Non-Abelian Stabilizer Codes for Quantum Error Correction

Mary Beth Ruskai

joint work with Harriet Pollatsek, Mount Holyoke College

Berkeley — Nov-Dec, 2002

Focus: Permutationally Invariant Codes as an example

Stabilizer formalism

S is Abelian subgroup of Pauli group, i.e., prods of $\sigma_x, \sigma_y, \sigma_z$

Code is invariant subspace of $\mathbf{C}_2^{\otimes n}$ — usually basis for trivial rep

Irred. reps. of S identify subspaces assoc with correctable errors
all 1-dim; correspond to e-spaces for sets of simult. e-vecs.

Cosets P/S — equivalence classes of errors
can choose one correctable error per coset (up to deg.)

Normalizer of S — gives logical Z and X (1-bit gates)

Advantages

- all ingred. needed to implement fault tolerant comp.
- easily constructed by extended classical code methods

Disadvantages

- best suited to correct all 1-bit, 2-bit, n-bit errors,
  rather than specific correlated errors.
- extra error subspaces in degenerate codes not nec useful
- based on classical idea of code distance

G non-Abelian group which leaves $\mathbf{C}^{2^n}$ invariant
typically elements g in algebra generated by Pauli group

Example: $G = S_n$ generated by E_{rs} exchange errors

$$E_{rs} = \frac{1}{2}[I \otimes I + X_r \otimes X_s + Y_r \otimes Y_s + Z_r \otimes Z_s]$$

$$E_{rs}\,|i_1 i_2 \ldots i_r \ldots i_s \ldots i_n\rangle = |i_1 i_2 \ldots i_s \ldots i_r \ldots i_n\rangle$$

Notation: $X_r = \sigma_x$ on bit r, $Y_s = \sigma_y$ on bit s etc.

$\mathcal{E} = \{E_1 \ldots E_m\}$ is set of errors want to correct
also in algebra gen by Pauli group

Assume G leaves $\mathcal{E}$ invariant, i.e, $gE_p g^{-1} = E_{p'}$ in $\mathcal{E}$
Example: $E_{25} X_2 E_{25} = X_5$

Key: Irred Reps of G give orthog decomp of $\mathbf{C}^{2^n}$
exactly what is needed for error correction

Get corresponding decomp of $\mathcal{E}$ into irred reps of G

But note: same irred rep can occur multiple times

Get code from lin combs of irred reps of same size and type

Sufficient condition for error correction $\langle E_p C_i | E_q C_j \rangle = \delta_{ij}\, \delta_{pq}$
can correct all errors $\{E_1, \ldots E_m\}$ and lin combs

General error correction condition $\langle E_p C_i | E_q C_j \rangle = \delta_{ij}\, d_{pq}$
where matrix $D = \{d_{pq}\}$ is indep of $i = 0, 1$.
can "diagonalize" via $E_p \mapsto F_p = \sum_q u_{pq} E_q$ $\quad U$ unitary
E_p multiples of Kraus ops — F_p give same noise Φ

Essential point — get orthogonal decomp of $\mathbf{C}_2^{\otimes n}$

For stabilizer codes d_{pq} block diagonal with (at worst)
blocks of form $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ for "degenerate" codes

Old proposal for 9-bit code based on M-dim irred rep of S_9
$\langle E_p C_i^m | E_q C_j^{m'} \rangle = \delta_{ij} \delta_{mm'} d_{pq}$ too much – not poss. for any n

Code is a pair of bases for trivial rep. (span 2-dim subspace)

$$g|C_0\rangle = |C_0\rangle \qquad g|C_1\rangle = |C_1\rangle \quad \forall\ g \in\ G$$

Let $\mathcal{E}'$ be subset of errors invariant under G, i.e.,

$$g\mathcal{E}'g^{-1} \subset \mathcal{E}' \ \forall\ g$$

Then $\{E_p|C_j\rangle : E_p \text{ in } \mathcal{E}'\}$ is invariant subspace of $\mathbf{C}^{2^n}$

Pf: $E_p|C_j\rangle = (gE_pg^{-1})g|C_j\rangle = E_{p'}|C_j\rangle$

- invar sets of errors take code to invar subspace of $\mathbf{C}_2^{\otimes n}$
- "Diag" of errors breaks $\mathcal{E}$ into invariant sets which generate bases for irred reps when acting on code

