FEYNMAN PATH INTEGRAL FOR AN INVERSE PROBLEM*

Brian De Facio and Arjuna Flenner

Department of Physics & Astronomy
Missouri University
Columbia, MO 65211

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SOME "PATH INTEGRAL" FORMULATIONS

1. SEQUENTIAL: PALEY, WIENER, ZYGMUND 1933; CAMERON, STORVICK 1980–; JOHNSON, SKOUG 1973– ; KALLIANPUR, KANNAN, KARANDIKAR 1985, ...

2. FRESNEL INTEGRAL: ALBEVERIO, HØEGH-KROHN 1976 ...

3. TROTTER PRODUCT: NELSON 1964; Lapidus 1981, ...

4. ANALYTICAL: CAMERON, STORVICK 1973– 1980– ; JOHNSON, SKOUG 1984; KALLIANPUR JOHNSON KANNAN, KARANDIKAR 1985; CHANG et al 2001

5. UNITARY GROUP: LAPIDUS 1980; JOHNSON, LAPIDUS 2000, ...

6. IMAGINARY RESOLVENT: LAPIDUS 1985, JOHNSON, LAPIDUS 2000, ...

EVOLUTION SYSTEMS

$$x(t) = \left(e^{-iHt} x\right)(0)$$

$$x(t) = \left(e^{-\beta H} x\right)(0)$$

$$H = T + V = H(p, q) \underset{QM}{=} \hbar \tilde{H}$$

AUTONOMOUS

$$L = L(q, \dot{q}) = T - V$$

AUTONOMOUS

$$H(p, q, t) = \sum_i p_i \dot{q}_i(p_i) - L(q, \dot{q}(p_i), t) + \left(\frac{dM}{dt}\right)(q, t)$$

$H \Rightarrow \dot{\text{D}}$YNAMICS

$L, S \Rightarrow$ SYMMETRY, **DYNAMICS**

$$S := \int_a^b L(q, \dot{q}, t')\, dt' = \text{ACTION } (a \to b)$$

$$S := \int_0^t dt' \; L(q,\dot{q}) = Action$$

$$S(b,a) = \int_a^b dt' (T - V) = Action(b \leftarrow a)$$

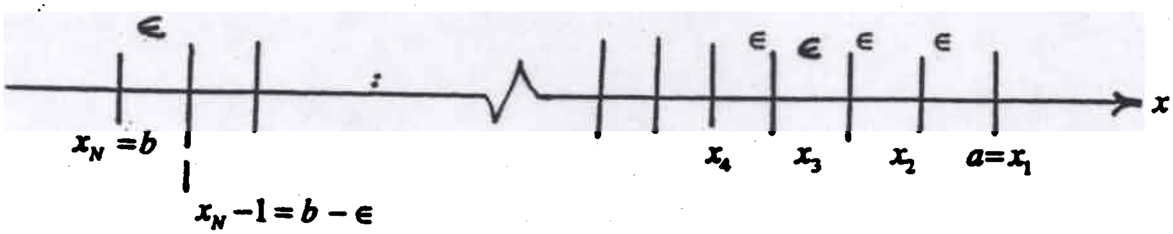
$$T = \tfrac{1}{2}\pi^2 = \tfrac{1}{2}\dot{\phi}^2$$

$$V = V(\phi), \in C[0,t]$$

$$K(b,a) = \int D\phi \, e^{iS(b,a)/\hbar}$$

$$\overset{=}{F} \lim_{\in \downarrow 0} \left\{ \frac{1}{A} \int \frac{dx_1}{A} \cdots \int \frac{dx_N}{A} e^{iS(b,a)/\hbar} \right\}$$

$$A = \frac{2\pi i \hbar \in}{m}$$



Note: $A \to 0, \in \downarrow 0$

$\nexists \dot{\phi} \quad a.e$

$D\phi \neq$ *countably additive*

$t > 0,\ t' \in [0, t]$

$C[0,t] = R^1$ **–valued, continous functions on** $[0,t]$

$C_0[0,t] =$ **Wiener Space**

$= \{x \mid x \in C\ [0,t], x(0) = 0\}$

m = Wiener measure on $C_0[0,t]$

Subset $A \subset C_0[0,t]$**, is scale-invariant, measurable**

if ρA = scale invar., meas.

$\forall \rho > 0.$

S = class of scale-invariant measurable sets is a σ **- algebra and**

$N \subset S$ **is called** **scale-invariant null** **if** $m(\rho N) = 0, \forall \rho > 0.$

A property which holds everywhere but on scale-invariant null sets is called **scale-invariant almost everywhere, s.-a.e**

CONTRACTION C_0 SEMI-GROUPS

$A \neq A(t)$, **explicitly**

$f = u(0) = IC$ **for evolution system**

$$u(t_1 + t_2) = T(t_1 + t_2) f(t_2)$$

uniqueness of u ⇒ semigroup property

$$T(t_1 + t_2) = T(t_1) T(t_2)$$

$$\forall t_1, t_2 > 0$$

$T(0) = 1 \Rightarrow t \to T(t)$ **differentiable on** $[0, \infty[$

$$\frac{d}{dt}(T(t) f) = A\, T(t) f$$

$$\Rightarrow u(t) = T(t) f$$

solves pde $\dfrac{du}{dt} = Au$

$t \geq 0, u(0) = f,\ f \in D_A$ **(dense)**

T is a linear operator if A is linear.

A C-semigroup is a strongly continous, I-parameter semigroup of bounded operators on a Banach space B**,** *s.t.*

1) $\|T(t)\|_B < \infty$

ie $\forall t>0, \sup_f \{\|T(t)f\|_B, f\in B, \|f\|\le 1\} < \infty.$

sup = g.l.b. , inf = l.u.b.

2) **T (t + s) f = T (t) T (s) f** $\quad \forall t,s\ge 0 \,\&\, f\in B$

3) **T (0) f = f ,** $\quad \forall f\in B$

4) **map:** $t \to T(t)f$ **is continous** $\forall t\ge 0 \;\&\; \forall f\in B$

A C- contraction semigroup is a C- semigroup for which

5) $\|T(t)f\| \le \|f\|, \forall t>0, \forall f\in B$

ie $\|T(t)\| \le 1, \forall t>0$

A C - semigroup has infinitesimal T (t) s.t.

$$Af = \lim_{t\to 0}\left(\frac{T(t)f-f}{t}\right)$$

$f\in D_A(dense).$

Hypothesis

(H 1.) $-\alpha$ is inf. gen. of a C_0 - contraction semigroup of operators on a separable Hilbert space.

(H 2.) a. $\forall\, p=1,2,\cdots,n, \mu_p$ on the Borel class of $[0,+\infty), \tilde{B}[0,+\infty)$, Can be signed or complex.

b. The associated positive total variation measures $|\mu_p|$ satisfy $|\mu_p|([0,T]) < \infty, \forall T \geq 0, \forall p$.

μ_p's are continous measures, but can be singular wrt Lebesque measure.

(H 3.) a. $\forall\, p, \beta_p : [0,+\infty) \to \mathcal{L}(\mathcal{H})$ satisfies the property that $[\beta_p(E)]^{-1}$ is in the Borel class of $[0,+\infty), \forall$ strong operator open subset $E \in \mathcal{L}(\mathcal{H})$.

b. $\int_0^T \|\beta_p(s)\| \mu_p(ds) < \infty, \forall T \geq 0$

c. $\forall$ fixed $p, \{\beta_p(s) : 0 < s < \infty\}$ is a commuting family in $\mathcal{L}(\mathcal{H})$.

