\boldsymbol{x}_n}) \end{array}$$



$$\boldsymbol{x}_n = (x_1, \cdots, x_n)$$

- **(Encode)**

$$\mathbf{E}_n : \rho_{\boldsymbol{x}_n} \mapsto \mathbf{E}_n(\rho_{\boldsymbol{x}_n}) \in \mathcal{S}(\mathbf{C}^{d'_n}) \qquad (d'_n \leq d^n)$$

- **(Decode)**

$$\mathbf{D}_n : \mathbf{E}_n(\rho_{\boldsymbol{x}_n}) \mapsto \mathbf{D}_n \circ \mathbf{E}_n(\rho_{\boldsymbol{x}_n}) \in \mathcal{S}(\mathbf{C}^{d \otimes n})$$

- Optimize $\mathbf{E}_n$ and $\mathbf{D}_n$ to minimize

$$\lim_{n\to\infty} \frac{1}{n} \log d'_n \qquad \text{compression rate}$$

subject to

$$\sum_{x_n} \mathrm{dist}(\rho_{\boldsymbol{x}_n}, \mathbf{D}_n \circ \mathbf{E}_n(\rho_{\boldsymbol{x}_n})) \to 0 \quad (n \to \infty)$$

- Here, we use Bure's distance

$$\mathrm{dist}(\rho, \sigma) = \mathrm{B}(\rho, \sigma) := 1 - \sqrt{\mathrm{tr}\, \rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}}} \qquad (1)$$

Pure State Source (Source is known)

Theorem 1 *If ρ_x is pure,*

$$\lim_{n\to\infty} \frac{1}{n} \log d'_n \geq S(\bar{\rho}) \tag{2}$$

and equality can be achived. Here,

eigen vector of $\bar{\rho}$

$$\bar{\rho} = \sum_x p_x \rho_x = \sum_j q_j |\, e_j \,\rangle\langle\, e_j \,|$$

eigenvalue of $\bar{\rho}$

$$S(\rho) = -\operatorname{tr} \rho \log \rho$$

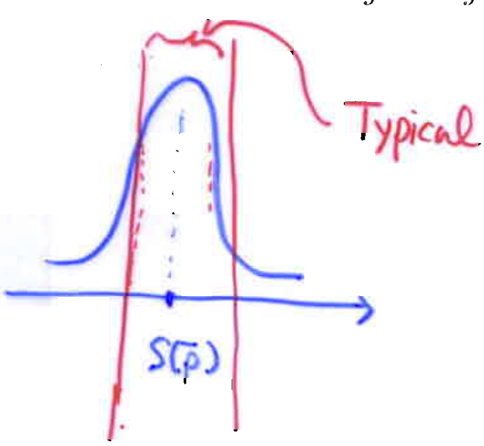
- (Asymptotically) optimal encoder is projection **entropy typical subspace**, and decoder is just isometric injection.
- Entropy typical subspace

$$\bigoplus_{v} \mathcal{U}^{\otimes n} \quad \bigoplus_{\mathbf{n}: H(\frac{n_i}{n}) < S(\bar{\rho}) + \epsilon} \mathcal{H}_{\mathbf{n}}. \tag{3}$$

where $\mathcal{H}_{\mathbf{n}}$ is eigenspace of $\bar{\rho}^{\otimes n}$ corresponding to the eigenvalue $\prod_{j=1}^{d} q_j^{n_j}$.

Shannon entropy

$H(q) := -\sum_j q_j \log q_j$



Fixed Length Universal Code

- Source is unknown
- Compress by the rate R.
- If $S(\bar{\rho}) < R$, successfully compressed by the rate R.
- Otherwise fails

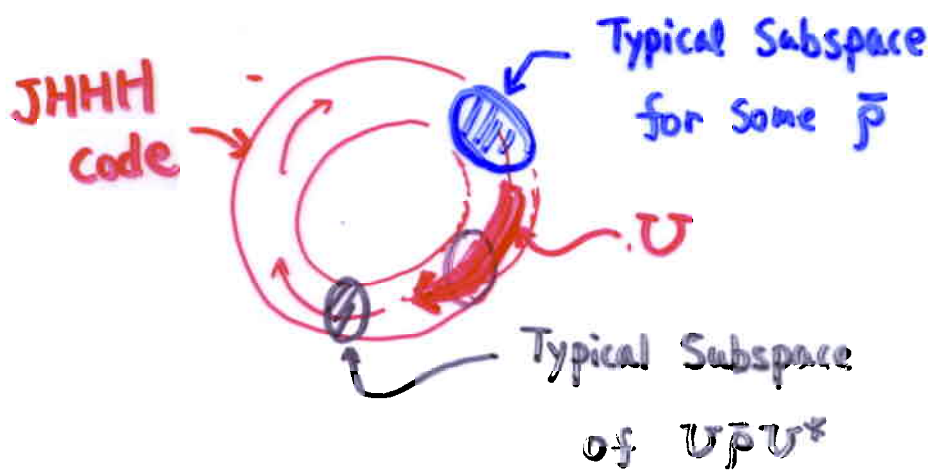
Assume $S(\bar{\rho})$ is known or can be estimated

