A Double Whammy Talk

Part 1. **A Quantum Hidden Subgroup Algorithm on the Circle:**

Speaker: Sam Lomonaco

Joint work with Lou Kauffman

Part 2. **Quantum Entanglement and Topology**

Speaker: Lou Kauffman

Joint work with Sam Lomonaco

This work supported by Defense Advance Research Projects Agency (DARPA) & Air Force Research Laboratory, Air Force Materiel Command, USAF Agreement Number F30602-01-2-0522.

- **This work also supported by National Institute for Standards and Technology (NIST).**

A Quantum Hidden Subgroup Algorithm on the Circle

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Outline

- Preamble
- Fourier analysis on the circle
- A lifting of Shor's quantom factoring algorithm
- The dual algorithm on the circle
- The corresponding discrete algorithm
- Conclusion

Preamble

Hidden Algebraic Structure

Definition. A map

$$\varphi : A \longrightarrow S$$

[Ambient Group: A] [Target Set: S]

is said to have **hidden algebraic structure** if there exist

- A subgroup K_φ of A, and [Hidden Subgroup]
- An injection $\iota_\varphi : A/K_\varphi \longrightarrow S$

  such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & S \\ & {\scriptstyle \nu}\searrow \quad \nearrow{\scriptstyle \iota_\varphi} & \\ & A/K_\varphi & \end{array}$$

[Hidden Natural Surjection: ν] [Set of Right Cosets: A/K_φ]

is commutative.

If K_φ is an invariant subgroup of A, then

$$H_\varphi = A/K_\varphi$$

[Hidden Quotient Group]

is a group, and $\nu : A \longrightarrow A/K_\varphi$ is an epimorphism. [Hidden Epimorphism]

Shor's Quantum factoring algorithm reduces the task of factoring an integer **N** to the task of finding the period **P** of the function

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z} \bmod N \\ n & \mapsto & a^n \bmod N \end{array}$$

Finding the period P is equivalent to finding the subgroup $P\mathbb{Z} \subset \mathbb{Z}$, i.e., the kernel of φ.

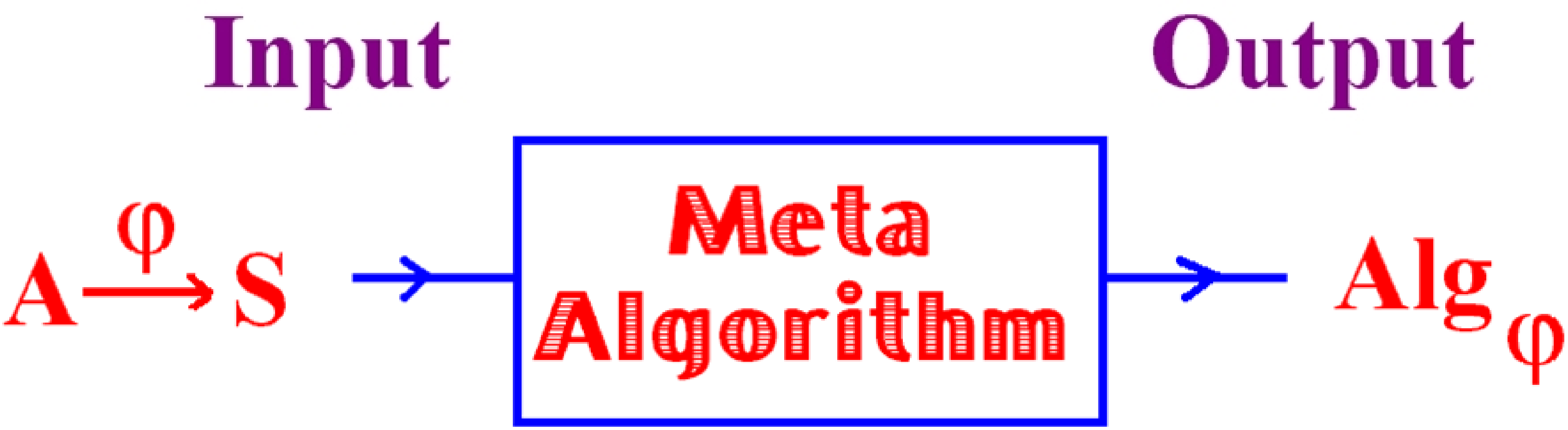
- **Lomonaco & Kauffman, Quantum Hidden Subgroup Algorithms:** A Mathematical Perspective, **AMS, CONM/305, (2002).** **http://xxx.lanl.gov/abs/quant-ph/0201095**

- **Lomonaco & Kauffman, A Continuous Variable Shor Algorithm, http://xxx.lanl.gov/abs/quant-ph/0210141**

- **Lomonaco & Kauffman, A Quantum Hidden Subgroup Algorithm on the Circle, (in preparation).**

The Quantum Hidden Subgroup Paper Shows how to create a

Meta Algorithm



- **Lomonaco & Kauffman, Quantum Hidden Subgroup Algorithms:** A Mathematical Perspective, **AMS, CONM/305, (2002).** **http://xxx.lanl.gov/abs/quant-ph/0201095**

- **Lomonaco & Kauffman, A Continuous Variable Shor Algorithm,** **http://xxx.lanl.gov/abs/quant-ph/0210141**

- **Lomonaco & Kauffman, A Quantum Hidden Subgroup Algorithm on the Circle, (in preparation).**

In this paper we created a

Continuous Variable Shor Algorithm

Recall that Shor's algorithm reduces to the task of finding the period P of a function

$$\varphi : \mathbb{Z} \to \mathbb{Z} \bmod N$$

So a CV Shor algorithm should be a HSG algorithm that finds the period P of a function of the form

$$\varphi : \mathbb{R} \to \mathbb{C}$$

- **Lomonaco & Kauffman, Quantum Hidden Subgroup Algorithms:** A Mathematical Perspective, **AMS, CONM/305, (2002).** [**http://xxx.lanl.gov/abs/quant-ph/0201095**](http://xxx.lanl.gov/abs/quant-ph/0201095)

- **Lomonaco & Kauffman, A Continuous Variable Shor Algorithm,** [**http://xxx.lanl.gov/abs/quant-ph/0210141**](http://xxx.lanl.gov/abs/quant-ph/0210141)

- **Lomonaco & Kauffman, A Quantum Hidden Subgroup Algorithm on the Circle, (in preparation).**

Fourier Analysis on the

Circle

The Circle as a Group

The **circle group** can be viewed as

- A **multiplicative group**, i.e., as the unit circle in the complex plane $\mathbb{C}$

$$\left\{ e^{2\pi i x} : x \in \mathbb{R} \right\}$$

$$e^{2\pi i x} \bullet e^{2\pi i y} = e^{2\pi i (x+y)}$$

where $\mathbb{R}$ denotes the additive group of reals.

The Circle as a Group

The circle group can also be viewed as

- An additive group, i.e., as

$$\mathbb{R}/\mathbb{Z} = \textit{reals} \bmod 1$$

$$x + y \bmod 1$$

where $\mathbb{Z}$ denotes the additive group of integers.

The Character Group

The character group $\widehat{A}$ of an abelian group A is defined as

$$\widehat{A} = Hom(A, Circle)$$
$$= \{\chi : A \to Circle : \chi \ a \ morphism\}$$

with group operation (in multiplicative notation),

$$\chi(a_1 + a_2) = \chi(a_1) \bullet \chi(a_1)$$

or (in additive notation) as

$$\chi(a_1 + a_2) = \chi(a_1) + \chi(a_1)$$

The Character Groups of $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$

- The character group of $\mathbb{Z}$ is

$$\widehat{\mathbb{Z}} = \left\{ \chi_x : n \mapsto e^{2\pi i n x} : x \in \mathbb{R} \right\} = \mathbb{R}/\mathbb{Z}$$