J. B. Keller, Amer. Math. Mon. Direct and inverse problems are pairs of problems which are dual to one another.

necessary:

Direct Scattering		Inverse Scattering
Given (G^+, V) find F in $F = \int G^+ V d^3x'$	$\overset{D}{\rightarrow}$ / $\underset{I}{\leftarrow}$	Given (G^-, F) find V: $V = \int G^- F d^3x'$. Null space of $(G^- F)$ reqd

Eg $V \in L^1 \cap L^2$ allows Direct scattering: $F = F(k, \hat{k})$ analytic in unit disk in $k \in \mathbb{C}^1$

L. Faddeev, R. Newton, R. Beals, R. Coifman, A. Uhlman, ...

$F(k, \hat{k})$ scattered pattern (amplitude)

$V(\vec{x})$ difference in scatterer and host medium (potential).

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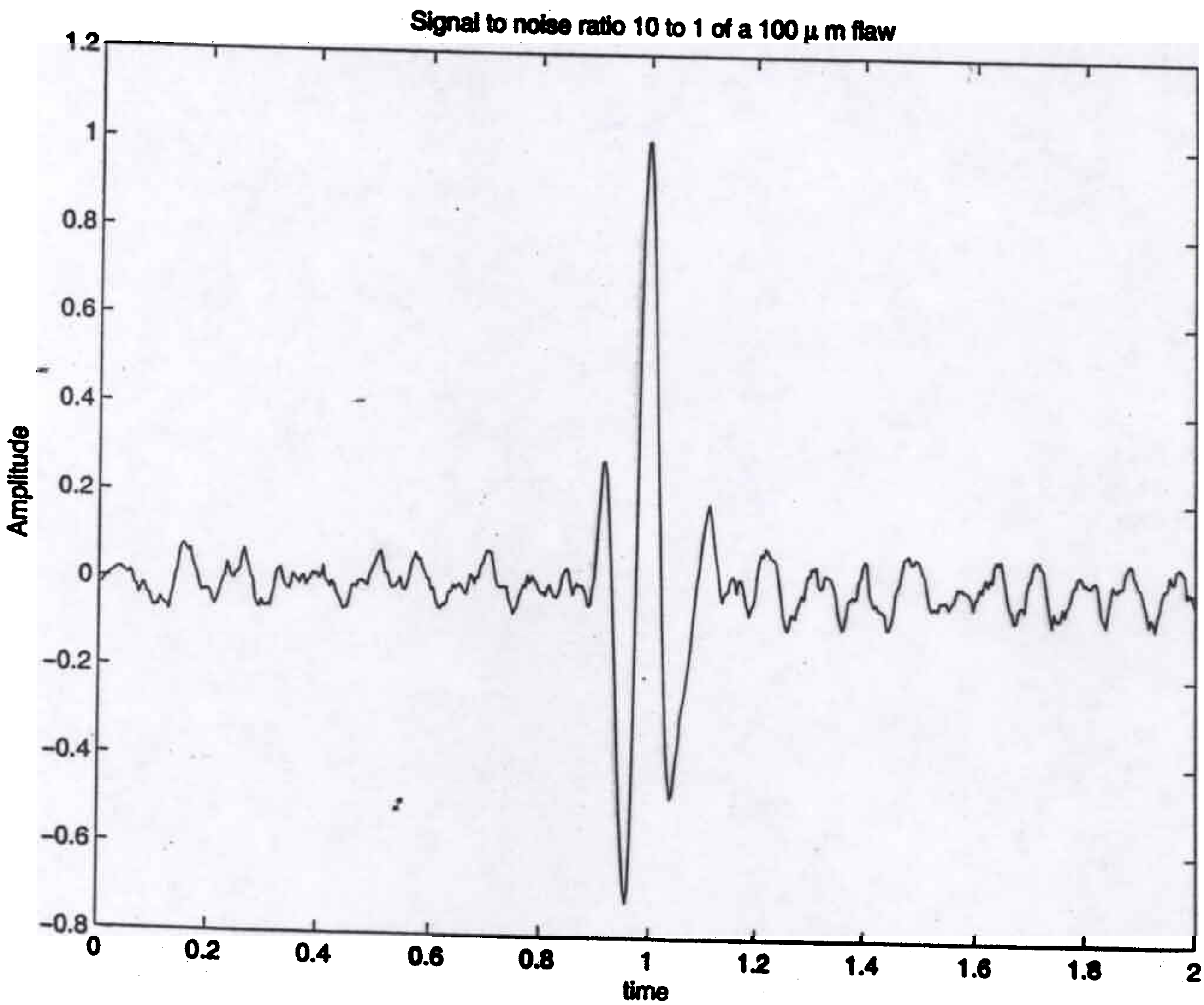
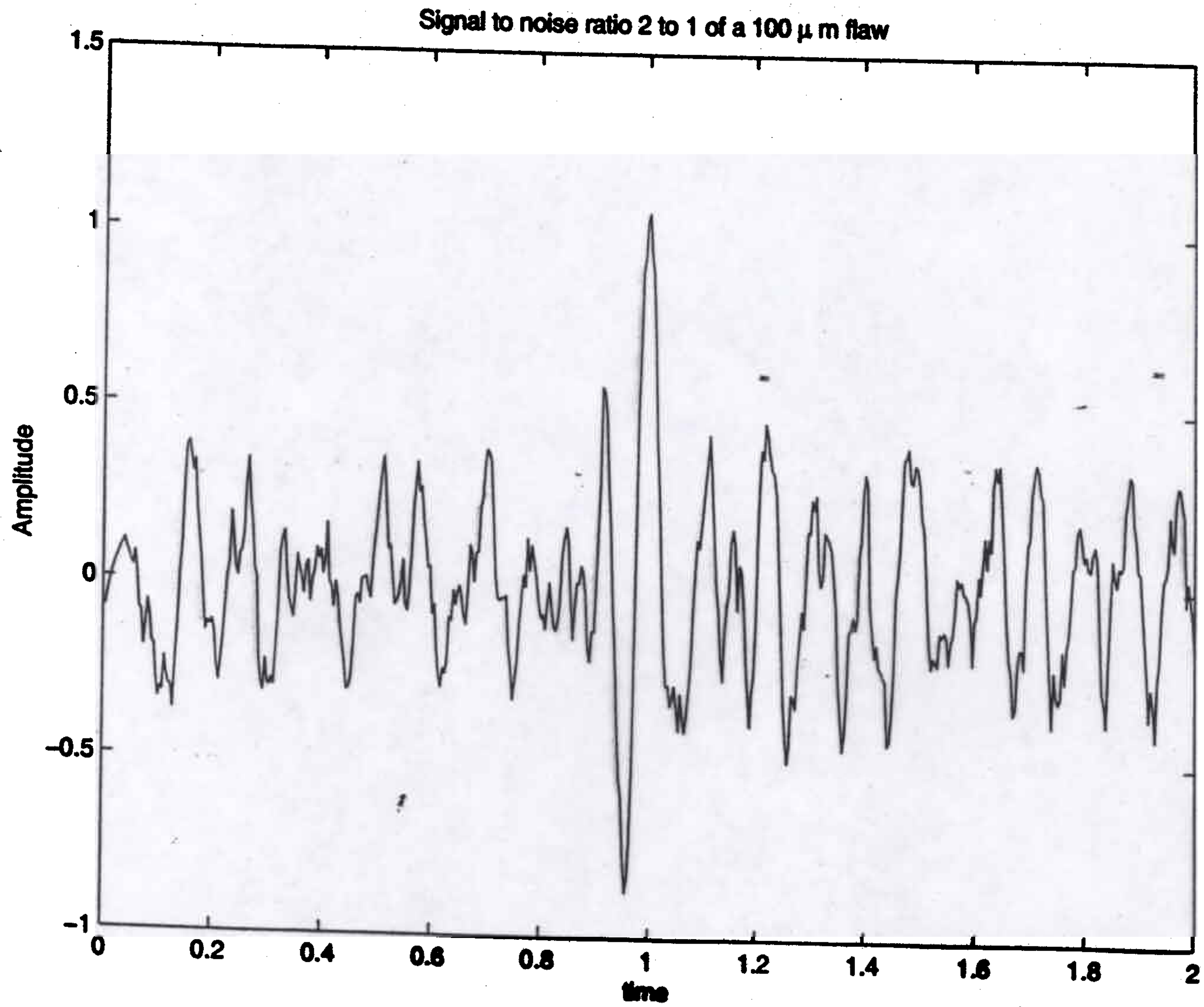
also Geophysics, Astronomy, NDE, MRI (NMR), Ultrasound, Spectros. at, mol, nucl

But noise is involved in measured values, and data is discrete and finite.