Jozsa-Horodeckis Code

- Encoder is the projection $Q^{\epsilon}_{n,R}$ onto the following subspace

$$\mathrm{span}_{\mathbf{C}}\{U^{\otimes n}|\,\phi\,\rangle \;:\; U \;:\; \text{a unitary}, \;|\,\phi\,\rangle \in \boxed{\underset{\text{Entropic Typical Subspace}}{\overset{\text{Entropic Typical Subspace}}{}} \!\!=\! \text{subspace (3)}}\}. \tag{4}$$

- This subspace covers all the possible entropy typical subspace whatever the eigenstates of $\bar{\rho}$ are. ($\Rightarrow$ state do not collapse)
- Achivement of the rate is proven by use of group representation theory



Variable Length Universal Code (I)

- Source is unknown
- Compress by the rate of $S(\bar{\rho})$.
- In classical case, given fixed length universal universal code, it is trivial to construct a variable length universal code. For we can observe state as many as needed. ($\Leftrightarrow$ $S(\bar{\rho})$ is estimated precisely)
- In quantum case, however, observation of the state collapse the state.
  In general

How to observe state without collaspe ?

- For a while, we consider only state estimation, and later come back to source coding.

State Estimation

- $\rho^{\otimes n}$ is given and ρ is unknown.
- Estimate state so that estimate of the state converges to ρ (in probability, in in norm ...).
- Describe the estimation by a POVM $\underbrace{\text{measurement}}_{\text{in } \mathcal{H}^{\otimes n}}$ $\{M^n_{\hat\rho}\}$ which returns estimate $\hat\rho$.

examples of $\{M^n_{\hat\rho}\}$

  - Do POVM measurement $\{m_\omega\}$ for n times. then, in $\mathcal{H}$ $\operatorname{tr}\rho m_\omega$ is estimated by $\dfrac{\#\text{ of }\omega}{n}$.
  - Estimate $\hat\rho$ by
    * solving system of equations
    $$\operatorname{tr} m_\omega \hat\rho = \frac{\#\text{ of }\omega}{n}$$
    * Maximally Likelihood Estimate etc.
    * In the following, uniform continuity of the map $\dfrac{\#\text{ of }\omega}{n} \to \hat\rho$ is assumed.
  - To optimize estimate, one should change measurement adaptively, and collective measurement should be used.

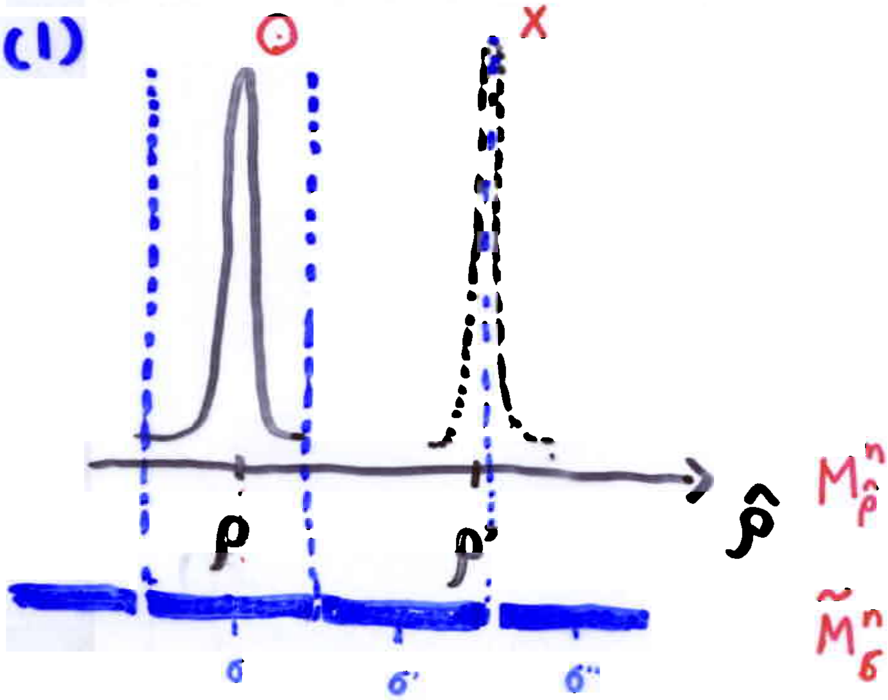
These kinds of estimation demolish state very much!