- The character group of $\mathbb{R}/\mathbb{Z}$ is

$$\widehat{\mathbb{R}/\mathbb{Z}} \cong \left\{ \chi_n : x \mapsto e^{2\pi i n x} : n \in \mathbb{Z} \right\}$$

$$\cong \left\{ \chi_n : x \mapsto nx \bmod 1 : n \in \mathbb{Z} \right\} = \mathbb{Z}$$

$$\boxed{\mathbb{Z} \Leftrightarrow \mathbb{R}/\mathbb{Z}}$$

Fourier Analysis on the Circle $\mathbb{R}/\mathbb{Z}$

The Fourier transform of $f:\mathbb{R}/\mathbb{Z}\to\mathbb{C}$ **is defined as the map**

$$\hat{f}:\mathbb{Z}\to\mathbb{C}$$

given by

$$\hat{f}(n)=\oint dx e^{-2\pi inx} f(x)$$

The inverse Fourier transform is defined as

$$f(x)=\sum_{n\in\mathbb{Z}} e^{2\pi inx}\hat{f}(n)$$

A Lifting of Shor's Quantum Factoring Algorithm

Needed

Mathematical Machinery

- Dirac Delta function $\delta(x)$ on $\mathbb{R}/\mathbb{Z}$

- For P a non-zero integer, we will also need on $\mathbb{R}/\mathbb{Z}$ the generalized function

$$\delta_P(x) = \frac{1}{|P|} \sum_{n=0}^{P-1} \delta\left(x - \frac{n}{P}\right)$$

Rigged Hilbert Space

- $H_{\mathbb{R}/\mathbb{Z}}$ denotes the rigged Hilbert space on $\mathbb{R}/\mathbb{Z}$ with orthonormal basis

$$\{|x\rangle : x \in \mathbb{R}/\mathbb{Z}\}$$, i.e., $$\langle x|y\rangle = \delta(x-y)$$

- The elements of $H_{\mathbb{R}/\mathbb{Z}}$ are formal integrals of the form

$$\oint dx\, f(x)|x\rangle$$

Finally, let $H_{\mathbb{Z}}$ denote the space of formal sums

$$\left\{ \sum_{n=-\infty}^{\infty} a_n |n\rangle : a_n \in \mathbb{C} \quad \forall n \in \mathbb{Z} \right\}$$

with orthonormal basis

$$\{ |n\rangle : n \in \mathbb{Z} \}$$

Periodic Functions on $\mathbb{Z}$

Let $\varphi : \mathbb{Z} \to \mathbb{C}$ be periodic function with hidden minimum period P .

O b j e c t i v e :

Find P

- **Step 0. Initialize** $\left|\psi_0\right\rangle = \left|0\right\rangle\left|0\right\rangle \in H_{\mathbb{R}/\mathbb{Z}} \otimes H_{\mathbb{C}}$

- **Step 1. Apply** $F^{-1} \otimes 1$

$$\left|\psi_1\right\rangle = \sum_{n\in\mathbb{Z}} e^{2\pi i n\cdot 0}\left|n\right\rangle\left|0\right\rangle = \sum_{n\in\mathbb{Z}}\left|n\right\rangle\left|0\right\rangle \in H_{\mathbb{Z}} \otimes H_{\mathbb{C}}$$

- **Step 2. Apply** $U_{\varphi} : \left|n\right\rangle\left|u\right\rangle \mapsto \left|n\right\rangle\left|u+\varphi(n)\right\rangle$

$$\left|\psi_2\right\rangle = \sum_{n\in\mathbb{Z}}\left|n\right\rangle\left|\varphi(n)\right\rangle$$

- **Step 3. Apply** $F \otimes 1$

$$
\begin{aligned}
|\psi_3\rangle &= \oint dx\, |x\rangle \sum_{n\in\mathbb{Z}} e^{-2\pi i n x} |\varphi(n)\rangle \in H_{\mathbb{R}/\mathbb{Z}} \otimes H_{\mathbb{C}} \\
&= \oint dx\, |x\rangle \sum_{n_1\in\mathbb{Z}} \sum_{n_0=0}^{P-1} e^{-2\pi i (n_1 P + n_0) x} |\varphi(n_1 P + n_0)\rangle \\
&= \oint dx\, |x\rangle \left(\sum_{n_1\in\mathbb{Z}} e^{-2\pi i n_1 P x} \right) \sum_{n_0=0}^{P-1} e^{-2\pi i n_0 x} |\varphi(n_0)\rangle \\
&= \oint dx\, |x\rangle \delta_P(x) \sum_{n_0=0}^{P-1} e^{-2\pi i n_0 x} |\varphi(n_0)\rangle \\
&= \sum_{n=0}^{P-1} \left|\frac{n}{P}\right\rangle \left(\frac{1}{P} \sum_{n_0=0}^{P-1} e^{-2\pi i n_0 x} |\varphi(n_0)\rangle \right) \\
&= \sum_{n=0}^{P-1} \left|\frac{n}{P}\right\rangle \left|\Omega\left(\frac{n}{P}\right)\right\rangle
\end{aligned}
$$

•Step 4. Measure

$$\left|\psi_3\right\rangle = \sum_{n=0}^{P-1} \left|\frac{n}{P}\right\rangle \left|\Omega\left(\frac{n}{P}\right)\right\rangle$$

with respect to the observable

$$A = \oint dy \frac{\lfloor Qy \rfloor}{Q} \left|y\right\rangle\left\langle y\right|$$

to produce a random eigenvalue m/Q and then proceed to find the corresponding n/P using the continued fraction recursion.
(We assume $Q \geq 2P^2$)

The Actual (Un-Lifted) Shor Algorithm

Make the following approximations by selecting a sufficiently large integer Q :

$$\mathbb{Z} \approx \mathbb{Z}_Q = \{k \in \mathbb{Z} : 0 \le k < P\}$$

$$\mathbb{R}/\mathbb{Z} \approx \mathbb{Z}_Q = \left\{\frac{r}{Q} \bmod 1 : r = 0, 1, \ldots, Q-1\right\}$$

$$\varphi : \mathbb{Z} \to \mathbb{C} \quad \approx \tilde{\varphi} : \mathbb{Z}_Q \to \mathbb{C}$$

$\tilde{\varphi}$ is only approximately periodic !

Run the algorithm in

$$\boldsymbol{H}_{\mathbb{Z}_Q} \otimes \boldsymbol{H}_S$$

and measure the observable

$$\boldsymbol{O} = \sum_{r=0}^{Q-1} \frac{r}{Q} \left| \frac{r}{Q} \right\rangle \left\langle \frac{r}{Q} \right|$$

The Dual Algorithm on the

Circle

Rigged Hilbert Space

- $H_{\mathbb{R}/\mathbb{Z}}$ **denotes the rigged Hilbert space on $\mathbb{R}/\mathbb{Z}$ with orthonormal basis**

$\{|x\rangle : x \in \mathbb{R}/\mathbb{Z}\}$ **, i.e.,** $\langle x|y\rangle = \delta(x-y)$

- **The elements of $H_{\mathbb{R}/\mathbb{Z}}$ are formal integrals of the form**

$$\oint dx\, f(x)|x\rangle$$

Finally, let $H_{\mathbb{Z}}$ denote the space of formal sums

$$\left\{ \sum_{n=-\infty}^{\infty} a_n |n\rangle : a_n \in \mathbb{C} \quad \forall n \in \mathbb{Z} \right\}$$

with orthonormal basis

$$\{ |n\rangle : n \in \mathbb{Z} \}$$

Periodic Admissible Functions on $\mathbb{R}/\mathbb{Z}$

Let $f:\mathbb{R}/\mathbb{Z}\to\mathbb{C}$ be an admissible periodic function of minimum rational period $\alpha\in\mathbb{Q}/\mathbb{Z}$

Proposition: If $\alpha = a_1/a_2$ with $\gcd(a_1,a_2)=1$, then $1/a_2$ is also a period of f .

Remark: Hence, the minimum rational period is the reciprocal of an integer modulo 1 .

