Qualifying Entanglement with Knot Theory

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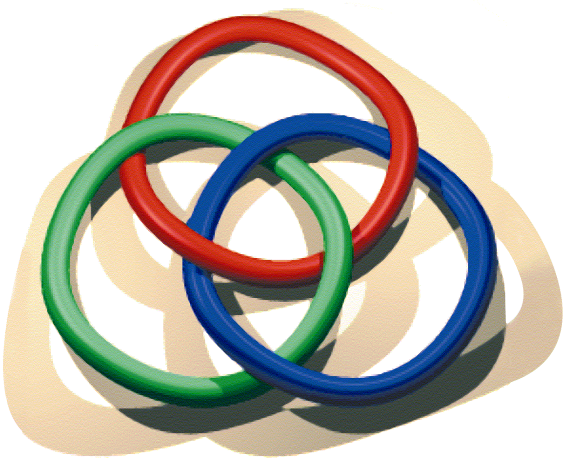
An Observation by “P.K.” Aravind



Aravind, in "Borromean entanglement of the GHZ state“ (1997), pointed at the similarities between Borromian rings and the entanglement properties of the GHZ state

$$\left|GHZ\right\rangle = \frac{1}{\sqrt{2}}\left(\left|000\right\rangle + \left|111\right\rangle\right)$$

- As a triplet it is connected
- Removing any one part disconnects the remainders.

Describing Correlations with Subsets of Power sets

A correlation scheme **C** for N items is a subset of the power set: **C** $\subseteq 2^{\{1,\ldots,N\}}$, such that S$\in$**C** *should be interpreted as* "S *by itself is correlated in* **C**".

Examples:

- Borromean rings:
  {{1}, {2}, {3}, {1,2,3}}
- 3-Chain:
  {{1},{2},{3}, {1,2},{2,3}, {1,2,3}}

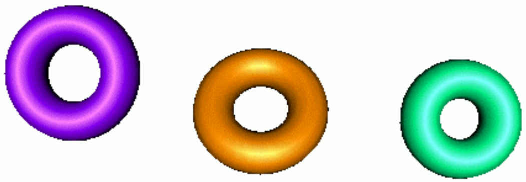
What schemes are possible?

All Possible Correlation Schemes for Three Items

(cases 1–3)

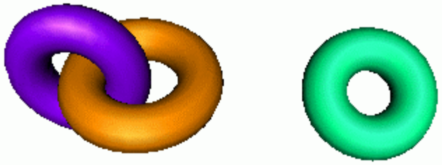
Three uncorrelated items:

{{1}, {2}, {3}}



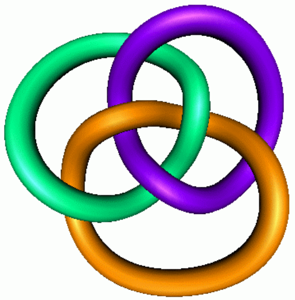
One correlated pair:

{{1}, {2}, {3},
{1,2}}



Borromean correlations:

{{1}, {2}, {3},
{1,2,3}}

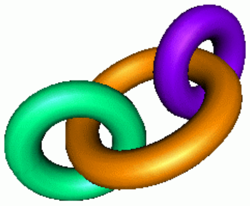


All Possible Correlation Schemes for Three Items

(cases 4–6)

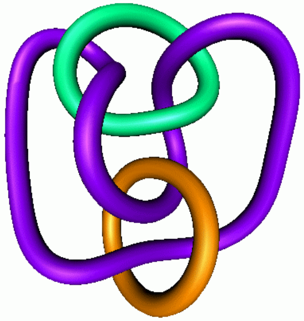
3-Chain:

{{1}, {2}, {3},
{1,2},{2,3},
{1,2,3}}



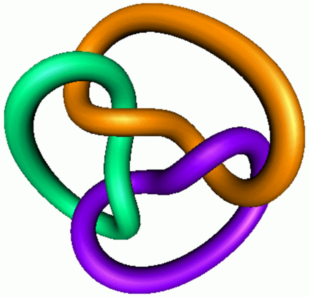
Unexpected one:

{{1}, {2}, {3},
{1,2},
{1,2,3}}



All correlated:

{{1}, {2}, {3},
{1,2},{1,3},{2,3},
{1,2,3}}



6 different Ways of Entangling Three Qubits

1) **Uncorrelated:**

$$|000\rangle$$

2) **Correlated Pair:**

$$\frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)$$

3) **Borromean GHZ:**

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

4) **Three Chain:**

$$\frac{1}{\sqrt{5}}(|000\rangle + |010\rangle + |011\rangle + |110\rangle + |111\rangle)$$