Example: $G = S_n$, $\mathcal{E}' = \{X_1 \dots X_n\}$ bit flips

Perm invar code $|C_0\rangle = \sum_{k=0}^{n} a_k W_k$, $|C_1\rangle = \sum_{k=0}^{n} b_k W_k$

where $\widehat{W}_k = \sum_{\mathcal{P}} |\underbrace{1\dots1}_{\kappa}\underbrace{0\dots0}_{n-\kappa}\rangle$ $W_k = \binom{n}{k}^{-1/2} \widehat{W}_k$

get cyclic matrix $D_X = \begin{pmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & & & \vdots \\ b & b & \dots & a \end{pmatrix}$ easy to diag.

$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k$ "average" takes code to another 1-dim rep

$\{X_1 - X_s : s = 2 \dots 9\}$ corr to basis for $(n-1)$- dim rep

span$\{X_1 \dots X_n\}$ = span$(\bar{X} \cup \{X_1 - X_s : s = 2 \dots 9\})$.

DFS (Decoherence Free subspace and subsystem) codes
also based on invariant subspace — but for interaction
$G =$ group generacted by suitable interaction ops.

Hamiltonian $\quad H = H_C + H_E + V_{CE}$

V_{CE} interaction between Computer and Environment

Want subspace in which interaction V_{CE} is diag in prod basis

Equiv to $V_{CE} = \sum_k F_k \otimes G_k$ and invariant subspace for $\{F_k\}$.

G = non-Abelian group generated by $\{F_k\}$ − DFS group

DFS subspace $\quad F_k|c_j\rangle = |c_j\rangle$

Code = basis for trivial 1-dim rep. of (non-Abelian) DFS group G

Additional correctable errors:

- $E_p E_q^\dagger$ anti-commute with $G \quad \Rightarrow$ satisfy orthog cond,
- look at errors assoc with other (higher-dim) irred reps.
  BUT testing may interfere with DFS isolation

Original focus of DFS subspace — eliminate effect of errors
by confining system to a subspace one which V_{CE} acts like $I \otimes W$.

But need universal gates which leave subspace invariant
can be done with exchange interaction

DFS subsystem — only require invariant subspace for code (not I)
goal: universal set of gates can use higher dim irred reps

non-Abelian stabilizer – focus use higer dim reps for error correction

In all cases, gates are in commutant.

$G = S_n$ group gen by all exchange errors E_{jk}

Can consider as DFS group for exchange interaction
corresponds to a QC completely isolated from outside world
errors can arise only from interaction of qubits with each other
spin qubits in QC linked to spatial "environment" by Pauli
exclusion even without explicit space/spin interaction
idealized, but shows connection with other models

Compare:

$G =$ Abelian subgp of Pauli gp — $GF(2)$ codes

$G =$ non-Abelian S_n — perm inv codes

$G =$ non-Abelian group of "probable errors"
goal: adapative codes for specific models of QC

$G =$ non-Abelian DFS gp — code is stable subspace,
but error correction has potential to destabilize

Construction of Permutationally Invariant Codes

For 9-bit code $|C_0\rangle = \widehat{W}_0 + \frac{1}{\sqrt{28}}\widehat{W}_6 \quad |C_1\rangle = \widehat{W}_9 + \frac{1}{\sqrt{28}}\widehat{W}_3$

Error condition $\langle E_p C_i | E_q C_j \rangle = \delta_{ij} d_{pq}$ gives block diag $D = d_{pq}$

$$\begin{pmatrix} D_0 & 0 & 0 & 0 & 0 \\ 0 & D_X & 0 & 0 & 0 \\ 0 & 0 & D_Y & 0 & 0 \\ 0 & 0 & 0 & D_Z & 0 \\ 0 & 0 & 0 & 0 & ?? \end{pmatrix} \qquad \begin{pmatrix} 37 & & 1 \text{ (deg)} \\ 9 & = & 1+8 \\ 9 & = & 1+8 \\ 9 & = & 1+8 \\ 228 & = & \end{pmatrix}$$

$$(\,37 \quad 9 \quad 9 \quad 9 \quad 228\,) \qquad = \quad 3 \cdot 27 + 2 \cdot 48 + 42$$

where D_0 is the identity and 36 degenerate exchange errors

each 1-bit block $\begin{pmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & & & \vdots \\ b & b & \dots & a \end{pmatrix}$ splits into $1 \oplus (n-1)$ as before

other irred reps may be able identify additional correctable errors

Notation: $\widehat{W}_k = \sum_{\mathcal{P}} |\underbrace{1\ldots1}_{\kappa}\underbrace{0\ldots0}_{n-\kappa}\rangle$