•Step 0. Initialize $$|\psi_0\rangle = |0\rangle|0\rangle \in H_{\mathbb{Z}} \otimes H_{\mathbb{C}}$$

•Step 1. Apply $F^{-1} \otimes 1$

$$|\psi_1\rangle = \oint dx\, e^{2\pi i x \cdot 0} |x\rangle|0\rangle = \oint dx |x\rangle|0\rangle \in H_{\mathbb{R}/\mathbb{Z}} \otimes H_{\mathbb{C}}$$

•Step 2. Apply $U_\varphi : |x\rangle|u\rangle \mapsto |x\rangle|u+\varphi(x)\rangle$

$$|\psi_2\rangle = \oint dx\, |x\rangle|\varphi(x)\rangle$$

- **Step 3. Apply** $F \otimes 1$

$$
\begin{aligned}
|\psi_3\rangle &= \sum_{n\in\mathbb{Z}} \oint dx\, e^{-2\pi i n x} |n\rangle |\varphi(x)\rangle \\
&= \sum_{n\in\mathbb{Z}} |n\rangle \oint dx\, e^{-2\pi i n x} |\varphi(x)\rangle \in H_{\mathbb{Z}} \otimes H_{\mathbb{C}}
\end{aligned}
$$

Letting $x_m = x - \frac{m}{a}$, we have

$$\oint dx\, e^{-2\pi i n x} \left| \varphi(x) \right\rangle = \sum_{m=0}^{a-1} \int_{\frac{m}{a}}^{\frac{m+1}{a}} dx\, e^{-2\pi i n x} \left| \varphi(x) \right\rangle$$

$$= \sum_{m=0}^{a-1} \int_{0}^{\frac{1}{a}} dx_m\, e^{-2\pi i n \left(x_m + \frac{m}{a} \right)} \left| \varphi\left(x_m + \frac{m}{a} \right) \right\rangle$$

$$= \left(\sum_{m=0}^{a-1} e^{-\frac{2\pi i n m}{a}} \right) \int_{0}^{\frac{1}{a}} dx\, e^{-2\pi i n x} \left| \varphi(x) \right\rangle$$

But $\displaystyle\sum_{m=0}^{a-1} e^{-\frac{2\pi i n m}{a}} = a\delta_{n=0 \bmod a} = \begin{cases} a & \text{if } n = 0 \bmod a \\ 0 & \textit{otherwise} \end{cases}$

Thus,

$$\begin{aligned}
|\psi_3\rangle &= \sum_{n\in\mathbb{Z}} |n\rangle \oint dx\, e^{-2\pi i n x} |\varphi(x)\rangle \\
&= \sum_{n\in\mathbb{Z}} |n\rangle \delta_{n=0 \bmod a} \int_0^{1/a} dx\, e^{-2\pi i n x} |\varphi(x)\rangle \\
&= \sum_{\ell\in\mathbb{Z}} |\ell a\rangle \left(\int_0^{1/a} dx\, e^{-2\pi i \ell a x} |\varphi(x)\rangle \right) \\
&= \sum_{\ell\in\mathbb{Z}} |\ell a\rangle |\Omega(\ell a)\rangle
\end{aligned}$$

•Step 4. Measure

$$\left|\psi_3\right\rangle = \sum_{\ell\in\mathbb{Z}} \left|\ell a\right\rangle \left|\Omega(\ell a)\right\rangle$$

with respect to the observable

$$A = \sum_{n\in\mathbb{Z}} n \left|n\right\rangle\left\langle n\right|$$

to produce a random eigenvalue ℓa

The

corresponding

discrete

algorithm

We now create a corresponding discrete algorithm

The approximations are:

$$\mathbb{Z} \approx \mathbb{Z}_Q = \{k \in \mathbb{Z} : 0 \le k < P\}$$

$$\mathbb{R}/\mathbb{Z} \approx \mathbb{Z}_Q = \left\{ \frac{r}{Q} \bmod 1 : r = 0, 1, \ldots, Q-1 \right\}$$

$$\varphi : \mathbb{Z} \to \mathbb{C} \quad \approx \tilde{\varphi} : \mathbb{Z}_Q \to \mathbb{C}$$