5) **Unexpected one:**

$$\frac{1}{\sqrt{3}}(|000\rangle + |110\rangle + |001\rangle)$$

6) **All correlated:**

$$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

Definition of Proper Correlation Schemes

A <u>correlation</u> <u>scheme</u> $\mathbf{C} \subseteq 2^{\{1,\ldots,N\}}$ for N items is *proper* if it obeys:

- If $A,B \in \mathbf{C}$ and $A \cap B \neq \emptyset$, then $A \cup B \in \mathbf{C}$,
- $\{1\}, \{2\}, \ldots, \{N\} \in \mathbf{C}$,

Removing an item k corresponds to removing all sets $A \in \mathbf{C}$ with $k \in A$.

Borromean example:
$\{\{1\},\{2\},\{3\},\{1,2,3\}\}$ α_3
$\{\{1\},\{2\}\}$

General Line of Thought

The just defined proper correlation schemes allow us to relate the splitting behavior of N rings to the entanglement properties of N qubits, and vice versa.

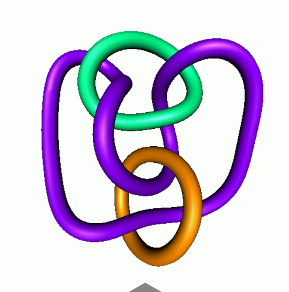
A correlated subset $A \subseteq \{1,\ldots,N\}$ stands for, respectively,

•… the fact that the rings of A are unsplittable, even if we leave out all other rings $\{1,\ldots,N\}\setminus A$.

•… the qubits labeled by the elements of A are entangled. More specifically, the parties in A can distill, with LOCC, entangled states $|0\ldots0\rangle + |1\ldots1\rangle$, without help from the other parties $\{1,\ldots,N\}\setminus A$.

If a subset is not correlated, then the rings in A can be split non-trivially — the qubits in A are not entangled.

Proving the Bijection



$\{\{1\}, \{2\}, \{3\},\{1,2\},\{1,2,3\}\}$

$$\frac{1}{\sqrt{3}}\left(|000\rangle + |110\rangle + |001\rangle\right)$$

It can be shown that:

- There is a bijection between the possible link-splitting configurations and proper correlation schemes
- Similarly, there is a one-to-one mapping between proper correlation schemes and possible entanglement configurations.

Every Entangled State corresponds to a proper Correlation Scheme

{{1}, {2}, {3},{1,2},{1,2,3}}

$$\frac{1}{\sqrt{3}}\left(|000\rangle + |110\rangle + |001\rangle\right)$$

Simple Proof:
Let A and B be subsets of the parties {1,…,N}.

If the parties of A can create an entangled state, the same holds for the parties of B, and A and B are not disjoint,
then A and B together can create a maximally entangled state as well.

This shows that indeed “If A,B$\in$**C**, A$\cap$B$\neq\emptyset$, then A$\cup$B$\in$**C**” holds.

Every Correlation Scheme can be implemented as an Entangled Qubits

$\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}$

$$\frac{1}{\sqrt{3}}\left(|000\rangle + |110\rangle + |001\rangle\right)$$

Constructive Proof:

Let **C** be a correlation scheme $\{A_1, A_2, \ldots, A_k\} \subseteq 2^N$

$$\rho_C = \sum_{i=1}^{k} w_i \sigma_i$$

- weights w_i that depends on the sizes $|A_i|$
- density matrices σ_i that implement the entanglement among the qubits of A_i

We have to take care that the different σ_i do not ‘wash’ each other out.

Details of the Entanglement Construction

For a set $A \subseteq \{1,\ldots,N\}$ and a subset $S \subseteq A$ we have the maximally entangled state Ψ defined by

$$\left|\Psi_S^A\right\rangle := \frac{1}{\sqrt{2}}\left(\left|0_S 1_{A\setminus S} 0\ldots 0\right\rangle + \left|1_S 0_{A\setminus S} 0\ldots 0\right\rangle\right)$$

The mixed state σ_i uses these Ψ to get entanglement over A_i,
but not over any other set:

$$\sigma_i := \frac{1}{2^{|A_i|}-2}\left(\sum_{S\subseteq A_i}\left|\Psi_S^{A_i}\right\rangle\left\langle\Psi_S^{A_i}\right| - \left|\Psi_\varnothing^{A_i}\right\rangle\left\langle\Psi_\varnothing^{A_i}\right| - \left|\Psi_{A_i}^{A_i}\right\rangle\left\langle\Psi_{A_i}^{A_i}\right|\right)$$

By picking the right weights w_i, the mixture $\Sigma_i w_i \sigma_i$ has distillable entanglement over all sets A_i, and nowhere else.