If carelessly done

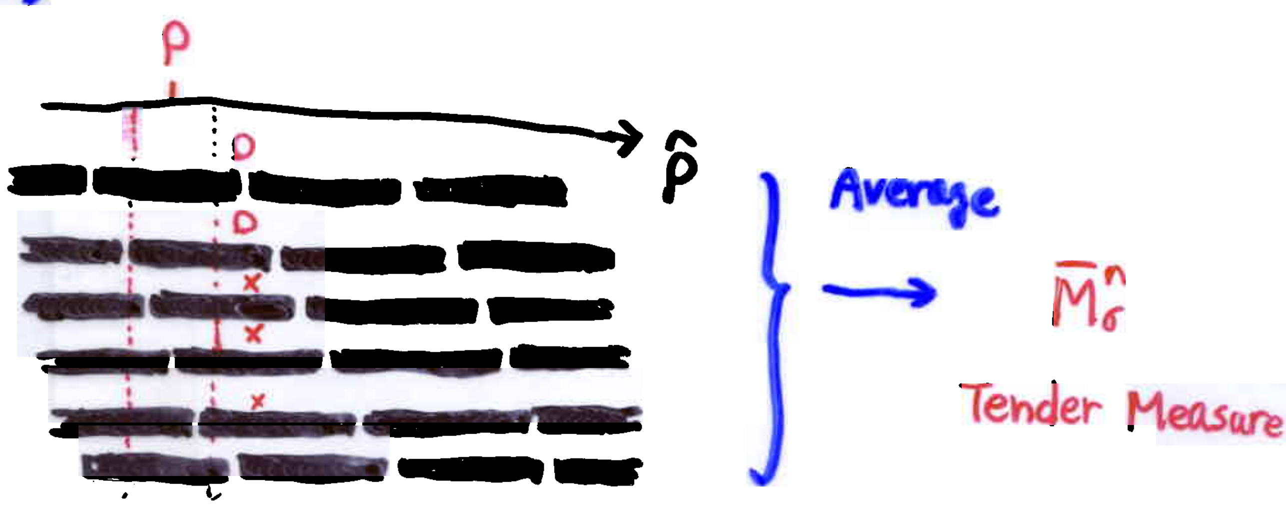
Tender Measurement

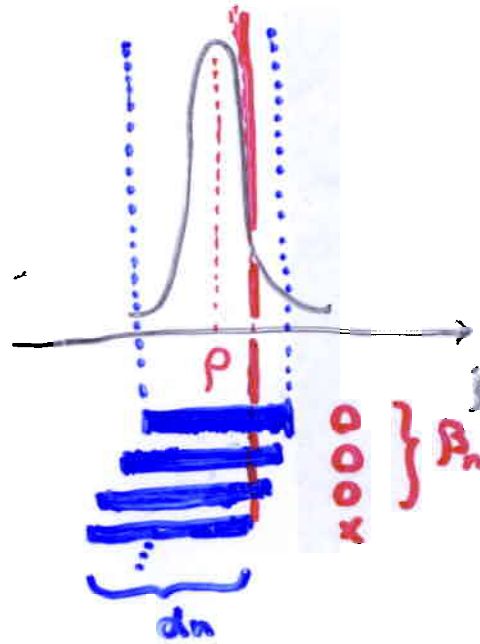
$=$ Coarse grading[1] + Average out[2]

(1)



(2)





Tendor Measurement

$$\bar{M}_\sigma^n := \frac{1}{\alpha_n} \sum_{\hat{\rho} \in U_{\delta,\sigma}} M_{\hat{\rho}}^n,$$

where $U_{\delta,\sigma}$ is δ-ball centered at σ, and

$$\alpha_n := \text{mes}\, U_{\delta,\sigma}$$

measure or volume.