$\tilde{\varphi}$ **is only approximately periodic !**

Run the algorithm in

$$H_{\mathbb{Z}_Q} \otimes H_S$$

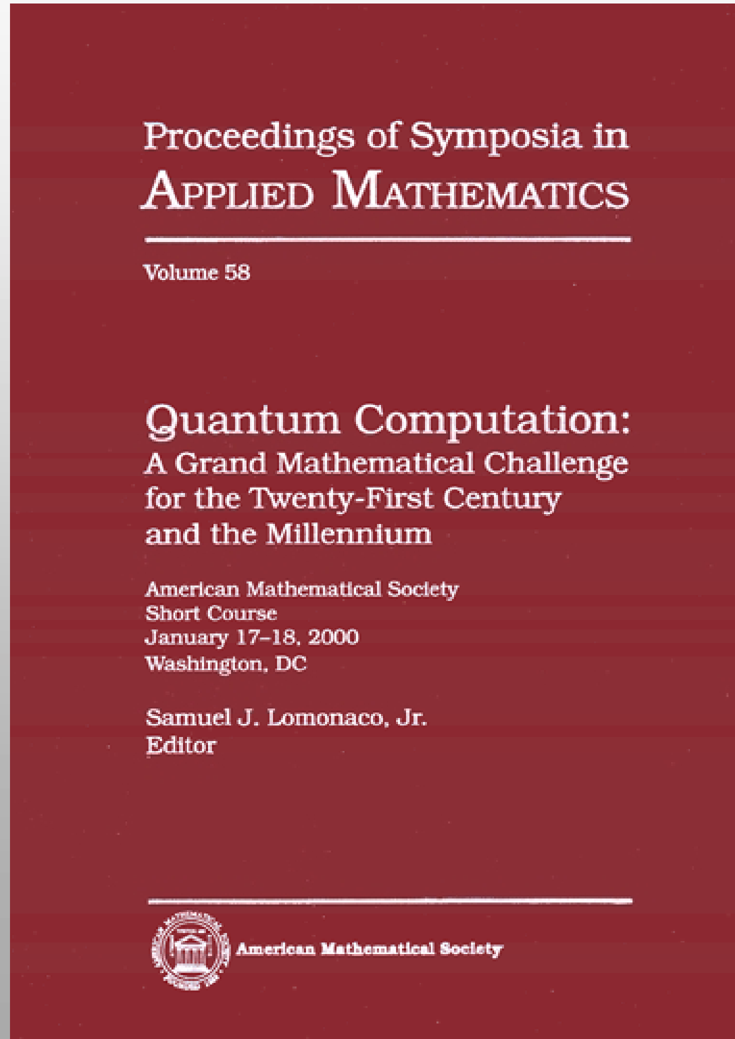
and measure the observable

$$O = \sum_{k=0}^{Q-1} k \left| k \right\rangle \left\langle k \right|$$

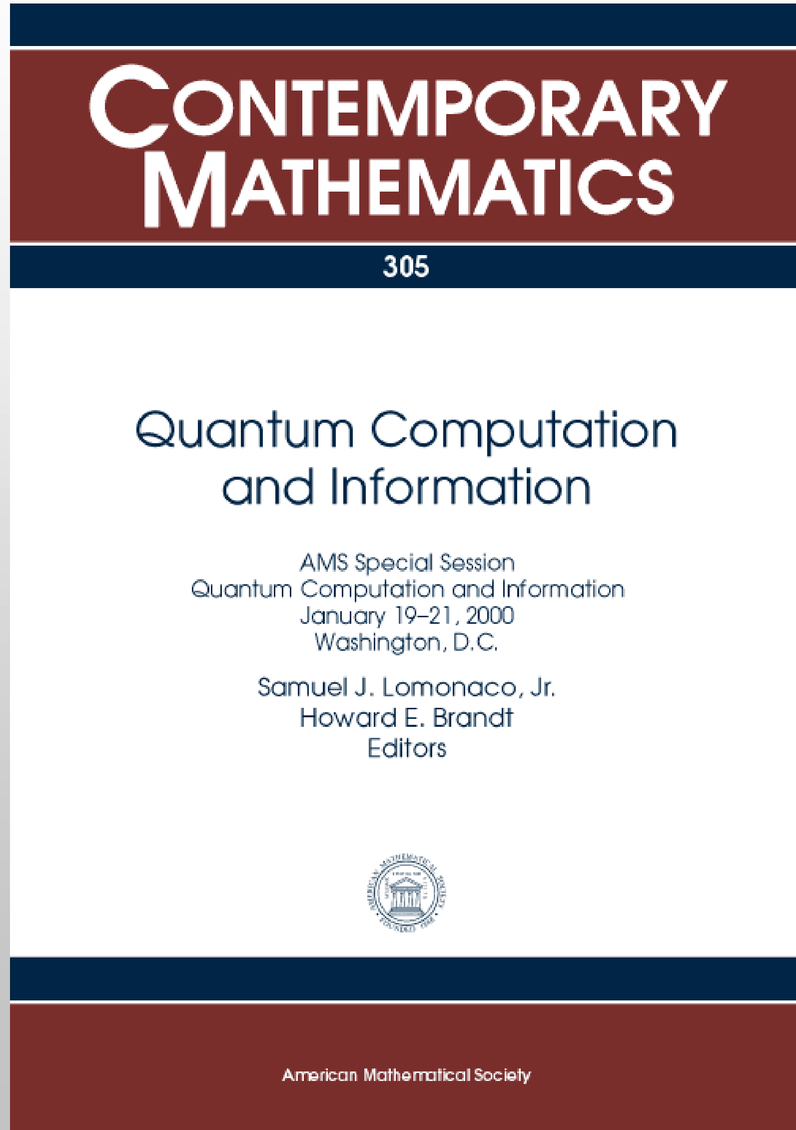
Conclusion

- **Shor's quantum factoring algorithm can be lifted to an algorithm on the integers $\mathbb{Z}$.**
- **This lifting gives some insight into the inner workings of the hidden subgroup algorithms.**
- **We have constructed an algorithm naturally dual to Shor's algorithm.**
- **Shor's quantum factoring algorithm can also be lifted to the reals $\mathbb{R}$, and to the compact circle $\mathbb{R}/\mathbb{Z}$**
- **Lifted continuous algorithms can be used to create new discrete algorithms.**
- **Implementation ?**

Quantum Computation: A Grand Mathematical Mathematical Challenge for the Twenty-First Century Century and the Millennium, **Samuel J. Lomonaco, Jr. (editor), AMS PSAPM/58, (2002).**



Quantum Computation and Information, Samuel J. Lomonaco, Jr. and Howard E. Brandt (editors), AMS CONM/305, (2002).



American Mathematical Society

Annual Meeting

Baltimore, MD

January 15-16, 2003

Special Session

Quantum Computation & Information

Mathematical Challenges

AMS Special Session Organizers

Samuel J. Lomonaco

Howard E. Brandt

Louis H. Kauffman

- Lomonaco, An entangled tale of quantum entanglement, PSAPM/58, AMS, (2002), 305-349.

New Journal of Physics
An Institute of Physics and Deutsche Physikalische Gesellschaft Journal

Quantum entanglement and topological entanglement

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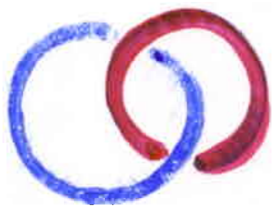
Topological Entanglement and Quantum Entanglement

L. Kauffman and S. Lomonaco

I.

L 

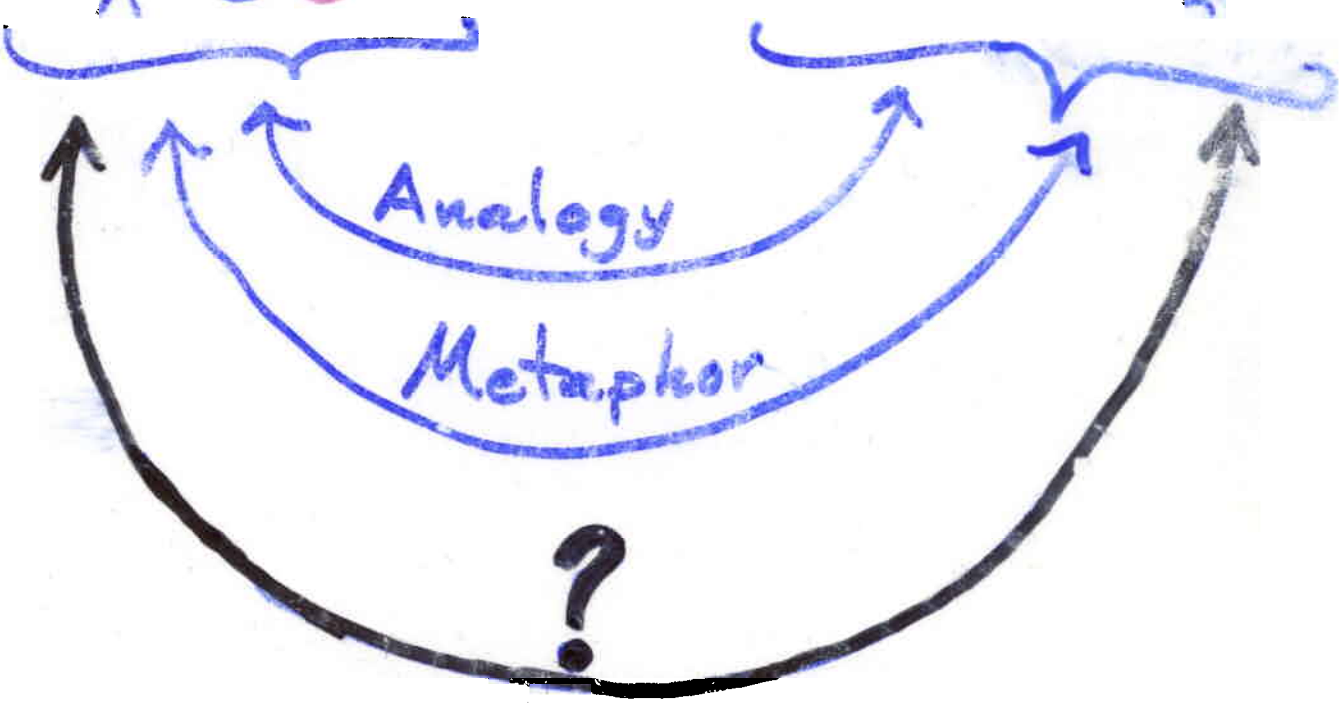
A topological linking (entanglement)