Signal to noise ratio 10 to 1 of a 100 µ m flaw
Amplitude
time
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



Signal to noise ratio 2 to 1 of a 100 μ m flaw
Amplitude
time
1.5
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

$$F = \int G^+ V d^3x' + N$$

$$V = \int G^- F d^3x' + N$$

meas. model not Fred. 2nd kind NOT

$$\psi^+ = \phi^+ + G^+ V \psi^+$$

V = wanted signal, N = unwanted signal ≡ noise

$$F(\omega_i) = \int_\Lambda \tilde{G}^+(\omega_i, \lambda) \Gamma(d\lambda)$$

$\tilde{G}^+(\omega_i, \lambda)$ = known kernel

$\Gamma(d\lambda)$ = unknown measure

$\tilde{G}^+ : \Gamma \longrightarrow F$ smoothing

$(\tilde{G}^+)^{-1} : F \longrightarrow \Gamma$ ill-posed, $\dim(\mathcal{N}) \geq 1$

$$F_i = \sum_{j=g} \tilde{g}^+_{ij} \Gamma_j ,$$

discrete direct problem

g = # i's

$$\Gamma = (\tilde{G}^{+\prime} \tilde{G}^-)^{-1} \tilde{G}^{-\prime} F ,$$

discrete inverse problem, SVD

$(\tilde{G}^{+\prime} \tilde{G}^-)$ = ill-conditioned ≠ ill-posed

ok up to 1000 × 1000, and only this large.

$g = 10^6 \times 10^6,\ 10^{10} \times 10^{10}, \ldots$

IF $\mathcal{G}$ IS SMALL ENOUGH $\nexists$ SOLUTION, BUT $\mathcal{G}$ LARGE ENOUGH $\Rightarrow \exists$ MANY SOLUTIONS.
The Tikhonov regularization is a <u>roughness penalty</u> which greatly reduces the number of solutions.

$$\mathcal{M}^+(\Lambda) := \text{Space + measures on } \Lambda$$

$$\Gamma \in \mathcal{M}^+(\Lambda), \quad \text{random measure}$$

$$G(\omega_i) = G_i = g_i \in \mathcal{G}$$
$$i = 1, 2, \ldots n$$

$$\pi(d\Gamma) \text{ on } \mathcal{M}^+(\Lambda) = \text{P.D.F. of } d\Gamma$$

now a measurement model for G_i to get a <u>likelihood function</u> $L(\Gamma)$. If $A, B \in \Lambda$ and $\Gamma(A) \cap \Gamma(B) = \phi$ then A, B are stochastically independent. Let $\Phi : \Lambda \to \mathbb{R}^1$ be measurable and let

$$\Gamma(\Phi) := \int_\Lambda \Phi(\lambda)\, \Gamma(d\lambda)$$

be a stochastic integral of the form

$$\log\left\{E\left[e^{i\Gamma(\Phi)}\right]\right\} = \iint_{\mathbb{R}^1_+ \times \Lambda} \left[e^{i\mu\Phi(\lambda)} - 1\right] \nu(d\mu\, d\lambda)$$

is a non-stationary generalization of the Lévy-Khinchine representation for some + measure $\nu(d\mu d\lambda)$ on $\mathbb{R}^1_+ \times \Lambda$ which satisfies the integrability condition

$$\iint_{\mathbb{R}^1_+ \times K} (1 \wedge \mu)\, \nu(d\mu\, d\lambda) < \infty$$

for compact $K \subset \Lambda$. The inverse Lévy measure, ILM, gives an algorithm for the construction of these random fields from the representation

$$\Gamma(\Phi) = \iint_{\mathbb{R}^1_+ \times \Lambda} \mu\,\Phi(\lambda)\, H(d\mu\, d\lambda) = \sum_{j \in \mathcal{J}} \mu_j\, \Phi(\lambda_j)$$

of $\Gamma(d\lambda)$ in terms of a Poisson measure $H(d\mu d\lambda)$ on $\mathbb{R}^1_+ \times \Lambda$ with mean Lévy measure

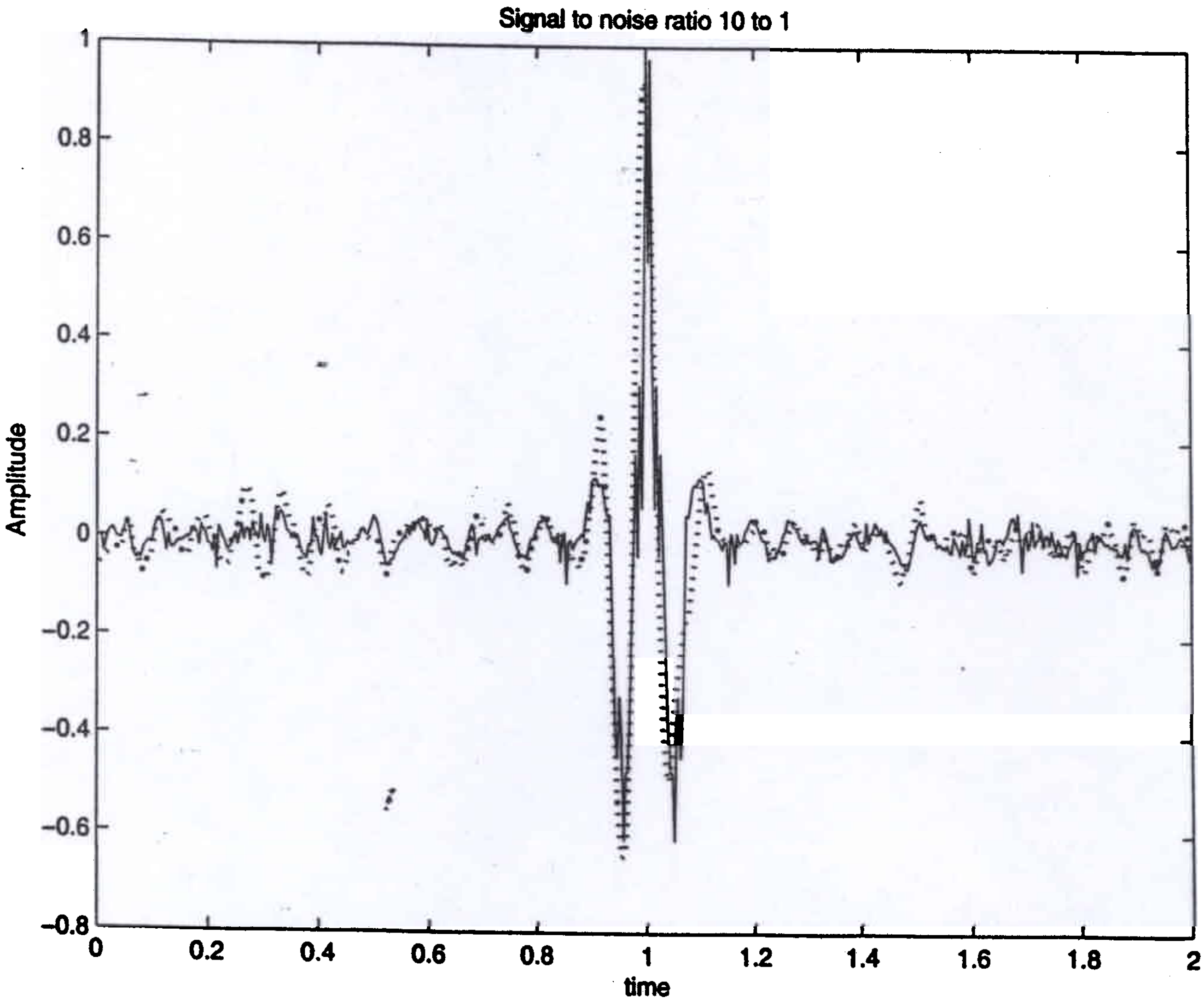
$$E\left[H(d\mu\, d\lambda)\right] = \nu(d\mu\, d\lambda).$$