Details of the Details of the Construction

Proving the distillability for A_i in $\rho_C = \Sigma_i w_i \sigma_i$:

Let ρ' be the restriction of ρ_C to A_i. [Dür, Cirac'00] proved:
"If ρ' has negative partial transpose for all non-trivial splittings of A_i, then there is distillable entanglement for A_i."

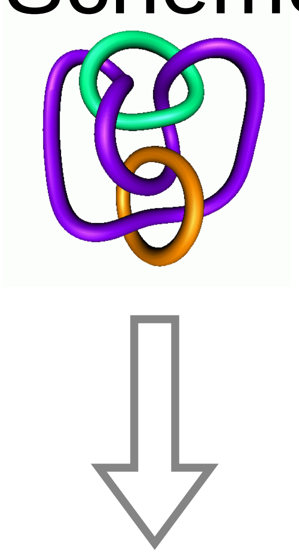
This negativity holds if the weights w_j decrease fast enough as a function of the sizes $|A_i|$.

Proving separability for sets S not in C:

For all A_i that do not lie within S, the σ_i do not matter.
By the restriction "if $A,B \in \mathbf{C}$, $A \cap B \neq \emptyset$, then $A \cup B \in \mathbf{C}$",
we can split S into S_1 and S_2, such that all sets $A_i \subset S$
are either in S_1 or in S_2.

Hence ρ, restricted to S, is separable along the this splitting.

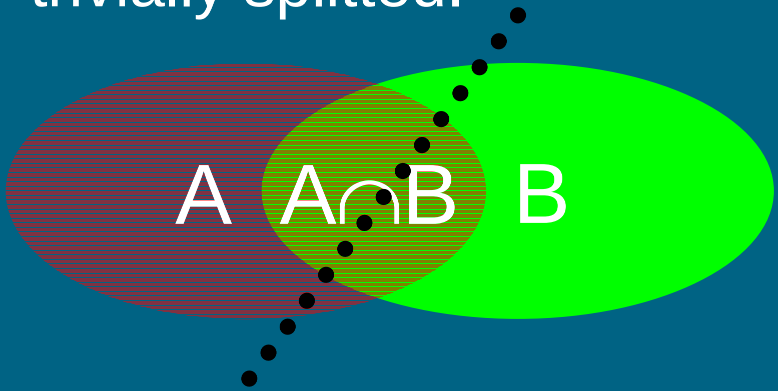
A Link Structure corresponds to a Correlation Scheme



{{1}, {2}, {3},{1,2},{1,2,3}}

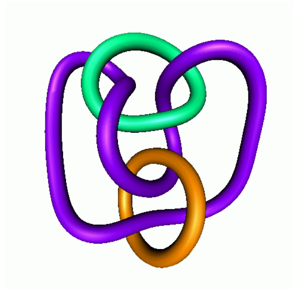
Contradiction Proof:
Let A and B be non-disjoint subsets of the links 1,…,N

If the links of A∪B can be non-trivially splitted, then A or B can be non-trivially splitted.



This shows that indeed "If A,B∈**C**, A∩B≠∅, then A∪B∈**C**" holds.

Every Correlation Scheme can be implemented as a Link Configuration



{{1}, {2}, {3},{1,2},{1,2,3}}

Constructive Proof:
Let **C** be a correlation scheme $\{A_1, A_2, \ldots, A_k\} \subseteq 2^N$.

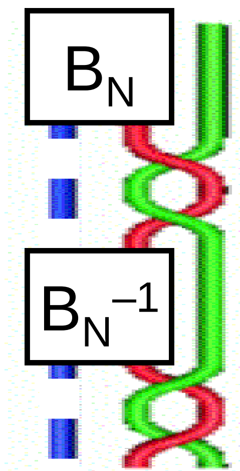
Using “Borromean braiding” we can add, one-by-one, the unsplittability for the sets A_i in an N-braid.

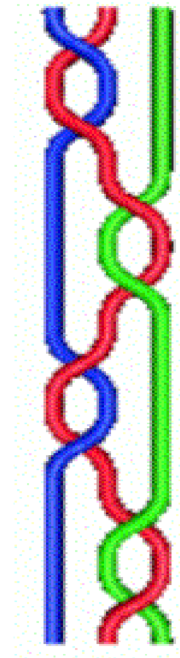
The closure of this braid will be an N-component link configuration that has **C** as its unplittability scheme.

