Action and Symmetries*

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Abstract

Since 1942 classical action functionals are key building blocks of functional integrals in physics. There exists another construction which starts with the generators $\{X_\alpha\}$ of the groups of transformations which exist on the configuration space of the system. The generators $\{X_\alpha\}$ serve as dynamical vector fields in Quantum Physics. Their use has been suggested by a stochastic construction (1980) of solutions of Schroedinger equations on riemannian manifolds. Given $\{X_\alpha\}$ and possibly other vector fields playing the role of external potentials we construct a functional integral solution of a parabolic PDE which can be identified with the Schroedinger equation of the system. Its semi-classical expansion yields the action function $\mathcal{S}$ of the system. We give two examples.

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This Fall I have been teaching a graduate course at Sharif University of Technology (Tehran) on functional integration. The lecture notes will serve as a first draft of a book I am writing with Pierre Cartier under the title

Functional Integration
Action and Symmetries

The book consists of three parts
I Basics (completed)
II Tools from Geometry (on the drawing board)
III Tools from Analysis (next on the drawing board)
The table of contents is available upon request.

I am looking forward to the lectures and discussions of this workshop, and their impact on this book.

Today, I shall approach functional integration not from the action functional of a system, but from its symmetries. I shall construct the Schroedinger equation of the system and characterize volume forms on function spaces. This material will be included in Section II, Tools from Geometry.

1) A basic theorem

Let $\mathbb{X}$ be a space of $L^{2,1}$ pointed paths x on $\mathbb{T}$ with values in a manifold $\boldsymbol{N}^D$:

$$\begin{cases} x : \mathbb{T} \longrightarrow \boldsymbol{N}^D, & \mathbb{T} = (t_a, t_b] \\ x(t_o) = \mathbf{x_o}, & in \;\; Q.M., t_o = t_b, \mathbf{x_o} = \mathbf{x_b}; \end{cases} \tag{1}$$

i.e. $\quad x \in \mathcal{P}_o \boldsymbol{N}^D$

$$||x||_{L^{2,1}} = \int_{\mathbb{T}} dt \, g_{\alpha\beta} \dot{x}^\alpha(t) \dot{x}^\beta(t) < \infty \tag{2}$$

Dynamical vector fields $\{X_{(\alpha)}\}$

Let $\{\sigma_{(\alpha)}(r)\}$, $\quad \alpha \in \{1, \ldots, D\}$ be D one-parameter groups of transformations on $\boldsymbol{N}^D$, and $X_{(\alpha)}$ their generators, i.e.

$$\frac{d}{dr}(\mathbf{x_o} \cdot \sigma_{(\alpha)}(r)) = X_{(\alpha)}(\mathbf{x_o} \cdot \sigma_{(\alpha)}(r)) \tag{3}$$

with

$$X_{(\alpha)}(\mathbf{x}) = \frac{d}{dr}(\mathbf{x} \cdot \sigma_{(\alpha)}(r))|_{r=0} \tag{4}$$

generically the vector fields $X_{(\alpha)}$ do not commute,

$$[X_{(\alpha)}, X_{(\beta)}] \neq 0 \tag{5}$$

So much for a classical system with configuration space (possibly a fibre bundle) $\boldsymbol{N}^D$ which admits a group of transformations $\{\sigma_{(\alpha)}(r)\}$ with generators $\{X_{(\alpha)}\}$. It becomes a quantum system when the generators are used to define a group of transformations on $\boldsymbol{N}^D$ parametrized by a space $\mathcal{P}_o\boldsymbol{R}^D$ of pointed paths z on $\boldsymbol{R}^D$.
Explicitly,

$$dx\,(t,z) = X_{(\alpha)}(x(t,z))\,dz^\alpha(t) \tag{6}$$

Spaces of pointed paths are contractible, and eq.(6) defines a map from
$P : \mathcal{P}_o\boldsymbol{R}^D \longrightarrow \mathcal{P}_o\boldsymbol{N}^D$ by $z \mapsto x$
A path z is a continuous, real, vector valued function on $\mathbb{T}$ such that
$z(t_o) = 0$
and, for a given $h_{\alpha\beta}$,

$$Q_o(z) := \int dt\, h_{\alpha\beta}\dot{z}^\alpha(t)\dot{z}^\beta(t) < \infty \tag{7}$$

When the generators do not commute, X is not a function of $z(t)$ but a functional of z, as well as a function of t. In general, one cannot solve (6) explicitly, but one can prove that the solution can be written

$$x(t,z) = \mathbf{x_o} \cdot \sum(t,z) \tag{8}$$

where $\sum(t,z)$ is an element of a group of transformations on $\boldsymbol{N}^D$:

$$\mathbf{x_o} \cdot \sum(t+t', z \times z') = \mathbf{x_o} \cdot \sum(t,z) \cdot \sum(t',z') \tag{9}$$

where z defined on $(t_o, t]$ is followed by z' on $(t, t']$.