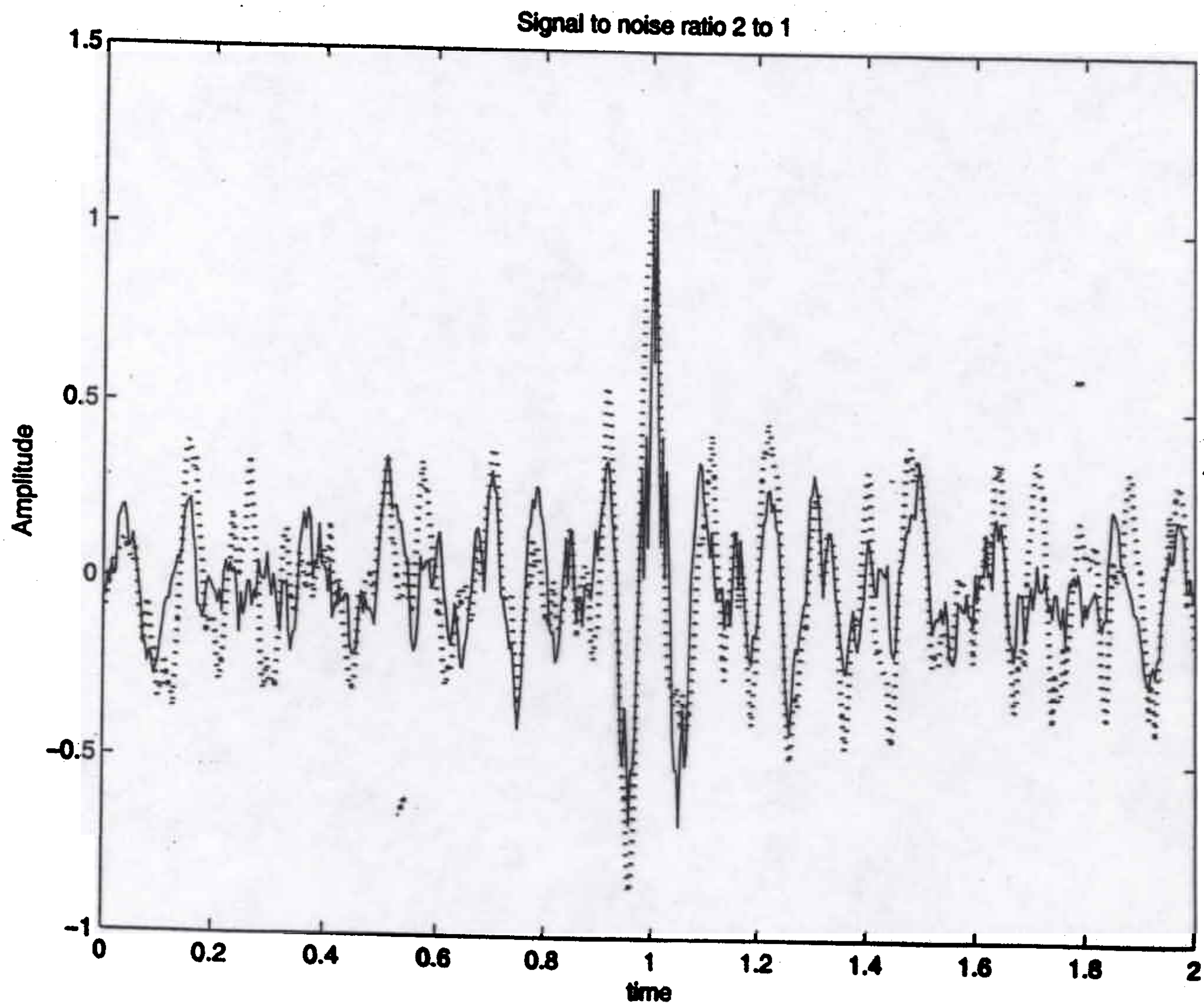
Here $\{\mu_j, \lambda_j\}_{j \in \mathcal{J}}$ represents a support of $H(d\mu d\lambda)$ which is at most countable. NOT ENOUGH

If $\nu(\mathbb{R}^1_+ \times \Lambda) < \infty$ then $H(d\mu d\lambda)$, and hence $\Gamma(d\lambda)$, has only finitely many points of support. Thus, the Levy P.D.F. acts as a "roughness penalty" of usual P.D.E. Regularization methods in a stochastic framework.

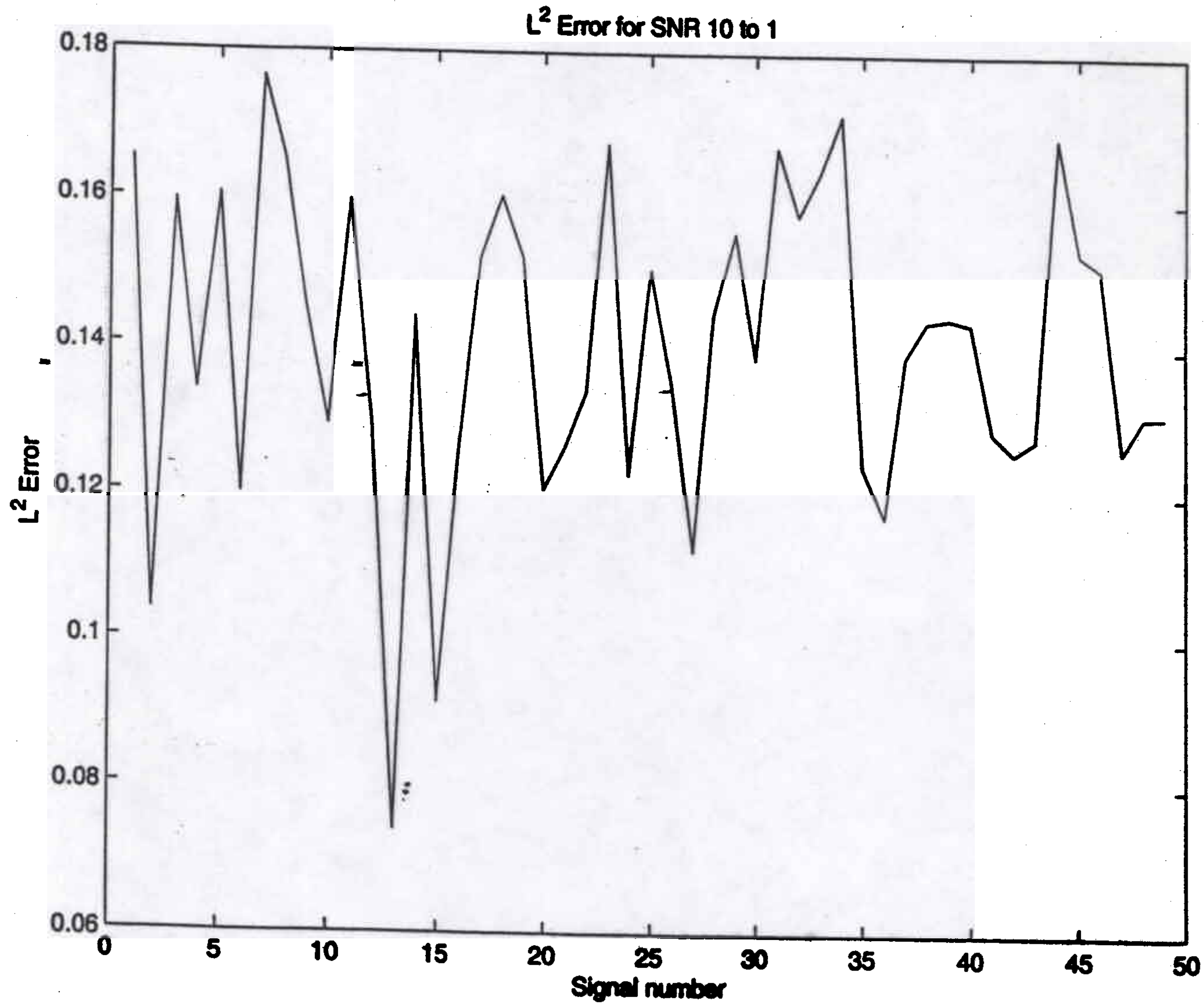
Now, Johnson & Lapidus added point measures to path integral measures where they act as "noise" impulses. Here we have found a very different finitely, discrete SAMPLING of the functions in the inverse Reconstruction process. We have used wavelet basis or frames and maximized the Karhunen-Loéve mutual (entropy): Information.

