PATH INTEGRALS, MOMENTUM REPRESENTATION AND STOCHASTIC ANALYSIS

J.C. Zambrini, GFM, Lisbon
(joint with N. Privault, La Rochelle)

MSRI, Berkeley, December 9, 2002

Program of EQM: Well defined, time-symmetric probability measures on path spaces, with quantum-like properties.

Technical tool: Stochastic analysis and probability theory.

Conceptual guidelines: Feynman's path integral approach.

Aim: Transfer concepts from stochastic analysis (much richer mathematical structure) to quantum physics and back.

Key element of the construction: (on $L^2(\mathbb{R}^3)$)

Feynman's "transition element" (or "amplitude") on $I = [s, u]$

$$
\begin{aligned}
< \varphi | \mathbb{I} \psi >_{S_L} &= \int \psi_s(x) K(x, u-s, z) \overline{\varphi}_u(z) dx dz \\
&= \int \underset{\Omega_x^z}{\int} \int \psi_s(x) \, e^{\frac{i}{\hbar} S_L[\omega(\cdot); u-s]} \, \overline{\varphi}_u(z) dx \mathcal{D}\omega dz
\end{aligned}
$$

where $K(x, u-s, z) = (e^{\frac{i}{\hbar}(u-s)H})(x, z)$, H = Hamiltonian observable

$$
\begin{aligned}
S_L &= \text{classical action along quantum paths} \in \Omega_x^z \\
\Omega_x^z &= \{\omega \in C(I; \mathbb{R}^3) \; s.t \; \omega(s) = x, \; \omega(u) = z\} \\
\mathcal{D}\omega &= \prod_{s \leqslant \tau \leqslant u} d\omega(\tau) \\
\langle ... \rangle_{S_L} &= \text{expectation of a "process"} \\
\mathbb{I} &= \text{identity operator}
\end{aligned}
$$

Feynman stresses: all his results hold for any states φ, ψ at ∂I ($\Rightarrow$ short notation $\langle . \rangle_{S_L}$)

$\longrightarrow$ "Pseudo probabilistic" reinterpretation of transition amplitude ($\in \mathbb{C}$, in general !) and elements, <u>beyond</u> Born's interpretation of single states.

Unusual feature of Feynman's approach ($w.r.t$ probability theory): Boundary conditions at ∂I : (ψ_s, φ_u).

"Probabilistic" interpretation of transition amplitude plausible if φ, ψ not unrelated but $\varphi_u = \exp\left(-\frac{i}{\hbar}(u-s)\psi_s\right)$.

Then integral kernel propagating the initial (Born) probability density $|\varphi_s(x)|^2 dx$:

$p_F(s, x, u, dz) = (\overline{\varphi}_s(x))^{-1} K(x, u-s, z)\overline{\varphi}_u(z)dz \quad \forall x \ s.t \ \varphi_s(x) \neq 0$

or, in a dual (or time-symmetric) way, the one propagation backward final

$|\psi_u(z)|^2 dz : p_B(s, dx, u, z) = \psi_s(x) K(x, u-s, z)(\psi_u(z))^{-1} dx$

$\nexists$ such stochastic (time-reversible) process $t \mapsto \omega(t) = z(t, \omega)$ (cf. Cameron 1960). But its profile is clear:

Example If $H = -\frac{\hbar^2}{2}\Delta + V$ in $L^2(\mathbb{R}^3)$ then $\omega(.) \longrightarrow$ diffusion with drift $i\hbar\frac{\nabla\overline{\psi}_t}{\overline{\psi}_t}$ (or $-i\hbar\frac{\nabla\psi_t}{\psi_t}$) and diffusion coefficient $i\hbar\mathbb{I}$

Roughly: Strategy of EQM $\longrightarrow$ look for Euclidean counterpart ("Wick rotation" $t \mapsto it$) of this whole dual framework; involving on I:

$$\begin{cases} -\hbar\frac{\partial\eta^*}{\partial t} = H\eta^* \\ \eta^*(q,s) = \eta^*_s(q) \end{cases} \quad \underline{\text{and}} \text{ its adjoint} \quad \begin{cases} +\hbar\frac{\partial\eta}{\partial t} = H\eta \\ \eta(q,u) = \eta_u(q) \end{cases}$$

Resulting measure P on Ω $s.t$ $P(X(t) \in A) = \int_A (\eta^*\eta)(q,t)dq$

If $A = \mathbb{R}^3 \longrightarrow$ Probability conservation.