Borromean Braiding

We can recursively define a braid B_N that gives the N-strand generalization of the Borromean braid for which removing one strand makes the braid trivial:

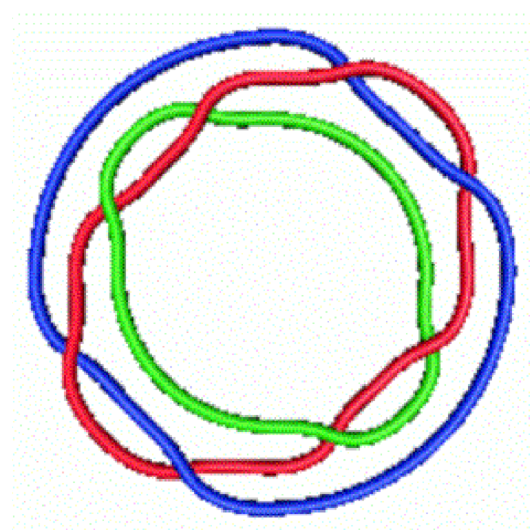
B_{N+1} =





Closing the N-braid gives a Borromean structure for N rings:

Example:
N=3



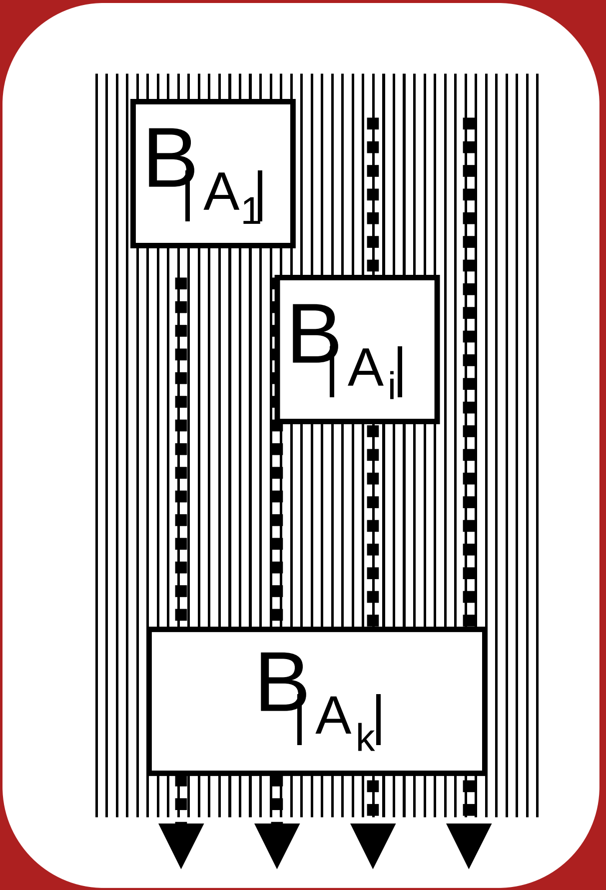
Details of the Braid Construction

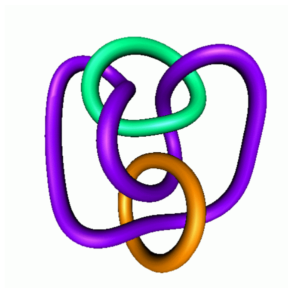
Let **C** be a correlation scheme $\{A_1,A_2,\ldots,A_k\} \subseteq 2^N$.

By 'weaving' the strands of A_i in a Borromean braid $B_{|A_i|}$, we make the braid unsplittable for A_i.

By the 'Borromean properties' and the requirement "if $A,B \in$ **C**, $A \cap B \neq \emptyset$, then $A \cup B \in$ **C**", these weavings $B_{|A_i|}$ do not interfere with each other.

(The hard part is proving that Borromean braids are unsplittable.)





$$\Updownarrow$$

{{1}, {2}, {3},{1,2},{1,2,3}}

$$\Updownarrow$$

$$\frac{1}{\sqrt{3}}\left(\left|000\right\rangle+\left|110\right\rangle+\left|001\right\rangle\right)$$

Final Result:

Bijection between Linked Rings, Correlation Schemes, and Distillable Entanglement

Comparison with Classical Correlations

The Correlation Scheme bijection also holds for classically correlated bits.