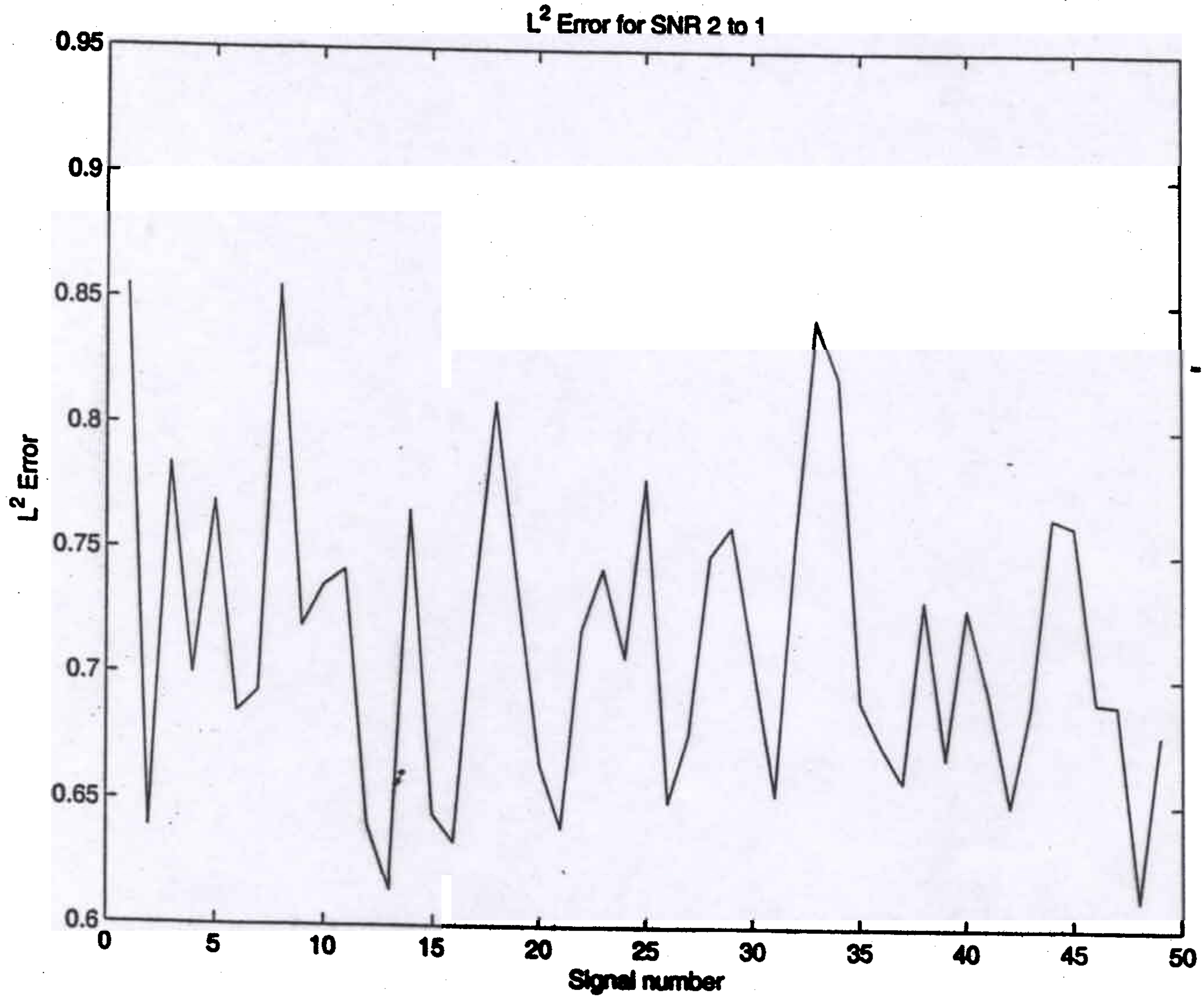
Next, we look at some of these calculations. Several sets of 49 reconstructions have been run and typical examples will be shown.