A functional integral on $\mathcal{P}_o\boldsymbol{R}^D$

$$\Psi(t,\mathbf{x_o}) : \int_{\mathcal{P}_o\boldsymbol{R}^D} \mathcal{D}_{s,Q_o}z \cdot \exp(-\frac{\pi}{s}Q_o(z))\phi(\mathbf{x_o} \cdot \sum(t,z)) \tag{10}$$

Here $\mathcal{D}_{s,Q_o}z \cdot \exp(-\frac{\pi}{s}Q_o(z))$, $s \in \{1, i\}$ is the gaussian volume form defined by

$$\mathcal{D}_{s,Q_o}z \cdot \exp(-\frac{\pi}{s}Q_o(z)) - 2\pi i < z', z >:= \exp(-\pi s W_o(z')) \tag{11}$$

The quadratic form (7) defines a differential operator D by

$$Q_o(z) =< Dz, z >; \tag{12}$$

the quadratic form W on the dual $\boldsymbol{R}_D$ of $\boldsymbol{R}^D$ is

$$W_o(z') = < z', Gz' > \tag{13}$$

where G is the unique Green function of D in the domain of integration $\mathcal{P}_o \boldsymbol{R}^D$.

$$DG = \mathbb{1} \quad , \quad GD = \mathbb{1} \tag{14}$$

A basic theorem

The functional integral (10) is the solution of the parabolic equation

$$\begin{cases} \frac{\partial \Psi}{\partial t} = \frac{s}{4\pi} h^{\alpha\beta} \mathcal{L}_{X_{(\alpha)}} \mathcal{L}_{X_{(\beta)}} \Psi \\ \Psi(t_o, \mathbf{x}) = \phi(\mathbf{x}) \;\; for any \; \mathbf{x} \in \boldsymbol{N} \end{cases} \tag{15}$$

where $\mathcal{L}_{\boldsymbol{X}}$ is the Lie derivative with respect to the generator $\boldsymbol{X}$ of a group of transformation.

2) An example: paths in non-cartesian coordinates

Equation (6) can be used for expressing non-cartesian differentials $\{dx^\alpha(t)\}$ in terms of cartesian differentials $\{dz^\alpha(t)\}$. In this case, x is not a functional of z, but simply a function of $z(t)$. Polar coordinates in $\boldsymbol{R}^2$ are sufficient for displaying the general construction of the relevant dynamical vector fields $\{X_{(1)}, X_{(2)}\}$.

Let us abbreviate $z^\alpha(t)$ to z^α, $x^1(t)$ to r, and $x^2(t)$ to θ; it follows from

$$z^1 = r\cos\theta\,, \qquad z^2 = r\sin\theta\,,$$

that eq.(6) reads

$$\begin{cases} dr &= \cos\theta \cdot dz^1 + \sin\theta \cdot dz^2 &=: X^1_{(1)} dz^1 + X^1_{(2)} dz^2 \\ d\theta &= -\frac{\sin\theta}{r} dz^1 + \frac{\cos\theta}{r} dz^2 &=: X^2_{(1)} dz^1 + X^2_{(2)} dz^2 \end{cases} \tag{16}$$

The dynamical vector fields are

$$\begin{cases} X_{(1)} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \\ X_{(2)} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \end{cases} \tag{17}$$

According to eq.(15) the Schroedinger equation in polar coordinates for a free particle is

$$\begin{aligned} \frac{\partial \Psi}{\partial t_b} &= \frac{s}{4\pi} \delta^{\alpha\beta} \mathcal{L}_{X_{(\alpha)}} \mathcal{L}_{X_{(\beta)}} \Psi \\ &= \frac{s}{4\pi} \delta^{\alpha\beta} X_{(\alpha)} X_{(\beta)} \Psi \end{aligned}$$

$$\frac{s}{4\pi}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\Psi \tag{18}$$