$$\psi = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

A quantum entanglement

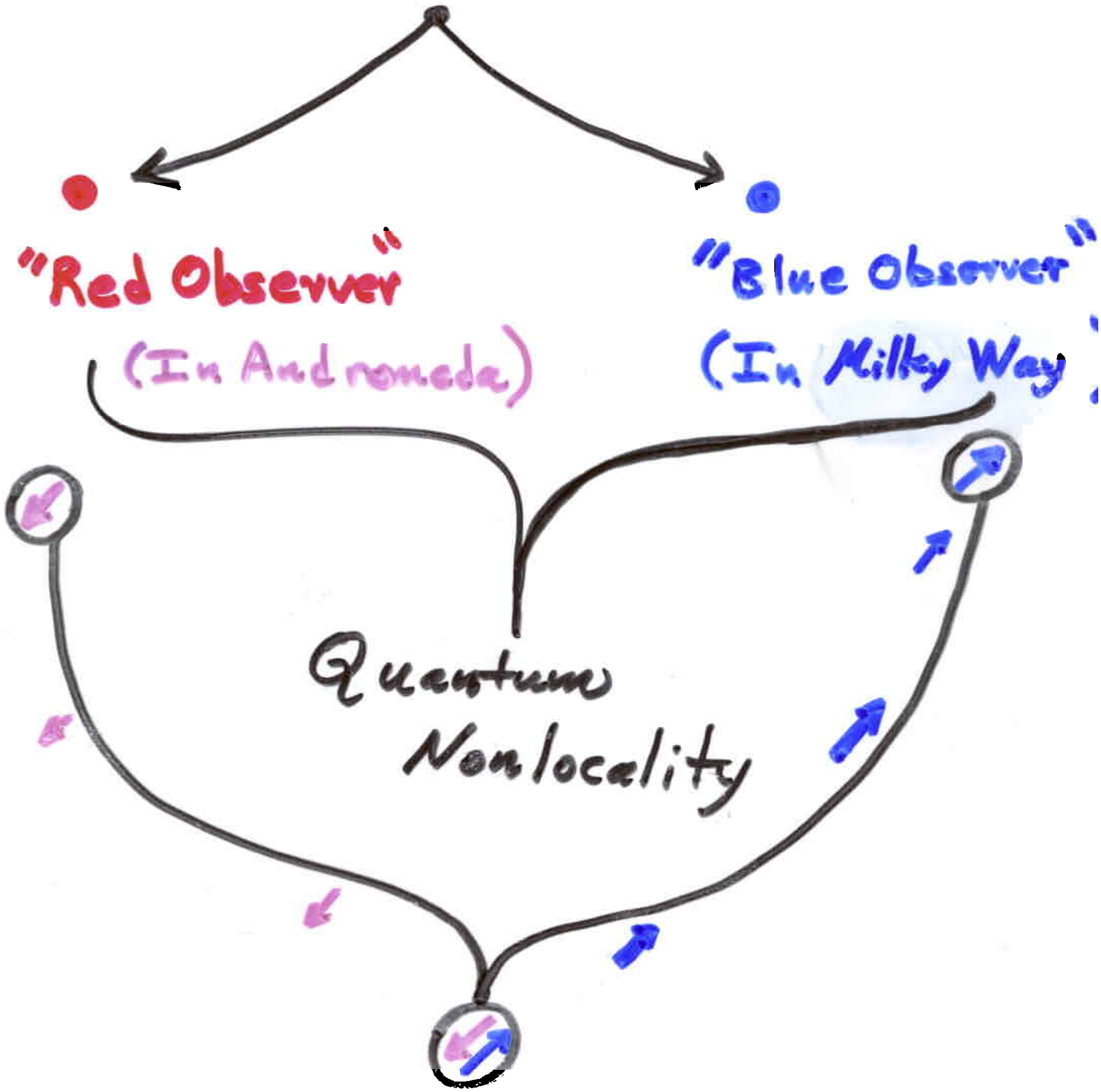
L ≠ O O

$\psi \neq \psi_1 \otimes \psi_2$

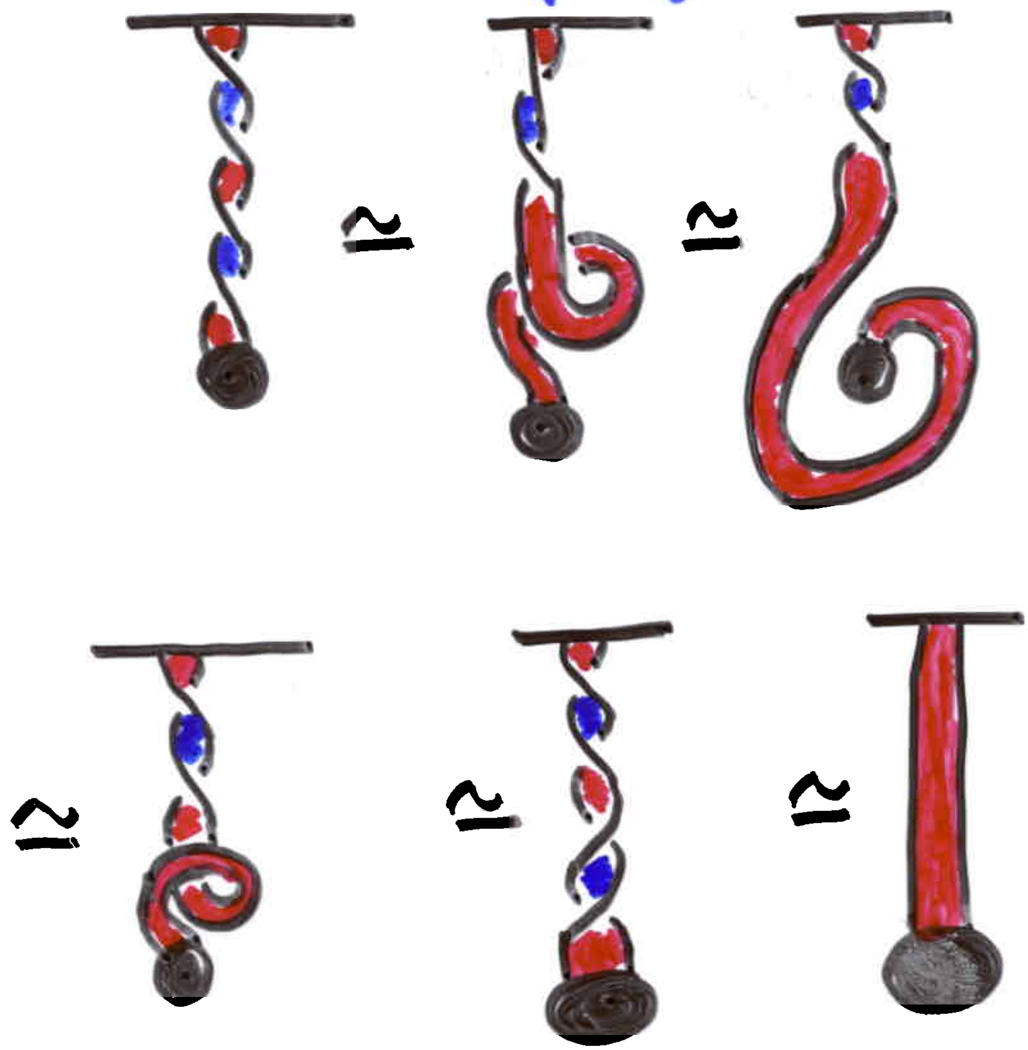


Quantum Entanglement

$$\Psi = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$



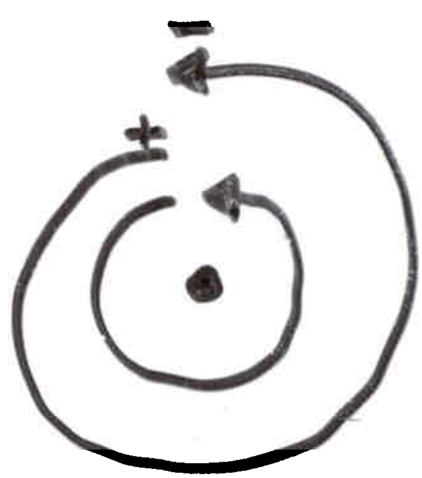
Topology



Dirac String Trick

$\longleftrightarrow$ 2-fold cover

$$SU(2) \xrightarrow{\;2-1\;} SO(3)$$



$\longleftrightarrow$ Change in phase of wavefunction for a fermion.

Example of P.K. Aravind

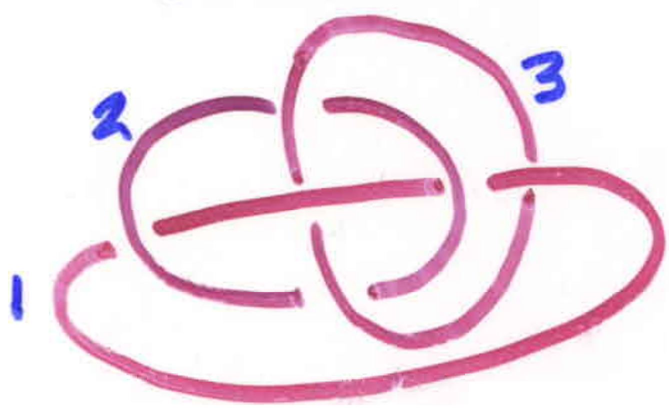
(In "Potentiality, Entanglement and Passion-at-a-Distance" Kluwer (1997) ed. by R.S. Cohen et al)

Borromean Rings and the GHZ state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\beta_1\rangle|\beta_2\rangle|\beta_3\rangle - |\alpha_1\rangle|\alpha_2\rangle|\alpha_3\rangle\right)$$

- 3 particles
- all spins in z-direction.

Measure any particle + state becomes disentangled.