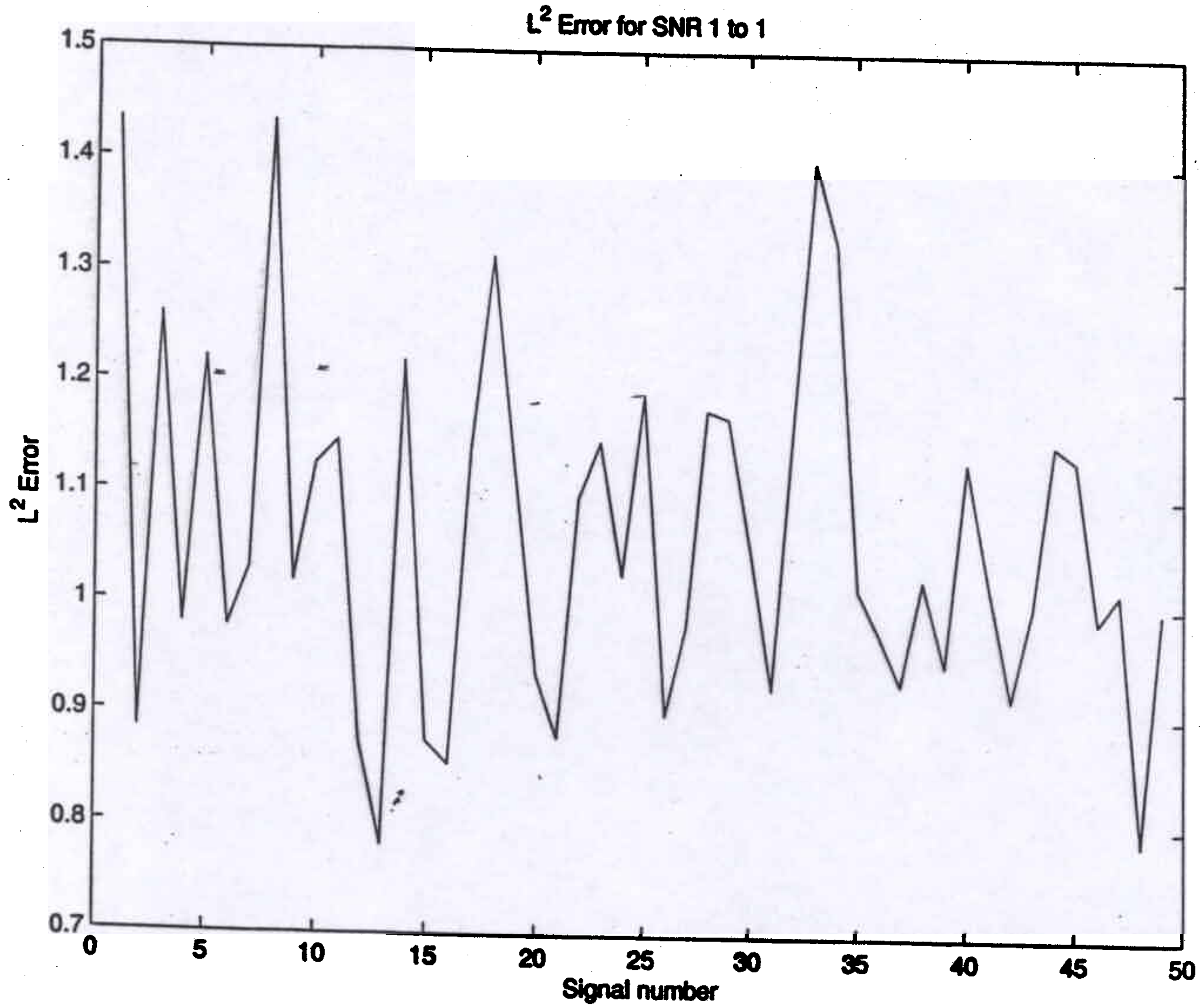

Signal to noise ratio 10 to 1
Amplitude
time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


Signal to noise ratio 2 to 1
Amplitude
time
1.5
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

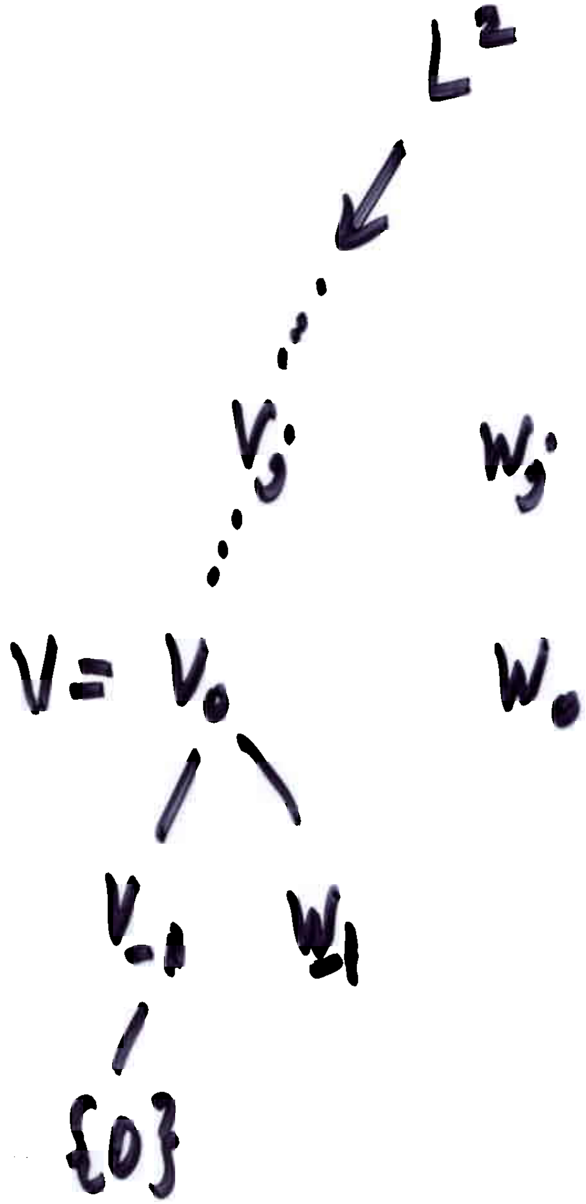
17


L2 Error for SNR 10 to 1
0.18
0.16
0.14
0.12
0.1
0.08
0.06
L2 Error
0
5
10
15
20
25
30
35
40
45
50
Signal number

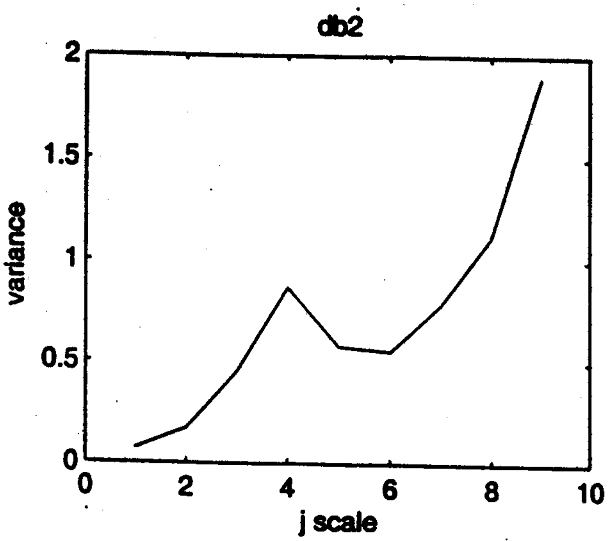

L² Error for SNR 2 to 1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
L² Error
0
5
10
15
20
25
30
35
40
45
50
Signal number

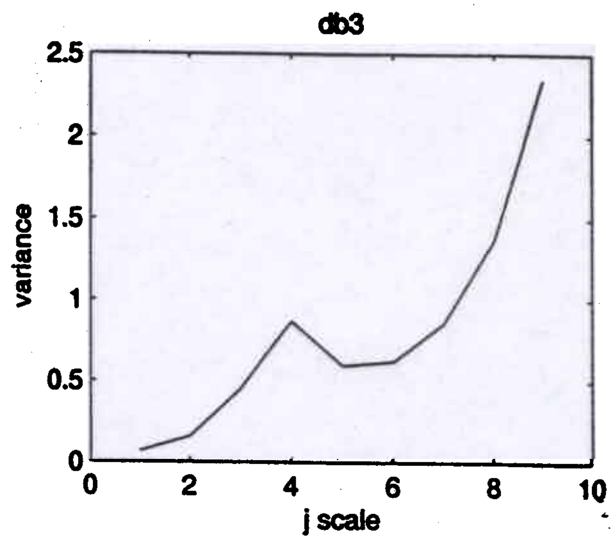

L² Error for SNR 1 to 1
L² Error
Signal number
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
5
10
15
20
25
30
35
40
45
50