For n odd seek perm inv codes of form

$$|C_0\rangle = \sum_{m=0}^{(n-1)/2} a_{2m}\widehat{W}_{2m} \quad |C_1\rangle = \sum_{m=0}^{(n-1)/2} a_{n-2m-1}\widehat{W}_{2m+1}$$

Then $(\otimes_k Z_k)\,|C_j\rangle = (-1)^j|C_j\rangle$ and $(\otimes_k X_k)\,|C_0\rangle = |C_1\rangle$

Want block diag form as for 9-bit code (depends on a_k)

Can assume wlog that each D_X, D_Y, D_Z decomposes
into direct sum of 1-dim and $(n-1)$-dim

Can NOT assume off-diag blocks are zero

DO know $\langle \bar{F}\widehat{W}_j, (G_r - G_s)\widehat{W}_k\rangle = 0$ for any j, k
which implies $\langle \bar{F}C_j, (G_r - G_s)C_k\rangle = 0$ for $j, k = 0, 1$
and any choice of $\bar{F}$ in $\{I, \bar{X}, \bar{Y}, \bar{Z}\}$ and $G = X, Y, Z$.
Different irred reps are orthogonal

Also get some zero terms from form of $|C_0\rangle, |C_1\rangle$

$$D_{00} = D_{11} \qquad \langle E_p C_i | E_q C_i \rangle \qquad \begin{pmatrix} \begin{matrix} d_{II} & 0 & 0 & d_{IZ} \\ 0 & d_{XX} & d_{XY} & 0 \\ 0 & d_{YX} & d_{YY} & 0 \\ d_{ZI} & 0 & 0 & d_{ZZ} \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} D_{XX} & D_{XY} & 0 \\ D_{YX} & D_{YY} & 0 \\ 0 & 0 & D_{ZZ} \end{matrix} \end{pmatrix}$$

$$D_{01} \qquad \langle E_p C_i | E_q C_j \rangle \qquad \begin{pmatrix} \begin{matrix} 0 & b_{IX} & b_{IY} & 0 \\ b_{XI} & 0 & 0 & b_{XZ} \\ b_{YI} & 0 & 0 & b_{YZ} \\ 0 & b_{ZX} & b_{ZY} & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & 0 & B_{XZ} \\ 0 & 0 & B_{YZ} \\ B_{ZX} & B_{ZY} & 0 \end{matrix} \end{pmatrix}$$

Need $D_{01} = 0$ so must require all $b_{IX} = 0$ and $B_{XZ} = 0$ etc.
Form of $|C_j\rangle$ ensures $D_{00} = D_{11}$ for diag terms and blocks
But requires that all skew diag $d_{IZ} = d_{XY} = 0$ and $D_{XY} = 0$.

Looks like lots of conditions. For real a_k reduce to only 3

Two distinct 7-bit perm invariant (non-additive) codes

$$
\begin{aligned}
|C_0\rangle &= \frac{1}{8}[\pm\sqrt{15}W_0 - \sqrt{7}W_2 \pm \sqrt{21}W_4 + \sqrt{21}W_6] \\
|C_1\rangle &= \frac{1}{8}[\pm\sqrt{15}W_7 - \sqrt{7}W_5 \pm \sqrt{21}W_3 + \sqrt{21}W_1]
\end{aligned}
$$