Two illustrations of "quantum-like" properties:

1) Interference of probability

$(\eta_t, \eta^*_t) = 2$ positive solutions of adjoint PDE

$\longrightarrow$ Process $X(t)$, $t \in I$. $\eta_t =$ time reversed of η^*_t.

$$\eta^*_t(q) \equiv p^{1/2}(q,t)\exp(-S(q,t)) \text{ for } \begin{cases} p(q,t) = \eta_t\eta^*_t \\ S(q,t) = \frac{1}{2}ln(\frac{\eta_t}{\eta^*_t}) \end{cases}$$

= Euclidean decomposition into real and imaginary parts of (real time) $\psi_t \in L^2$.

Linearity of PDE $\longrightarrow$ If η_t^* is solution, $\eta_t^*(q-l)+\eta_t^*(q+l)$ also, for $l = cste$. New $X(t)$ $s.t$

$$p(q,t)dq = \{p(q-l,t)+p(q+l,t)+2p^{1/2}(q-l,t).p^{1/2}(q+l,t)\cdot \cos h[S(q-l,t)-S(q+l,t)]\}dq$$

$\downarrow$

Euclidean Interference of Probability !

2) Euclidean quantum symmetries

If $(\eta_t, \eta_t^*) \longrightarrow X(t), t \in I$, look for $(\eta_t^\alpha, \eta_t^{*\alpha}) \longrightarrow X^\alpha(t)$ $\forall \alpha \in \mathbb{R}$

Euclidean "unitarity" on $\mathcal{H} = L^2(\mathbb{R})$:

$$1 = \int_{\mathbb{R}} \eta\eta^* dq = \int_{\mathbb{R}} \eta\eta^* \frac{\eta_\alpha}{\eta} \frac{\eta_\alpha^*}{\eta^*} dq \equiv E\left[h^\alpha h_*^\alpha(X(t),t)\right]$$

In probability: $X(t) \mapsto X^\alpha(t)$: "Doob's h-transform".

Ex: $H = H_0 = -\frac{\hbar^2}{2}\Delta$

$$\eta^\alpha(q,t) = e^{\frac{1}{\hbar}(\alpha q - \frac{\alpha^2}{2}t)}\eta(q-\alpha t, t) \quad , \quad \forall \alpha \in \mathbb{R}$$

Simplest solution $\eta = 1$, Feynman's drift $= 0 \to X(t) =$ Brownian. Then $\eta^\alpha = e^{-\frac{\alpha}{\hbar}N}1$ for $N = t\frac{\partial}{\partial q} - q$.

$$h_\alpha(q,t) = \eta^\alpha(q,t) = e^{\frac{1}{\hbar}\left(\alpha q - \frac{\alpha^2}{2}t\right)} =$$

$$= 1 + \frac{\alpha}{\hbar}q + \frac{\alpha^2}{2\hbar^2}\left(q^2 - \hbar t\right) + \frac{\alpha^3}{3!\hbar^3}\left(q^3 - 3\hbar t q\right) + \cdots$$

Sucessive $\frac{\partial}{\partial \alpha}_{|\alpha=0}$ $\to$ martingales of the Brownian.

Quantum counterpart:

$Q(t)$, $Q^2(t) + i\hbar t$, $Q^3(t) + 3i\hbar t Q(t), \ldots$ (Heisenberg's sense)

are free constant observables ($: Q^n(t) :$!)

EQM's method in momentum picture

In Q (position) picture, spectral family $E^Q(\lambda) = \chi_{]-\infty,\lambda]}(q)$ and $P = -i\hbar\frac{\partial}{\partial q}$ $(\mathcal{H} = L^2(\mathbb{R}), (\cdot|\cdot))$

Fourier: $\psi \mapsto (U\psi) = \widehat{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-\frac{i}{\hbar}p.q}\psi(q)dq$

$\widehat{P}\widehat{\psi} = UPU^{-1}\widehat{\psi} = p\widehat{\psi}\,,\ \widehat{Q}\widehat{\psi} = UQU^{-1}\widehat{\psi} = i\hbar\frac{\partial}{\partial p}\widehat{\psi}$

p (momentum) picture physically equivalent:

$$(\widehat{\psi}_t|\widehat{A}\widehat{\psi}_t) = (\psi_t|A\psi_t) \qquad\qquad A = \text{observable.}$$

But $E^Q(\lambda) \mapsto U^{-1}E^Q(\lambda)U = E^P(\lambda) \rightarrow$

Momentum process P(t) ?