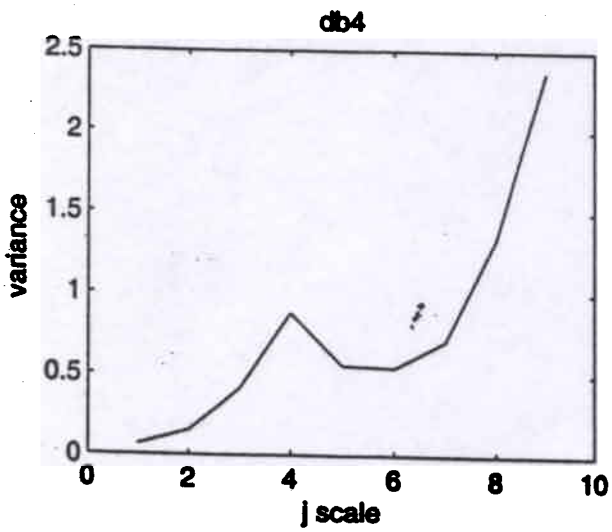
20W

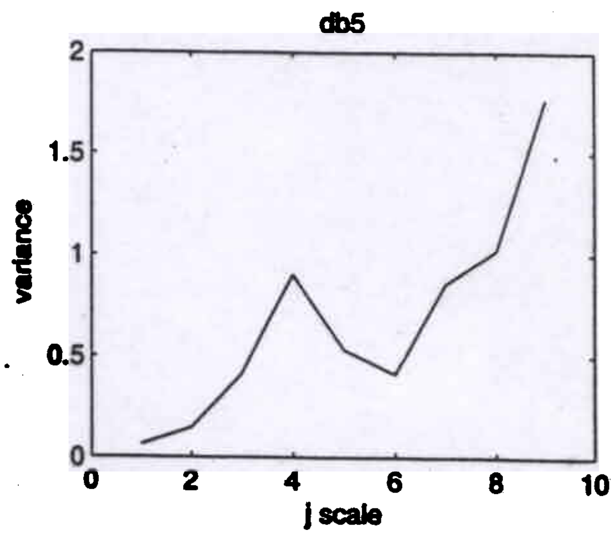


$$V_j \subset V_{j+1}$$

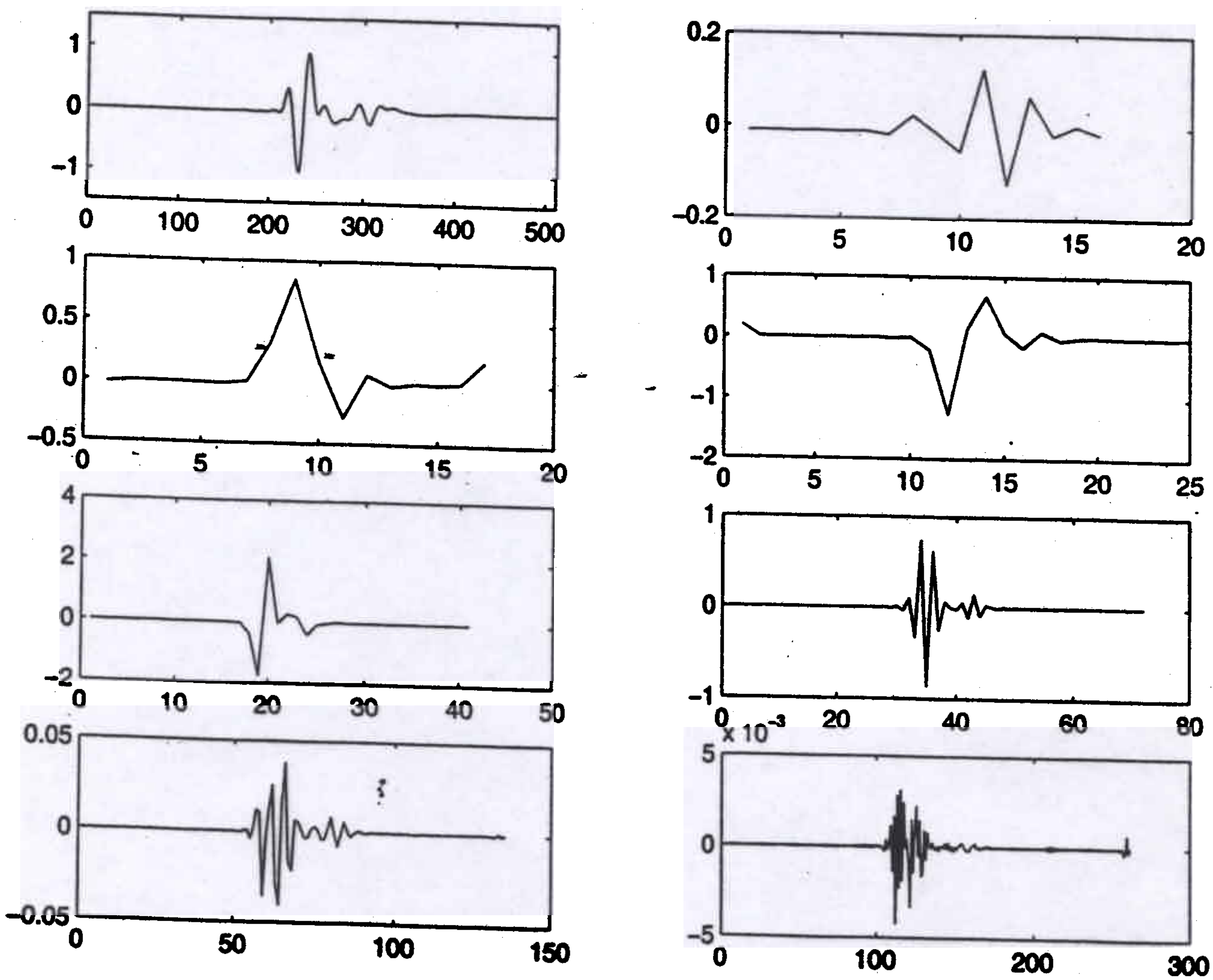

db2
variance
j scale
0
0.5
1
1.5
2
0
2
4
6
8
10


db3
variance
j scale
0
0.5
1
1.5
2
2.5
0
2
4
6
8
10


db4
variance
j scale
0
0.5
1
1.5
2
2.5
0
2
4
6
8
10

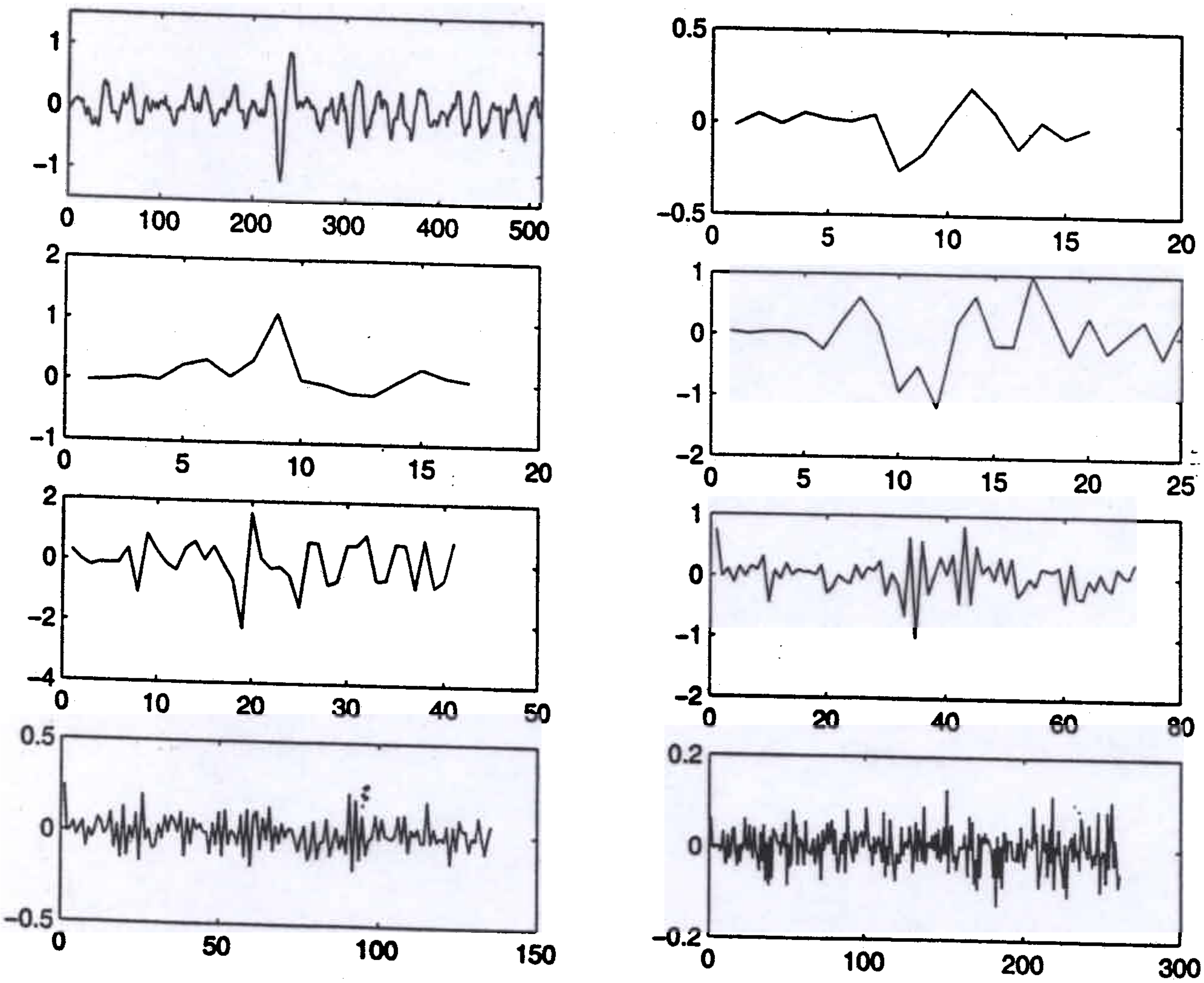

db5
variance
j scale
0
0.5
1
1.5
2
0
2
4
6
8
10

10:1

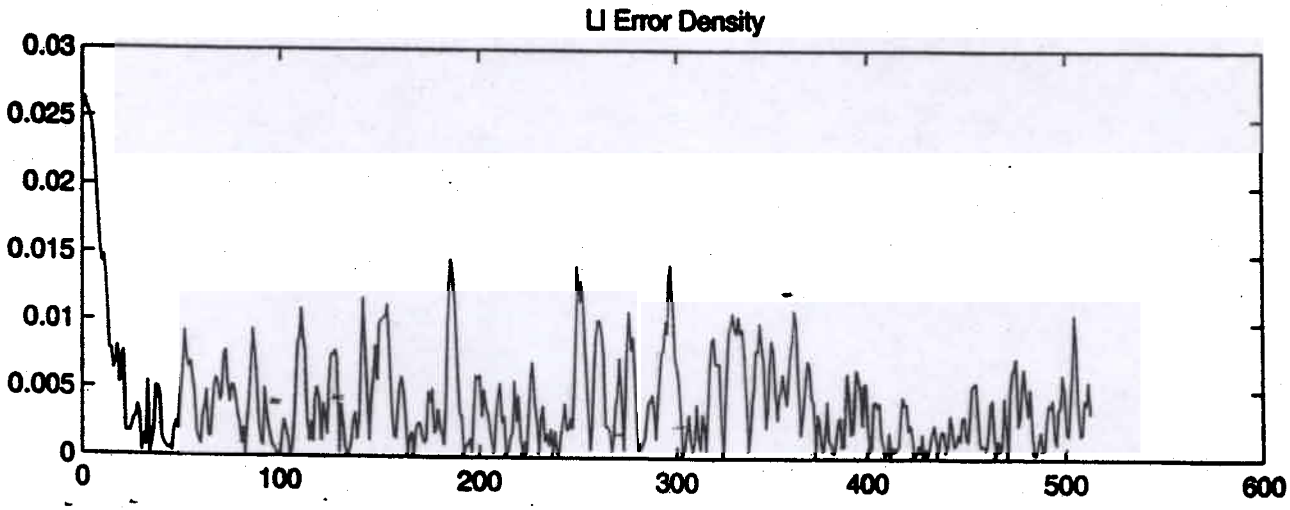

1
0
−1
0 100 200 300 400 500
0.2
0
−0.2
0 5 10 15 20
1
0.5
0
−0.5
0 5 10 15 20
1
0
−1
−2
0 5 10 15 20 25
4
2
0
−2
0 10 20 30 40 50
1
0
−1
0 20 40 60 80
0.05
0
−0.05
0 50 100 150
x 10^-3
5
0
−5
0 100 200 300

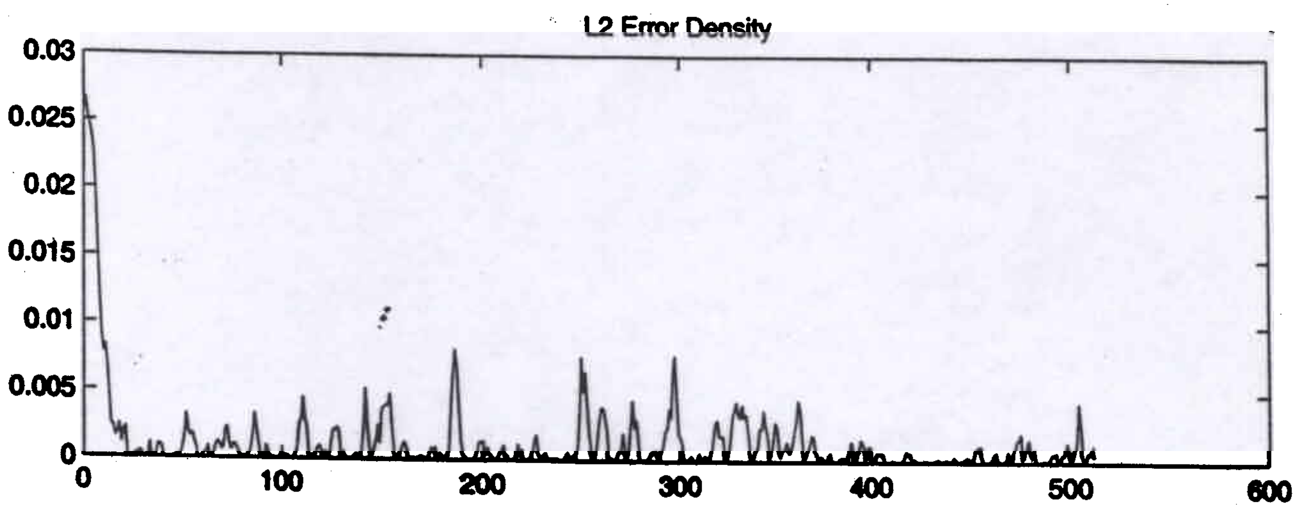
$n = 0$ $n \neq 0$

2:1



$n \equiv 0$ $\qquad$ $n \neq 0$


Ll Error Density
0.03
0.025
0.02
0.015
0.01
0.005
0
0
100
200
300
400
500
600


L2 Error Density
0.03
0.025
0.02
0.015
0.01
0.005
0
0
100
200
300
400
500
600

CONCLUSIONS

A promising formulation of an ultrasound (+ acoustic) inverse problem based on Feynman functionals. Lack of time kept us for showing the maximization of Karhunen – Loève information to choose the reconstruct.

∴ it is non-parametric, used wavelets – cont. for Cab's large in wt phase plane and edge sets. The Mallat – Daubechies gave means + variances + kurtosis.