3) An example. Frame bundles over Riemannian manifolds

On a principal bundle, the dynamical vector fields $\{X_{(\alpha)}\}$, necessary for the construction (10) of the solution $\Psi(t,\mathbf{x}_o)$ of the parabolic eq. (15), are readily obtained from connections. On a riemannian manifold, there exists a unique metric connection[1] such that the torsion vanishes. Whether unique or not, a connection defines the horizontal lift of a vector on the base space. After trivialization of the frame bundle, a point $\rho(t)$ is a pair: $x(t)$ on the base space and $u(t)$ a frame on the typical fibre.
$\rho(t)=(x(t),u(t))$.
The connection σ defines the horizontal lift of a vector $\dot{x}(t)$, namely

$$\dot{\rho}(t)=\sigma(\rho(t))\cdot\dot{x}(t) \tag{19}$$

A frame $u(t)$ is a map from $\boldsymbol{R}^D$ into $T_{x(t)}\boldsymbol{N}^D$; let $u(t)^{-1}:\dot{x}(t)\mapsto\dot{z}(t)\in\boldsymbol{R}^D$. Eq (19) can be rewritten

$$\begin{aligned}\dot{\rho}(t)\ &=\sigma(\rho(t))\cdot u(t)\circ u(t)^{-1}\cdot\dot{x}(t)\\ &=: x_{(\alpha)}(\rho(t))\cdot\dot{z}^{\alpha}(t)\end{aligned} \tag{20}$$

The dynamical vectors $\{X_{(\alpha)}\}$ generate the horizontal lifts of straight lines through the origin of $\boldsymbol{R}^D$. Indeed. let $z_{(\alpha)}(t)=\delta_\alpha^\beta\, e_{(\beta)}=e_{(\alpha)}$, then eq. (20) reads $\dot{\rho}_{(\alpha)}(t)=X_{(\beta)}(\rho_{(\alpha)}(t))\delta_\alpha^\beta=X_{(\alpha)}(\rho_{(\alpha)}(t))$.

On the frame bundle of a riemannian manifold, eq. (6) reads
$d\rho(t)=X_{(\alpha)}(\rho(t))\dot{z}^{\alpha}(t)$
The construction from (6) to (15) gives a parabolic equation on the bundle, its projection on the base space gives the parabolic expression with the Laplace-Beltrami operator.

4) Generalizations

• It is straightforward to replace eq. (6) by $dx(t,z)=X_{(\alpha)}(x(t,z))\,dz^\alpha+Y(x(t,z))\,dt$. Again, the solution of this equation can be written
$x(t,z)=\mathbf{x}_o\cdot\sum(t,z)$;

[1] This unique connection is the usual riemannian connection, characterized by the vanishing of the covariant derivative of the metric tensor. For the definition of a metric connection and the equivalence of the two characterizations, see for instance [Y. Choquet-Bruhat p.381]

the group of transformations $\sum$ parametrized by $\mathcal{P}_o \boldsymbol{R}^D$ is now defined by $(D+1)$ dynamical vector fields. Eq. (15) becomes

$$\begin{array}{ll} \frac{\partial \Psi}{\partial t} & = \frac{s}{4\pi} h^{\alpha\beta} \mathcal{L}_{X_{(\alpha)}} \mathcal{L}_{X_{(\beta)}} \Psi + \mathcal{L}_Y \Psi \\ \Psi(t_o, \mathbf{x}) & = \phi(\mathbf{x}) \quad \text{for any } \mathbf{x} \in \boldsymbol{N} \end{array} \tag{21}$$

• The number of generators $\{X_{(\alpha)}\}$ need not be equal to the dimension D of the manifold $\boldsymbol{N}^D$. Eq. (6) reads then

$$dx^\alpha(t,z) = X^\alpha_{(i)}(x(t,z))\, dz^i + Y^\alpha(x(t,z))\, dt \tag{22}$$

5) Volume forms

Integral characterizations

Equation (11) gives an integral characterization of the gaussian volume form; namely

$$\int_{\mathcal{P}_o \boldsymbol{R}^D} \mathcal{D}_{s,Q_o} z \exp(-\frac{\pi}{s} Q_o(z) - 2\pi i < z', z >:= \exp(-\pi s W_o(z') \tag{23}$$

A volume form on spaces of Poisson paths can be defined by a similar equation. Indeed, a Poisson path $x \in \mathbb{X}_n$ is characterized by n jump-times $\{t_1, \ldots, t_n\}$ and is interpreted as the sum $\delta_{t_1} + \delta_{t_2} + \ldots + \delta_{t_n}$. The space $\mathbb{X}$ is the union of all $\mathbb{X}_n$. Let a be a not necessarily real constant. Let $dv(t)$ be the dimensionless volume element on the time interval $\mathbb{T} = [t_a, t_b]$, with $T = t_b - t_a$.

$$dv(t) = a\, dt \quad , \quad vol(\mathbb{T}) = aT \tag{24}$$

$$vol(X_n) = a^n T^n / n! \tag{25}$$

$$vol(\mathbb{X}) = \exp(vol\mathbb{T}) \tag{26}$$

We can even say

$$\mathbb{X} = \exp \mathbb{T}$$

because addition of time intervals gives products of the corresponding spaces $\mathbb{X}_n$.