Borromean Rings

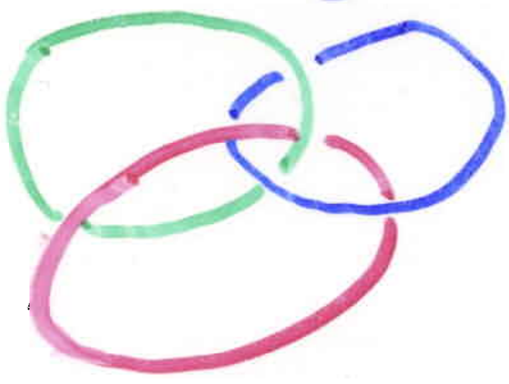
But if you change basis

$$|\psi\rangle = \frac{|\beta_{1x}\rangle}{\sqrt{2}}\left(\frac{|\beta_2\rangle|\beta_3\rangle - |\alpha_2\rangle|\alpha_3\rangle}{\sqrt{2}}\right)$$

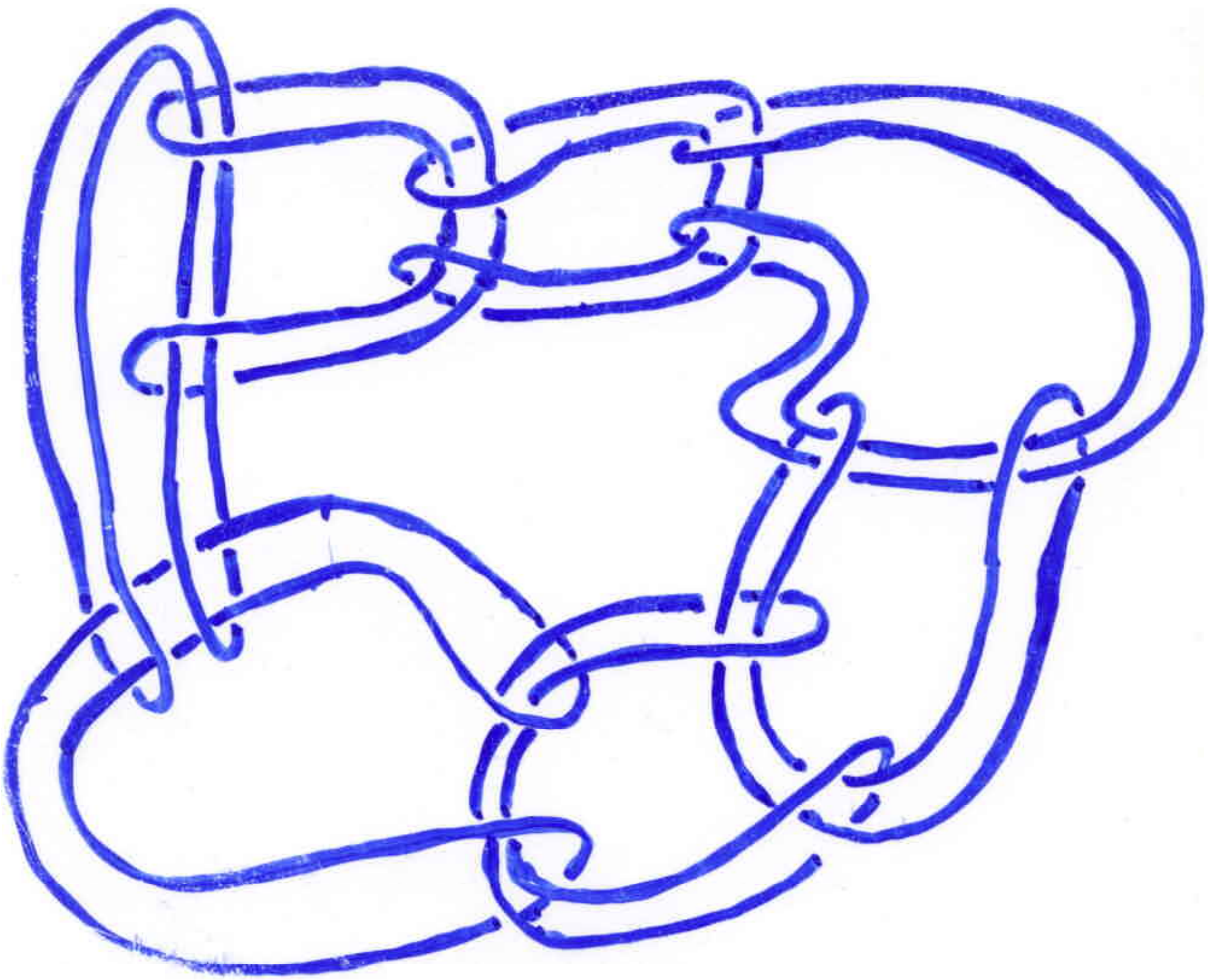
$$+ \frac{|\alpha_{1x}\rangle}{\sqrt{2}}\left(\frac{|\beta_2\rangle|\beta_3\rangle + |\alpha_2\rangle|\alpha_3\rangle}{\sqrt{2}}\right)$$

where $|\beta_{1x}\rangle$ & $|\alpha_{1x}\rangle$ denote spin-up & spin-down states of particle 1 in x direction.

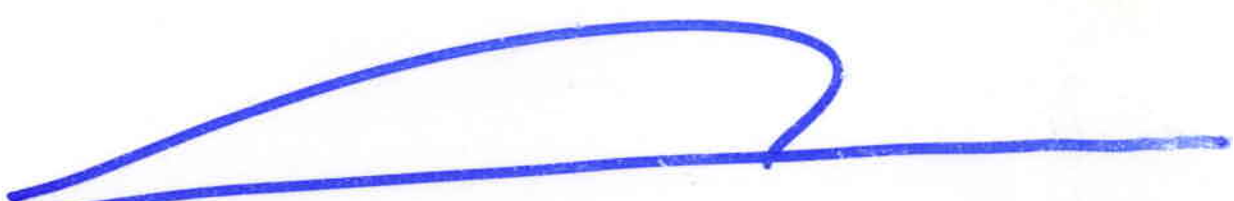
And analogy is now to



} each pair is linked once !

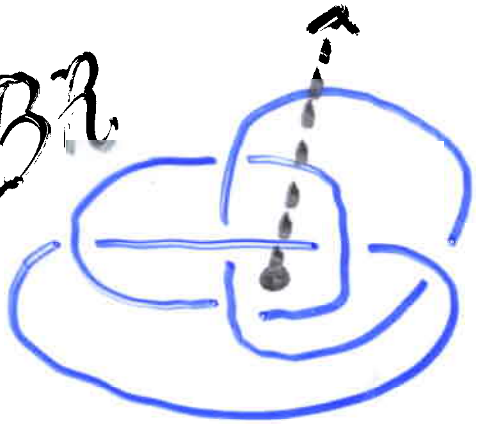


$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\beta_1\rangle \cdots |\beta_N\rangle - |\alpha_1\rangle \cdots |\alpha_N\rangle\right)$$



Question: Investigate the collection of linking patterns that describe the entanglement of a given state.

BR



27'

$\sigma_1 \quad \sigma_1^{-1}$

$\sigma_2 \quad \sigma_2^{-1}$

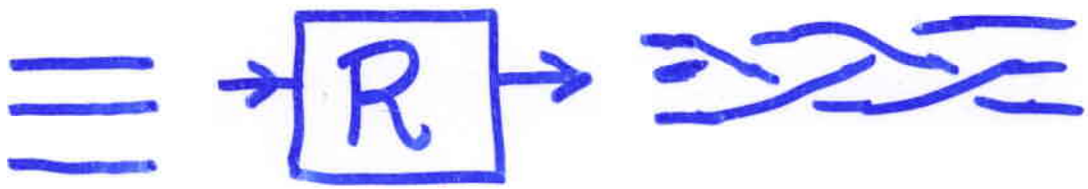
$$B = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$$

Borromean Braid

211

B

It is interesting to note that the Borromean rings are the closure of the braid B, a well-known sinnet in weaving of hair. Linking numbers cannot detect the entanglement of the braid B.



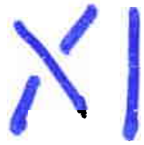
Needed: A quantum weaving machine

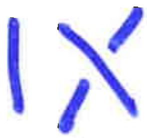
On topological side

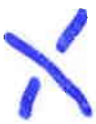
R can be an elementary braid.

So: How to associate a unitary operator to an elementary braid?

 $\rightsquigarrow R$

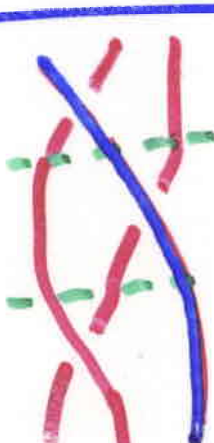
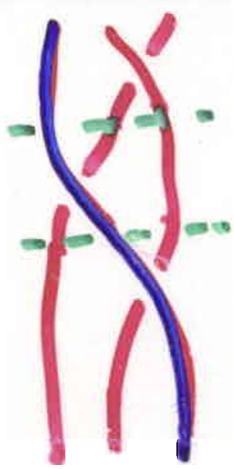
 $\rightsquigarrow R \otimes I$

 $\rightsquigarrow I \otimes R$ etc.