Computation time was decr. by 1000 – 1500 without of loss in the reconstructions.

The unknown meas $\Gamma(d\lambda)$ is not a Lévy process altho the point process $H(d\mu\, d\lambda)$ in the rep.

$$\Gamma(d\lambda) = \int_{R_+^1} \mu \, H(d\mu\, d\lambda)$$

assigns indep. random. var. to $H(d\mu_k\, d\lambda_k)$ to disjont sets $A_k \in \mathbb{R}_+^1 \times \Lambda$ BECAUSE the Liklihood function $L[\Gamma(d\lambda)]$ induces dependence among differ. $\{H(d\mu_k\, d\lambda_k)\}$'s !

Other good features of this approach:

1) <u>Flexible</u> diff. Lévy meas $\nu(d\mu\, d\lambda)$ can give smooth or rough respons. as grain size in high tech. metals.

2) <u>Tractibility</u> $\forall$ Lévy meas. & $\forall$ meas. model computations are straightforward.

3) <u>Occam's razor</u> (stat. parsimony) There almost-surely will be some $\nu(d\mu\, d\lambda)$ s.t. M = # sample points with $E[M] = 2, 3$ per space variable $\Rightarrow$ huge compress. These behaviors arise from Karh. Loève infor. max. & corresp. to <u>sampling</u> high info. configs.

These corresp Johnson–Lapidus $q_i \, \delta(x - x_i)$ in FT space in AMS Memoir monograph.

In addition, we are comparing Kullback–Liebler mutual infor. functionals to the Karhunen–Loève case discussed here.

Thank you.