We can define the Fourier transform of a measure $\mathcal{D}_{a,T} x$ on $\mathbb{X}$ by

$$\int_{\mathbb{X}} \mathcal{D}_{a,T} x \cdot \exp(i < x, f >) = \exp \int_{\mathbb{T}} dv(t) \exp i f(t) \tag{27}$$

$$\text{Hint:} \qquad < x, f >= f(t_1) + \ldots + f(t_n) \tag{28}$$

If a is a real constant, the Poisson path can be described by a sequence of waiting times T_k between jumps:

$$Pr(t_k \leq T_k \leq t_k + dt) = a \exp(-at_k)dt \tag{29}$$

For the case where the decay rate a varies with time see [See Kit Foong].

In general a volume form $\mathcal{D}_{\Theta,Z}$ on a Banach space $\boldsymbol{X}$ can be defined implicitly by an integral characterization,

$$\int_{\boldsymbol{X}} \mathcal{D}_{\Theta,Z}\, x \cdot \Theta(x, x') = Z(x') \tag{30}$$

A key feature of the integral representation of volume forms on function spaces is the use of a translation invariant symbol, $\mathcal{D}_{s,Q_o}z$, $\mathcal{D}_{a,T}z$, $\mathcal{D}_{\Theta,Z}$, as the case may be.

Differential characterizations

As in the construction of integral characterizations, we begin with a finite dimensional construction which suggests an infinite dimensional generalization.

Let $\mathcal{L}_X$ be the Lie derivative with respect to a vector field X on $\boldsymbol{N}^D$, either a (pseudo-) riemannian manifold $(\boldsymbol{N}^D, g)$ or a symplectic manifold $(\boldsymbol{N}^{2N}, \Omega)$, the volume form ω (ω_g or ω_Ω) satisfies the equation

$$\mathcal{L}_X \omega = D(X) \cdot \omega \tag{31}$$

where the volume form ω is a top form (a D-dimensional form on $\boldsymbol{N}^D$) and $D(X)$ is a function on $\boldsymbol{N}^D$ depending on the vector field X on $\boldsymbol{N}^D$.
Indeed,

$$\mathcal{L}_X \omega_g = Div_g(X) \cdot \omega_g, \quad Div_g X := X^\alpha_{;\alpha} \tag{32}$$

and

$$\mathcal{L}_X \omega_\Omega = Div_\Omega(X) \cdot \omega_\Omega, \quad Div_\Omega X := X^\alpha_{,\alpha} \tag{33}$$

If X on $(\boldsymbol{N}^D, g)$ is a Killing vector field with respect to isometries, then

$$\mathcal{L}_X \omega_g = 0 \tag{34}$$

If X on $(\boldsymbol{N}^{2N}, \Omega)$ is a hamiltonian vector field, then

$$\mathcal{L}_X \omega_\Omega = 0 \tag{35}$$

Eqs. (34,35) suggest that given the dynamical vector fields $\{X_\alpha\}$ on $\boldsymbol{N}^D$ introduced in section 1, a volume form on $\boldsymbol{N}^D$ could be defined by

$$\mathcal{L}_{X_\alpha}\omega = 0 \tag{36}$$

Forms and densities in ordinary and Grassmann Variables

In the previous equations, the volume form ω is a top form (a D-dimensional form on $\boldsymbol{N}^D$). Top forms do not exist on infinite dimensional spaces nor on Grassmann manifolds (graded manifolds, supermanifolds, etc.).

Are forms the only useful concept for defining volume forms? No, in the thirties, densities were extensively used. Densities fell in disfavor, possibly because in contrast to forms they do make an algebra. On the other hand, complexes (ascending and descending) can be constructed with densities as well as with forms.

By forms (exterior differential forms) one means totally antisymmetric covariant tensors. By densities (linear tensor densities) one means totally antisymmetric contravariant tensor-densities of weight one.

We recall properties of forms and densities on ordinary D-dimensional manifolds $\boldsymbol{M}^D$ which can be established in the absence of a metric tensor because they are readily useful in Grassmann calculus.