Know one perm invariant 9-bit code — probably more

$$|C_0\rangle = \tfrac{1}{2}W_0 + \tfrac{\sqrt{3}}{2}W_6 \qquad |C_1\rangle = \tfrac{1}{2}W_9 + \tfrac{\sqrt{3}}{2}W_3$$

$$\overline{Z}|C_0\rangle = \tfrac{\sqrt{3}}{2}W_0 - \tfrac{1}{2}W_6 \qquad \overline{Z}|C_1\rangle = \tfrac{\sqrt{3}}{2}W_9 - \tfrac{1}{2}W_3$$

$$\overline{X}|C_0\rangle = \tfrac{1}{4}W_1 + \tfrac{1}{\sqrt{2}}W_5 + \tfrac{\sqrt{7}}{4}W_7 \qquad \overline{X}|C_1\rangle = \tfrac{1}{4}W_8 + \tfrac{1}{\sqrt{2}}W_4 + \tfrac{\sqrt{7}}{4}W_2$$

$$i\overline{Y}|C_0\rangle = -\tfrac{1}{4}W_1 + \tfrac{1}{\sqrt{2}}W_5 - \tfrac{\sqrt{7}}{4}W_7 \qquad i\overline{Y}|C_1\rangle = -\tfrac{1}{4}W_8 + \tfrac{1}{\sqrt{2}}W_4 - \tfrac{\sqrt{7}}{4}W_2$$

Compare: 7-bit CSS code

Breaks $\mathbf{C}_2^{\otimes 7}$ into $2^6 = 64$ orthog 2-dim subspaces

Code and 1-bit errors use $1 + 3 \cdot 7 = 22$ of these subspaces

What about additional 42 subspaces ??

Correct all errors of form $X_j Z_k \quad (j \neq k)$ exactly $7 \cdot 6 = 42$

Shor 9-bit code:

$$|C_0\rangle = |000\,000\,000\rangle + |000\,111\,111\rangle + |111\,000\,111\rangle + |111\,111\,000\rangle$$
$$|C_1\rangle = |111\,111\,111\rangle + |111\,000\,000\rangle + |000\,111\,000\rangle + |000\,000\,111\rangle$$

$2^8 = 256$ orthog 2-dim subspaces; code + 1-bit errors use 28

Can correct pair of bit flips, phase or Y_k only in same 3-bit block

Most other correctable errors "oddball"

Doesn't systematically correct correlated errors in diff blocks

Partial perm symmetry – can't correct most exchange errors

may induce logical phase error from trying to correct exchange

	Shor	perm inv
27 1-bit	correct all, use 21	correct all, use 27
36 exchange	no, may get phase	in S, deg, use 0
36 2-bit flip	same block, correct 9	no
36 2-bit phase	same block in S corr 9 by deg, use 0	no
	226 more errors most not 2-bit	228 more errors at least 81 gen 2-bit
	classify by cosets S	classify by irred reps
stabilizer	$X_1X_2X_3X_4X_5X_6,\ Z_1Z_2$ $X_4X_5X_6X_7X_8X_9,\ Z_2Z_3$ $Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9,$	S_9 generated by E_{1k} 8 generators
irred rep	each occurs once all 1-dim	multiple occurence higher dim

What other errors might perm inv codes correct ??

Decomp of subspaces spanned by vectors of weight k (# of 1's)

k	$n = 7$	$n = 9$
0	$1 = 1$	$1 = 1$
1	$7 = 1 \oplus 6$	$9 = 1 \oplus 8$
2	$21 = 1 \oplus 6 \oplus 14$	$36 = 1 \oplus 8 \oplus 27$
3	$35 = 1 \oplus 6 \oplus 14 \oplus 14$	$84 = 1 \oplus 8 \oplus 27 \oplus 48$
4	$35 = 1 \oplus 6 \oplus 14 \oplus 14$	$126 = 1 \oplus 8 \oplus 27 \oplus 48 \oplus 42$
5	$21 = 1 \oplus 6 \oplus 14$	$126 = 1 \oplus 8 \oplus 27 \oplus 48 \oplus 42$
6	$7 = 1 \oplus 6$	$84 = 1 \oplus 8 \oplus 27 \oplus 48$
7	$1 = 1$	$36 = 1 \oplus 8 \oplus 27$
8		$9 = 1 \oplus 8$
9		$1 = 1$

Need 8 1-dim $\quad + \quad$ 6 $(n-1)$-dim to correct all 1-bit errors

$n = 7$ leaves 4 14-dim $\quad + \quad$ 2 other 14-dim — not much

$n = 9$ leaves 2 1-dim, 8-dim + 6 27-dim + 4 48-dim + 2 42-dim

E_{jk} exchange

$$\frac{1}{2}(I_j \otimes I_k + Z_j \otimes Z_k + X_j \otimes X_k + Y_j \otimes Y_k) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