 $\rightsquigarrow R \Rightarrow$  $\rightsquigarrow R'$

and  $\simeq$  $\Rightarrow R' = R^{*} = R^{-1}$

(since R is unitary)

 $\simeq$ 

$$(R \otimes I)(I \otimes R)(R \otimes I)$$
$$\|$$
$$(I \otimes R)(R \otimes I)(I \otimes R)$$

The Yang-Baxter Equation

An Example

$$R = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\begin{array}{cccc} 00 & 01 & 10 & 11 \\ a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & b \end{array}$$

a, b, c, d unit complex nos. (so $\bar{a} = a^{-1} \ldots$)

$\mathrm{Det}(R) = -abcd$

(So $-abcd = 1$ if want $\mathrm{Det}(R) = 1$.)

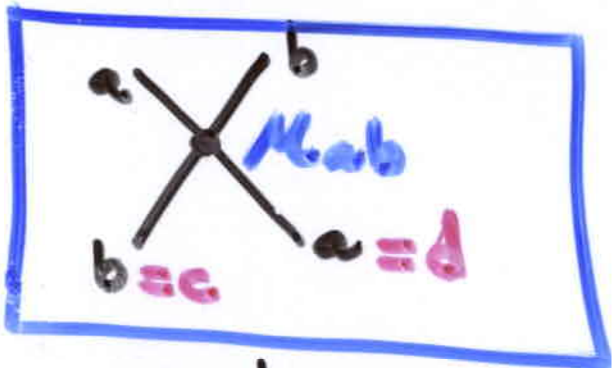
$$R|00\rangle = a|00\rangle$$
$$R|01\rangle = c|10\rangle$$
$$R|10\rangle = d|01\rangle$$
$$R|11\rangle = b|11\rangle$$

Claim:
- R is unitary
- R satisfies the Yang-Baxter Equation

More generally, let

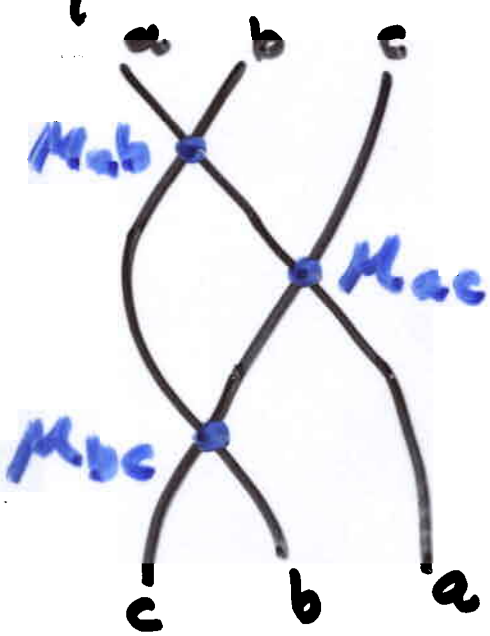
$M = (M_{ab})$ be any $n \times n$ matrix whose entries $M_{ab} \in \mathbb{C}$ with $|M_{ab}| = 1$.

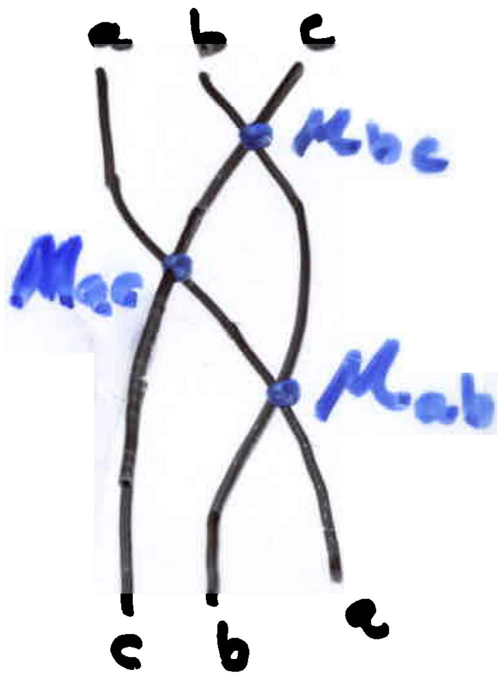
Let $R^{ab}_{cd} = M_{ab}\, \delta^a_d\, \delta^b_c$



$$(R^*)^{ab}_{cd} = \overline{M}_{ba}\, \delta^a_d\, \delta^b_c \Rightarrow R^* = R^{-1}$$

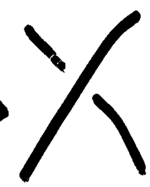
and R satisfies the Yang-Baxter Equation.





$$R = \begin{pmatrix} a & & & \\ & & c & \\ & d & & \\ & & & b \end{pmatrix} \leftrightarrow$$

$$R^* = \begin{pmatrix} \bar{a} & & & \\ & & \bar{d} & \\ & \bar{c} & & \\ & & & \bar{b} \end{pmatrix} \leftrightarrow$$



$$R^2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & & & \\ & cd & & \\ & & dc & \\ & & & b^2 \end{pmatrix}$$

$$R^2 \neq I$$

<u>Lemma</u>. Let $\psi = |0\rangle + |1\rangle$

$\phi = R(\psi \otimes \psi)$. Then

ϕ is entangled if $ab \neq cd$.

<u>Pf</u>. $R(\psi \otimes \psi) = R\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$

$= a|00\rangle + c|10\rangle + d|01\rangle + b|11\rangle$

$(x|0\rangle + y|1\rangle) \otimes (z|0\rangle + w|1\rangle)$

$= \underline{xz}|00\rangle + \underline{yz}|10\rangle + \underline{xw}|01\rangle + \underline{yw}|11\rangle$

//

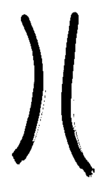
Special Case

$$R = \begin{pmatrix} a & & & \\ & & c & \\ & c & & \\ & & & a \end{pmatrix}.$$

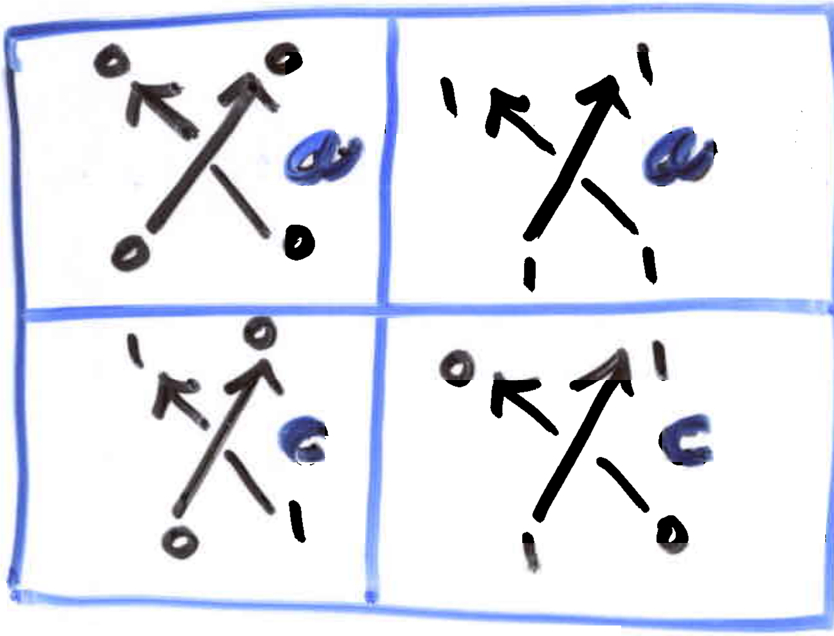
Then R entangles $(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$ if $a^2 \neq c^2$. Hence entangles if $c^2/a^2 \neq 1$.