Ascending complex of forms on $\boldsymbol{M}^D$

Let $\mathcal{A}^p$ be the space of p-forms on $\boldsymbol{M}^D$ and d the exterior differentiation

$$d : \mathcal{A}^p \longrightarrow \mathcal{A}^{p+1} \tag{37}$$

Since $dd = 0$, the graded algebra $\mathcal{A}^\bullet$ is an ascending complex w.r.t. the operator d

$$\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^D \tag{38}$$

Descending complex of densities on $\boldsymbol{M}^D$

Let $\mathcal{D}_p$ be the space of p-densities on $\boldsymbol{M}^D$ and $\bigtriangledown$ the divergence operator, also labeled b

$$\bigtriangledown\cdot : \mathcal{D}_p \longrightarrow \mathcal{D}_{p-1}, \ \bigtriangledown\cdot \equiv b. \tag{39}$$

Since $bb = 0$, $\mathcal{D}_\bullet$ (which is not a graded algebra) is a descending complex w.r.t. the divergence operator

$$\mathcal{D}_0 \xleftarrow{b} \mathcal{D}_1 \xleftarrow{b} \cdots \xleftarrow{b} \mathcal{D}_n \tag{40}$$

Metric-dependant and dimension-dependant transformations.
$C_g : \mathcal{A}^p \longrightarrow \mathcal{D}_p$ is a metric-dependant map. One can map $\mathcal{D}_p$ into $\mathcal{A}^{n-p}$ on an orientable manifold by a dimension-dependant transformation (using the alternating symbol on $\boldsymbol{R}^n$). The star operator (Hodge-de Rham) combines a metric-dependent and a dimension-dependant operator. By isolating its metric-dependant component C_g, one can construct an ascendiong complex on $\mathcal{D}_\bullet$ corresponding to the descending complex on $\mathcal{A}^\bullet$ w.r.t. to the metric transpose

$$\delta : \mathcal{A}^{p+1} \longrightarrow \mathcal{A}^p \tag{41}$$

Indeed $\beta := C_g d C_g^{-1}$ is such that

$$\beta : \mathcal{D}_p \longrightarrow \mathcal{D}_{p+1} \tag{42}$$

We are shelving temporarily this paragraph since there are no metric tensor on Grassmann manifolds.

Grassmann case

As a rule of thumb, it is most often sufficient to insert the word "graded" in the corresponding ordinary situation. For example an ordinary form is an antisymmetric covariant tensor, a Grassmann form is a graded antisymmetric covariant tensor: $\omega \ldots \alpha\beta \ldots = -(-1)^{\tilde{\alpha}\tilde{\beta}} \omega \ldots \beta\alpha \ldots$ where $\tilde{\alpha} \in \{0, 1\}$ is the grading of α. Therefore a Grassmann form is symmetric in the interchange of two Grassmann indices.
Two properties of forms on real variables remain true for forms on Grassmann variables, namely

$$dd\omega = 0 \tag{43}$$

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{\tilde{\omega}\tilde{d}} \omega \wedge d\theta \tag{44}$$

where $\tilde{\omega}$ and $\tilde{d} = 1$ are the parities of ω and d, respectively; the parity of a real p-form is even for $p = 0 \; mod\, 2$, odd for $p = 1 \; mod\, 2$; the parity of a Grassmann p-form is always even.
A form on Grassmann variables is a symmetric tensor; therefore the ascending complex $\mathcal{A}^\bullet(\boldsymbol{R}^{0|\nu})$ of Grassmann forms w.r.t. d does not terminate at ν-forms.
Two properties of densities on real variables remain true for densities $\mathcal{F}$ on Grassmann variables, namely

$$(\nabla \cdot)(\nabla \cdot)\mathcal{F} = 0 \tag{45}$$

$$(\nabla\cdot)(X\mathcal{F}) = (\nabla\cdot X)\cdot\mathcal{F} + (-1)^{\tilde{X}\tilde{\nabla}} X\nabla\cdot\mathcal{F},\ X \quad \text{a vector field} \tag{46}$$

A density is a tensor of weight 1; multiplication by a tensor of weight zero is the only possible product which maps a density into a density.

A density on Grassmann variables is a symmetric contravariant tensor; therefore the descending complex $\mathcal{D}_\bullet(\boldsymbol{R}^{0|\nu})$ of Grassmann densities with respect to $\nabla\cdot$ does not terminate at ν-densities.

Representations of fermionic and bosonic creation and annihilation operators are easily constructed on the ascending complex of forms and the descending complex of densities.

- on $\boldsymbol{M}^D$ for the fermionic case
- on $\boldsymbol{R}^{0|\nu}$ for the bosonic case.

They provide representations of supersymmetric Fock spaces.

Section 5 on volume forms is clearly a report of current work still on the drawing board. The pleasure of a workshop is to discuss unfinished work with colleagues.

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