Ex: Position picture: $V(q) = \alpha q$

$i\hbar\frac{\partial\psi}{\partial t} = H\psi = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2}\psi + V\psi$

"Diffusion" $\omega(t)$ $s.t$

$\int_A |\psi_t|^2(q)dq = P(\omega(t) \in A)$

Momentum picture: $p = \hbar k$

$i\hbar\frac{\partial\widehat{\psi}}{\partial t} = (\frac{p^2}{2} + \alpha i\hbar\frac{\partial}{\partial p})\widehat{\psi} = \widehat{H}\widehat{\psi}$

"Process" (?) $p(t) = \hbar k(t)$ $s.t$

$\int_{\widetilde{A}} |\widehat{\psi}_t|^2(p)dp = \widetilde{P}(p(t) \in \widetilde{A})$

Nobody knows probabilistic construction valid $\forall$ picture !

$\rightarrow$ Implications in stochastic analysis / quantum physics & quantum field theory.

Back to $H = -\frac{\hbar^2}{2}\Delta + V(q)$ (Position picture) $\mathcal{H} = L^2(\mathbb{R}^3)$

Hyp: $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ admits Lévy-Khintchine

$$V(q) = ic\cdot q + \frac{1}{2}|q|^2 - \int_{\mathbb{R}} \left(exp(-iq\cdot k) - 1 + iq\cdot k\cdot \mathbb{I}_{\{|k|<1\}} \right) \nu(dk)$$

for ν a Lévy measure on $\mathbb{R}$ s.t. $\int_{\mathbb{R}^3} (|k|^2 \wedge 1)\nu(dk) < \infty$

let $\xi_t = \mathbb{R}^3$ - valued Lévy process with characteristic exponent V:

$$E\left[e^{-\frac{i}{\hbar}\xi_t\cdot q}\right] = e^{-\frac{t}{\hbar}V(q)} \quad , \quad q \in \mathbb{R}^3, \;\; t \in \mathbb{R}$$

ξ_t representable in terms of Wiener process and Poisson process (Lévy-Itô).

Let $f : \mathbb{R}^3 \times \mathbb{R}^3 : (p,q) \rightarrow f(p,q) \in \mathbb{R}$ classical observable (polyn. bounded in q).

Pseudo-differential operator of symbol $f(p,q) : \forall \phi \in \mathcal{S}(\mathbb{R}^3)$

$$f(p, \hbar\nabla)\phi(p) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} e^{-i\frac{p}{\hbar}\cdot q} f(p,q)(U^{-1}\phi)(q)dq$$

so

$$\begin{aligned} -\frac{1}{\hbar}V(\hbar\nabla)\phi(p) &= c\cdot\nabla\phi + \frac{\hbar}{2}\Delta\phi \\ &+ \int_{\mathbb{R}^3} \left(\phi(p+y) - \phi(p) - y\cdot\nabla\phi\cdot \mathbb{I}_{\{|y|\le 1\}}\right)\nu(dy) \end{aligned}$$

Semigroup $P_t\phi(p) = E_p[\phi(\xi_t)]$.

P_t = pseudo-differential operator whose symbol = characteristic exponent so $P_t\phi(p) = e^{-\frac{t}{\hbar}\underbrace{V(\hbar\nabla)}_{\text{Generator}}}\phi(p)$

NB: ξ_t = forward Lévy process. Time reversed one $\xi_t^* = -\xi_{v-t}$, $t \in [0, v]$ $(p \xrightarrow{TR} -p)$ with backward discretized increment in integral term of generator and $-\frac{\hbar}{2}\Delta$.

Typical Hamiltonian $\hat{H}\phi(p) = \frac{p^2}{2}\phi + V(\hbar\nabla)\phi$.