- Average distortion is evaluated as :

$$\text{Average distortion} \leq 1 - \frac{1}{\alpha_n} \sum_{\sigma} (\text{tr} \sum_{\hat{\rho} \in U_{\delta,\sigma}} M_{\hat{\rho}}^n \rho)^{\frac{3}{2}}.$$

- Fix ρ and $\{\gamma_n\} (\gamma_n \to 0)$, consider

$$\beta_n := \text{mes}\{\sigma \,:\, \text{tr} \sum_{\hat{\rho} \in U_{\delta,\sigma}} M_{\hat{\rho}}^n \rho^{\otimes n} > 1 - \gamma_n\}$$

then, if $\{\gamma_n\}$ is appropriately chosen,

$$\text{Average distortion} \leq 1 - \frac{\beta_n}{\alpha_n}(1 - \gamma_n) \to 0$$

- This urgument is true even if δ converges to 0 very slowly as n increases.

n^{-s} . $0 < s < \frac{1}{2}$

Efficiency of Tender Measurement

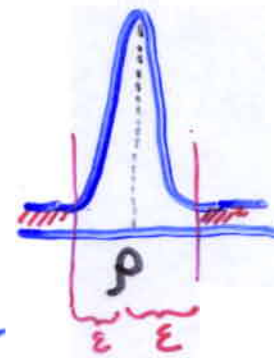
- Mean of diviation

$$\lim_{n\to\infty} n\mathrm{E}_\rho[\mathrm{dist}(\hat{\rho},\rho)]$$

$\mathrm{E}_\rho[\mathrm{dist}(\hat{\rho},\rho)] \sim \frac{1}{n}$

- Error exponent

$$\lim_{\epsilon\downarrow 0}\overline{\lim_{n\to\infty}} \frac{-1}{n\epsilon^2}\log \mathrm{Prob}\{\mathrm{dist}(\hat{\rho},\rho) > \epsilon\}$$



$\mathrm{Prob}\{\mathrm{dist}(\hat{\rho},\rho) > \varepsilon\} \sim e^{-n\varepsilon^2}$

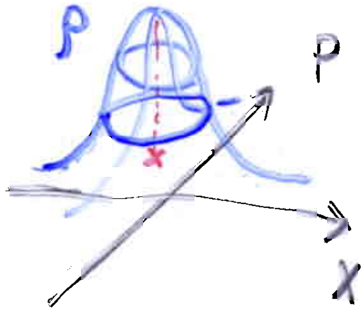
Conjecture 1

(1) In terms of mean diviation, optimal separable/collective measurement is impossible by tender measurement, and mean deviation of tendor measurement decrease much slower than $\frac{1}{n}$.

(2) In terms of error exponent, optimal separable/collective measurement is possible by tender measurement.

((1)$\cdots$ 70% (2)$\cdots$ 99%)

- (2) is true for the estimation of a scalar parameter of ρ (with appropriate regularity conditions)
- (1) is true for our tendermeasurement.



Tender measurement of X & P

⇒ breakes uncertainty principle ??

Paradox ??

Given $\rho^{\otimes n}$, to which we apply tender measurement. This does not change the state. So, we can do the same measurement again. Still, this does not change the state. So, we can repeat this for many times, and finally obtain perfect estimate of ρ.

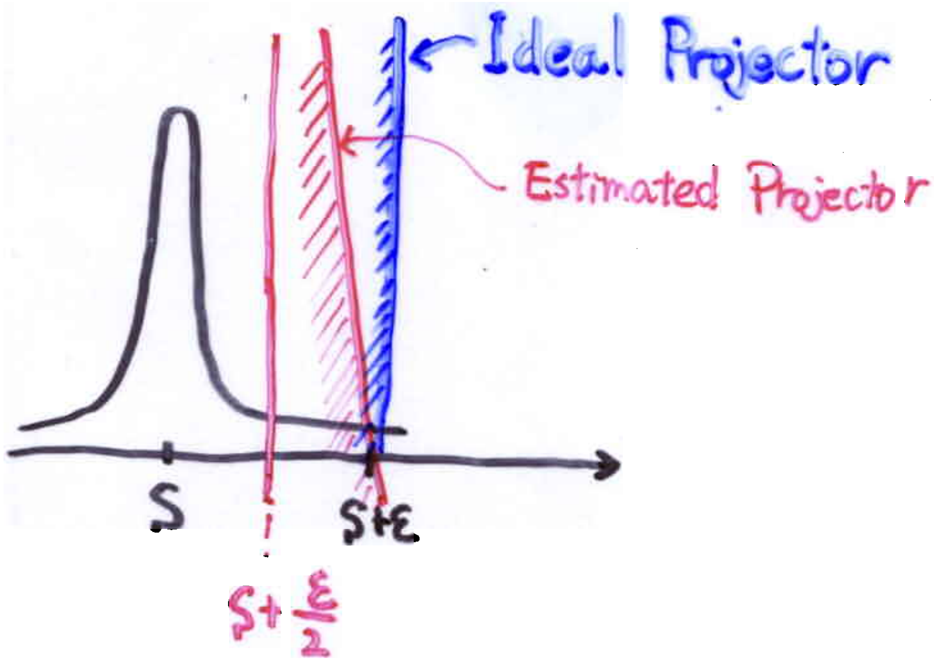
This cannot happen because ...