In this setting, the all-correlated state has “measure 1” over all distributions.

Quantum states, however, have no entanglement with non-zero probability.

Counting the Number of Possible Correlation Schemes

For N=1,2,3,4, one can check by hand:

$a_1 = 1$
$a_2 = 2$
$a_3 = 6$
$a_4 = 47$

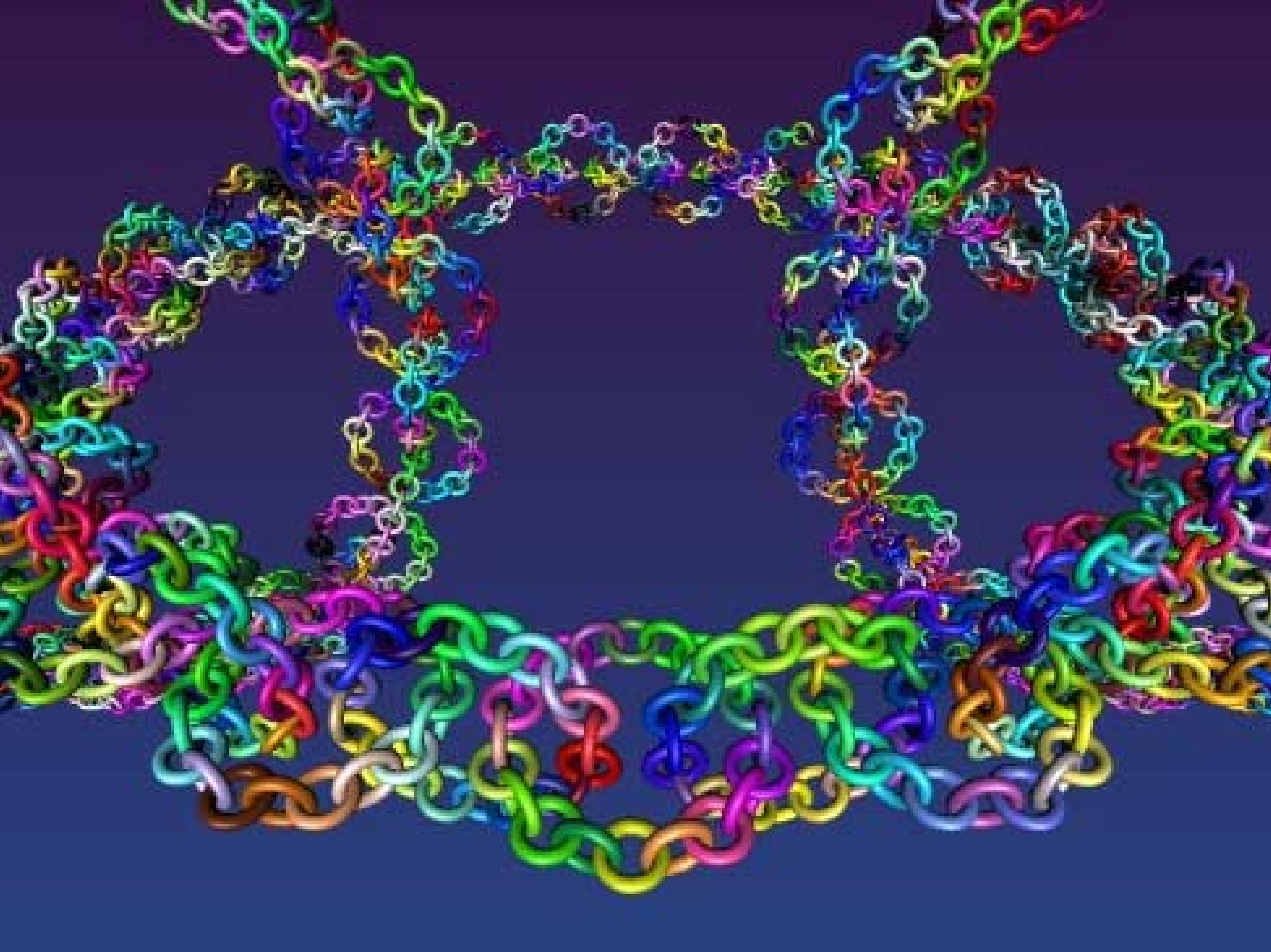
Computer calculations give us the next two values

$a_5 = 3095$

$a_6 = 26015236$

a_7 = undoable

Not much is known about this sequence (does not show up in Sloane's on-line dictionary), except that it grows double exponential.



invent