Drop the $\sqrt{-1}$ in $\hat{H} \rightarrow \hat{H}$ not always symmetric! But probability conservation O.K with $\hat{H}$ and adjoint $\hat{H}^{+}$:

$$-\hbar \frac{\partial \hat{\eta}^*}{\partial t} = \hat{H}^{+} \hat{\eta}^* \quad \text{and} \quad + \hbar \frac{\partial \hat{\eta}}{\partial t} = \hat{H} \hat{\eta}$$

Look for Markovian diffusion with jumps $p(t)$ s.t.

$$\tilde{P}(p(t) \in \tilde{A}) = \int_{\tilde{A}} \hat{\eta}^* \hat{\eta}(p,t) dp$$

$(\hat{\eta}_t, \hat{\eta}_t^*) > 0, t \in I$. Then generators of $p(t)$:

$$\hat{D} f(p,t) = \frac{1}{\hat{\eta}_t} \left(\frac{\partial}{\partial t} - \frac{1}{\hbar} \hat{H} \right) (\hat{\eta}_t f), \qquad f \in Dom(D)$$

$$\hat{D}_* f(p,t) = \frac{1}{\hat{\eta}_t^*} \left(\frac{\partial}{\partial t} + \frac{1}{\hbar} \hat{H}^{+} \right) (\hat{\eta}_t^* f) \qquad f \in Dom(\hat{D}_*)$$

Proposition:

$\hat{D} = \frac{\partial}{\partial t} + \mathcal{L}_{\hat{\eta}_t}$, where

$$
\begin{aligned}
\mathcal{L}_{\hat{\eta}_t} f &= \frac{\partial f}{\partial t} + c\nabla f + \hbar \nabla \log \hat{\eta}_t \cdot \nabla f + \frac{\hbar}{2}\Delta f \\
&+ \int_{\mathbb{R}^3} \left\{ f(p+y,t) - f(p,t) - y \cdot \nabla f \cdot \mathbb{I}_{\{|y|<1\}} \right\} \underbrace{\frac{\hat{\eta}_t(p+y)}{\hat{\eta}_t(p)} \nu(dy)}_{\text{Lévy kernel}} \\
&+ \int_{\mathbb{R}^3} \frac{\hat{\eta}_t(p+y) - \hat{\eta}_t(p)}{\hat{\eta}_t(p)} y \cdot \nabla f \cdot \mathbb{I}_{\{|y|<1\}} \nu(dy)
\end{aligned}
$$

Proof: Def. D and $\hat{H}$.

When $q \mapsto$ (symbol of $\mathcal{L}_{\hat{\eta}_t}$)$(p,q)$ continuous nonnegative definite, then $\mathcal{L}_{\hat{\eta}_t}$ = (forward) generator of a Markovian diffusion with jumps, $p(t)$. Time reversibility via $\mathcal{L}_{\hat{\eta}_t^*}$.

$p(t)$ admits Lévy-Itô representation with compensator = Lévy kernel and $p(t) << $ Lévy ξ_t (ξ_t^*).

$p(t)$ solves stochastic integro-differential equations driven by ξ_t w.r.t increasing filtration $\mathcal{P}_t$, $t \in I$ and by ξ_t^* w.r.t decreasing $\mathcal{F}_t$.

(Very) special examples:

- Poisson bridge between $a \in \mathbb{N}$ at time s and $b \in \mathbb{N}$ at time u.

- Brownian bridge between $a \in \mathbb{R}^3$ at s and $b \in \mathbb{R}^3$ at U.

Euclidean momentum picture

Under Fourier transform: position observable $\longrightarrow$ gradient and momentum $\longrightarrow$ multiplication operator.

Proposition

For $\hat{H}$ as before, $\forall t \in I$, a.s

$$D\left(\hbar \frac{\nabla \hat{\eta}}{\hat{\eta}}(p(t), t)\right) = p(t), \;\; Dp(t) = \frac{\nabla_q V(\hbar \nabla)\hat{\eta}}{\hat{\eta}}(p(t), t)$$

Under expectation $\longrightarrow$ Euclidean Ehrenfest Theorem.

Purely diffusive case (position picture) $H = -\frac{\hbar^2}{2}\Delta + V$,

$$DX(t) = \hbar \frac{\nabla \eta}{\eta}(X(t)), \;\; D\left(\hbar \frac{\nabla \eta}{\eta}(X(t), t)\right) = \nabla V(X(t))$$

Relation $X(t) \leftrightarrow p(t)$: Euclidean version of position $\leftrightarrow$ momentum pictures

Prospect

Time-symmetric measures counterparts of any quantum picture exist. Allow to export new concepts

$$\left.\begin{array}{l}\text{Feynman's path}\\ \\ \text{integral approach}\end{array}\right\} \longleftrightarrow \quad \text{Stochastic Analysis}$$

References

1. N. Privault, J-C Z.: "Markovian bridges and reversible diffusion processes with jumps" (Preprint).
2. K.L. Chung, J-C Z.: "Introduction to random time and quantum randomness", 2^th Edition expanded, World Scientific, to be published.