(1) In case that we use mean deviation as a measure, degree of demolishment per mesurement and speed of convergence of error is in trade-off.

(2) In case that we use, even tender measurement destroys state too much.

Variable Length Universal Code (II)

- Use tendor measurement and estimate $\bar{\rho}$.
  - It is shown that this does not change state by almost the same argument as state estimation case.
- Use Schmacher code or JHHH code.
  - evaluation of performance of Schumacher code is a bit harder, because noise on estimated eigenstate must be considered, too.



Optimal Variable Length Universal Cod

- There are many kinds of Variable Length Universal Code which achives von Neumann entropy, depending on tendor measurement and on conventional code that you use. (and other kinds of codes, like Jozsa-Presnell code quant-ph/0210196.)

e^{-n}

- Which one is optimal ?
  measure : minimum exponent of overflow probability,

$$\min_{U} \overline{\lim_{n\to\infty}} \frac{-1}{n} \log \mathrm{Prob}_{U\bar{\rho}U^*}\{\text{compression rate } > R\}$$

eigenvector of $\bar{\rho}$ is unknown

- Note that compression rate gives consistent estimate of von Neumann entropy of the source. Hence, by application of hypothesis test theory by Nagaoka-Ogawa, we obtaion the upper bound to the efficiency:

$$\boxed{\begin{aligned}&\min_{U} \overline{\lim_{n\to\infty}} \frac{-1}{n} \log \mathrm{Prob}_{U\bar{\rho}U^*}\{\text{compression rate} > R\}\\ &\le \inf_{\sigma:S(\sigma)>R} \mathrm{D}(\rho\|\sigma)\end{aligned}}$$

- 'tender measurement'+'conventional code' type scheme cannot be optimal. For this destroys state too much to give optimal large deviation exponent of estimamate of von Neumann entropy at the second measurement.

- so, we must do this only by one measurement.
- Optimal protocol should have invariancy by action of unitary group, for eigenstates of $\bar{\rho}$ is unknown.

Optimal protocol (Hayashi-Matsumoto quant-ph/0202001, 0209124)

- Consider encoder $Q^{\epsilon}_{n,R}$of JHHH code.
- Define

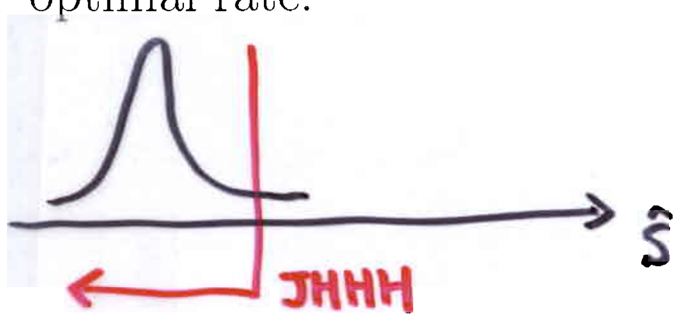
$$P_{n,R} := Q^{\epsilon}_{n,R} - Q^{\epsilon}_{n,R-frac1n}.$$

- Then, the measurement

$$\{P_{n,\frac{k}{n}}\}_k$$

gives consistent estimate of entropy.

- Construct tender mesurement from this measurement.
- Then, this tender measurement gives optimal estimate of source entropy, and output state is compressed by optimal rate.



$\{P_{n,\frac{k}{n}}\}_k$

+
Tender
||
HM

HM Code is
Group Invariant
For JHHH is group invaria

- Entanglement Concentration & Source Coding.

Schumacher Code "$\Longleftrightarrow$" "Standard Protcol" of Entanglement Concentration

Projection onto Typical Subspace

Projections onto eigenspaces of $\rho^{\otimes n}$

Sharper & more distructive

HM Code "$\Longleftrightarrow$" ?

⋮

Universal distortion free concentration.

Conclusions & Discussions

(1) Tender Measurement is useful.
in construction of universal code.

(2) To achieve optimal, group symmetric structure must be used in universal code

(3) By parallel construction, the optimal universal concentration protcol is given.

(4) The properties of tender measurements are yet to be known.

- Tradd-off between efficiency & distortion

- classical limit of quantum mechanics ?
  ($\overset{?}{=}$ macro scopic body + tender measurement

etc.

(5) universal code for markovian source.
c.f. P J code.