Here $R^2 = \begin{pmatrix} a^2 & & & \\ & c^2 & & \\ & & c^2 & \\ & & & a^2 \end{pmatrix}$

Thus $a^2 = c^2 \Rightarrow R^2 = a^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

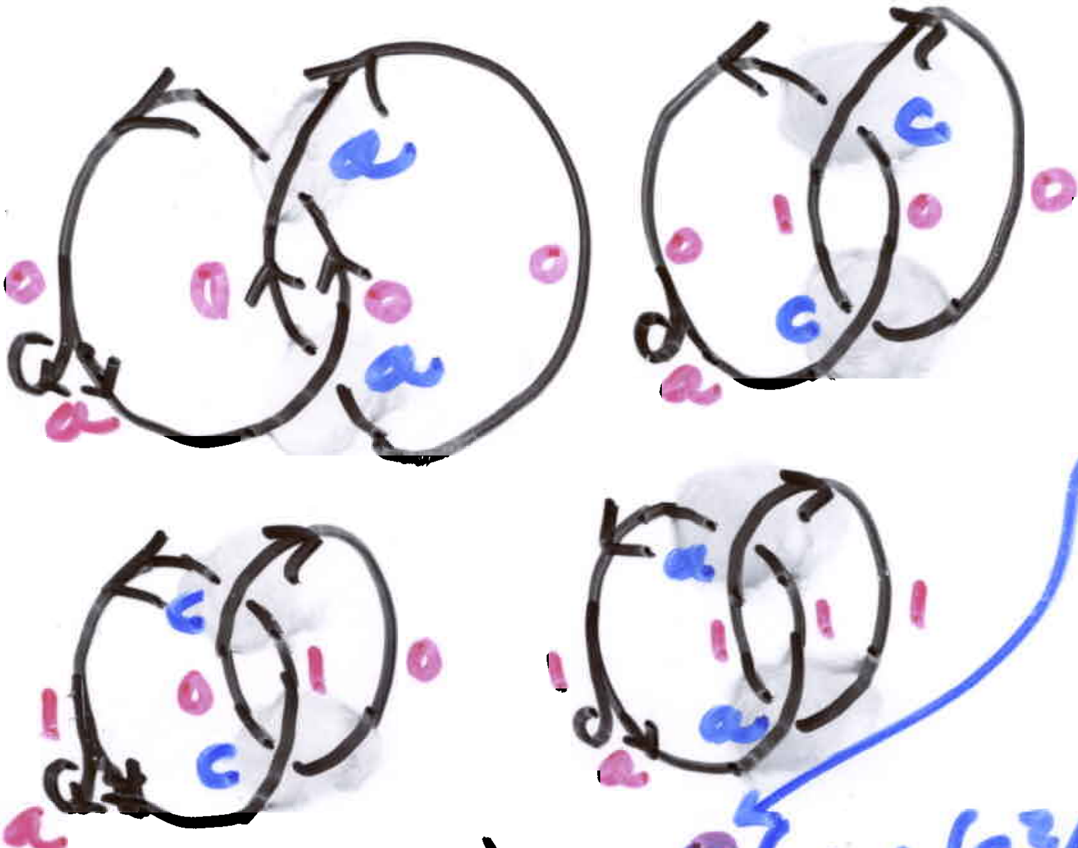
and  and  only differ by global phases.

So here the entanglement condition and the ability to detect linking are coincident.



$R|00\rangle = a|00\rangle$

$$R = \begin{pmatrix} a & & c & \\ & c & & \\ & & & a \end{pmatrix}$$



Here $3 = w(K)$

$$a(2a^2 + 2c^2) = 2a^3\left(1 + (c^2/a^2)\right)$$

Let $Z_K = \dfrac{\text{sum of contrib}}{2a^{w(K)}}$

$$\Rightarrow \boxed{Z_K = 1 + (c^2/a^2)^{\text{linking\#}(K)}}$$

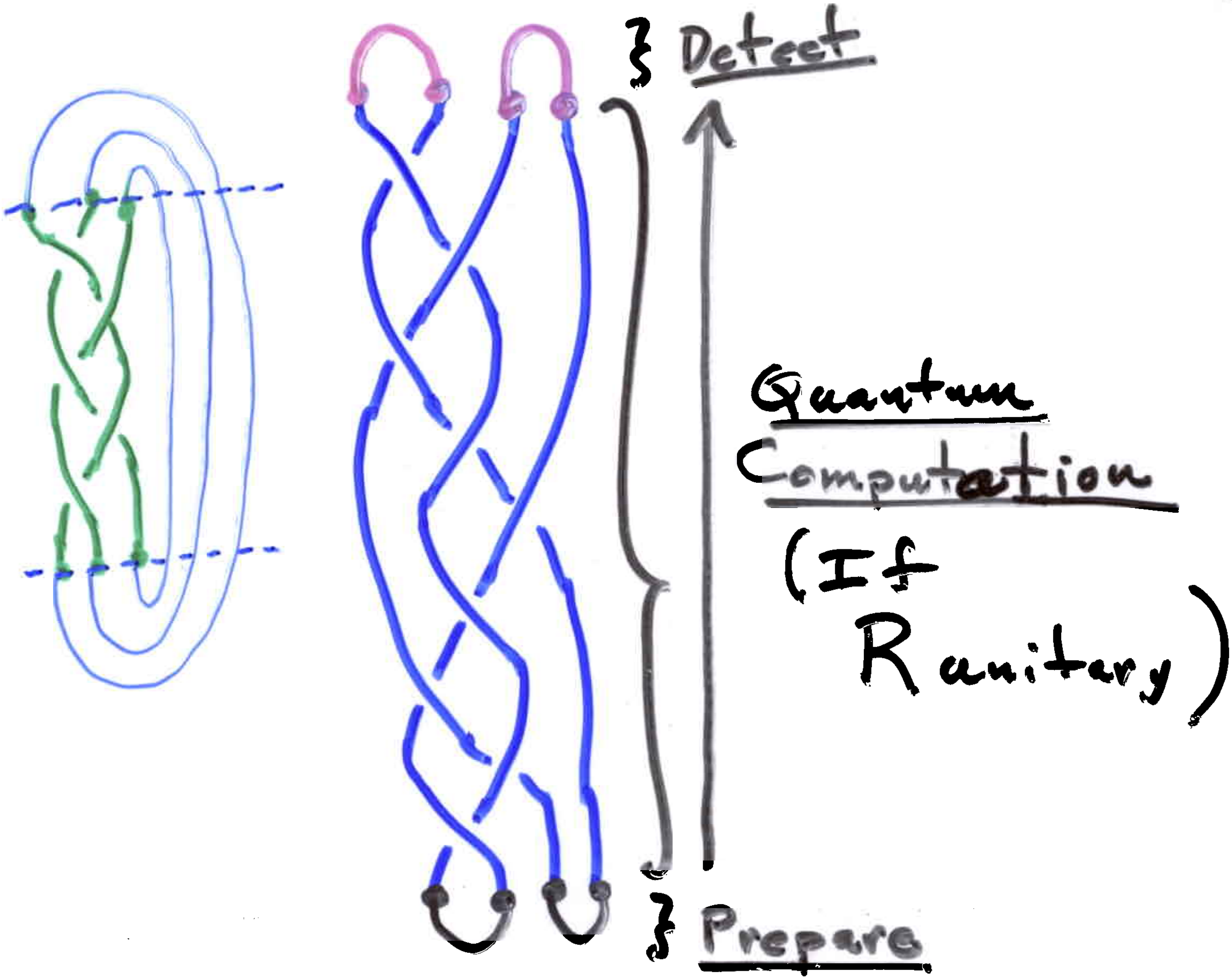
Other Unitary Solns to YBE

Heather Dye (UIC) has shown that all 4x4 unitary solns are obtained by conjugation from the types

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 0 \end{bmatrix}$$

and some numerical special cases ...

(based on paper by Jarmo Hietarinta)



Knots as Quantum Computers

$$U_1 = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} = \delta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$U_2 = \begin{bmatrix} \delta^{-1} & \sqrt{1-\delta^{-2}} \\ \sqrt{1-\delta^{-2}} & \delta - \delta^{-1} \end{bmatrix} = \delta \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix}$$

$$\left.\begin{aligned} U_1^2 &= \delta U_1 \\ U_2^2 &= \delta U_2 \\ U_1 U_2 U_1 &= U_1 \\ U_2 U_1 U_2 &= U_2 \end{aligned}\right\} \text{Temperley Lieb Algebra}$$

$a^2 + b^2 = 1$

$a = \delta^{-1}$

$b = \sqrt{1-\delta^{-2}}$

$$\left.\begin{aligned} \sigma_1 &\longmapsto A + A^{-1} U_1 \\ \sigma_2 &\longmapsto A^{-1} + A U_2 \end{aligned}\right\} \text{unita}$$

Want $A \in$ unit circle in cx plane

$\delta = -A^2 - A^{-2}$

$\sqrt{1-\delta^{-2}}$ real

$\Rightarrow 2\,\mathrm{Re}(A^4) + 1 \geq 0$

UNITARY BRAID RERESENTATION