F_{jk} exchange + phase if exchanged

$$\frac{1}{2}(I_j \otimes I_k + Z_j \otimes Z_k - X_j \otimes X_k - Y_j \otimes Y_k) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

G_{jk} flip iff identical

$$\frac{1}{2}(I_j \otimes I_k - Z_j \otimes Z_k + X_j \otimes X_k - Y_j \otimes Y_k) \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

H_{jk} flip and phase iff identical

$$\frac{1}{2}(I_j \otimes I_k - Z_j \otimes Z_k - X_j \otimes X_k + Y_j \otimes Y_k) \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

If one could correct all $E_{jk}, F_{jk}, G_{jk}, H_{jk}$ then could correct all two bit errors of form $X_j X_k, Y_j Y_k, Z_j Z_k$

On perm invariant vectors, these operators are equivalent, i.e., if $E_{jk}|\psi\rangle = |\psi\rangle$ for all $j < k$ then

since	$E_{jk} + F_{jk} = I + Z_j Z_k$	$F_{jk}|\psi\rangle = Z_j Z_k|\psi\rangle$
since	$E_{jk} + G_{jk} = I + X_j X_k$	$G_{jk}|\psi\rangle = X_j X_k|\psi\rangle$
since	$E_{jk} + H_{jk} = I + Y_j Y_k$	$H_{jk}|\psi\rangle = Y_j Y_k|\psi\rangle$

For perm invariant codes, ability to correct $F_{jk},\ G_{jk},\ H_{jk}$ equiv. to ability to correct $Z_j Z_k,\ X_j X_k,\ Y_j Y_k$ respectively

BUT can only correct those lin combs in 27-dim rep

$n = 7$ 1-bit errors use all 8 1-dim reps; and all 6 6-dim reps left for 2-bit errors: 4 (2 pair) 14-dim + 2 other 14-dim

$n = 9$ 1-bit errors use 8 of 10 1-dim reps; and 6 of 8 8-dim reps leaves: 2 1-dim, 2 8-dim, 6 (3 pair) 27-dim + higher-dim

Can not correct all 2-bit errors above without sacrificing some 1-bit

$n = 9$ can correct 27-dim parts of F_{jk}, etc.
What do they look like?? Take orthog comp of

$$\sum_{jk} F_{jk} \text{ and } T_r = \sum_{s=2}^{n} F_{1s} - \sum_{s \neq 1,r} F_{rs}$$

For $n = 4 = 1 \oplus 3 \oplus 2$ this orthog comp is spanned by
$2F_{12} - F_{13} - F_{14} - F_{23} - F_{24} + 2F_{34}$,
$F_{12} + F_{13} - 2F_{14} - 2F_{23} + F_{24} + F_{34}$

Question: What can 9-bit correct with pair of "extra" 1-dim and 8-dim reps – alas, not $Z_j Z_k$ or $X_k X_k$

Many practical applications: most important correlations are nearest neighbors, e.g., $Z_j Z_{j+1}$ highly localized

Not amenable to group structure – generates all $Z_j Z_k$

reps of S_n must include highly delocalized errors, e.g. $\overline{Z} = \sum_k Z_k$ or even $\overline{Z}_{rs} = \sum_{j<k} Z_j Z_k$

Higher dim reps can be chosen to have some local but must still have a few delocalized errors

$$Z_2 Z_3 = \frac{1}{6}\overline{Z}_{rs} + \sum_{j=1}^{n-1} u_j Z_j^{n-1} + \sum_{m=1}^{27} u_j Z_j^{27}$$

$$Z_2 Z_7 = \frac{1}{6}\overline{Z}_{rs} + \sum_{j=1}^{n-1} v_j Z_j^{n-1} + \sum_{m=1}^{27} v_j Z_j^{27}$$

	Abelian stabilizer	non-Abelian stabilizer
irred reps	all exactly once	only some can have multiple bases
	all 1-dim	higher dim
code is basis for	any irred rep	trivial rep
classify errors	by cosets	by irred reps
ident errors	e-vals of S chars of rep	need more ops
Z and X gates	normalizer of S	in commutant of G

Challenges

- Need to implement unusual superpositions
- E-vals or characters of irred not enough to fully distinguish
- How to implement other gates on code words