Path Integrals and Non-Commutative Probability Theory

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Matrix Models in Physics

There are many random systems in physics where the dynamical variables are matrices:

Yang-Mills theory

Chern-Simons gauge theory

Nonlinear Sigma Model

Euler's equations of hydrodynamics

Rigid body

All these systems are believed to simplify as the dimension of the matrices go to infinity: the basis independent observables have small fluctuations.

It is useful to start by recalling the simplification in the large dimension limit of vector models.

The Hydrogen Atom in n-dimensions

Consider the familiar hamiltonian of the hydrogen atom but generalized to n dimensional space:

$$H = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} + V(|x|), \quad V(x) = -\frac{Ze^2}{|x|}. \tag{1}$$

The classical limit $\hbar \to 0$ is a terrible approximation: there is no ground state.

But the limit $n \to \infty$ is also a kind of classical limit: fluctuations in rotation invariant observables are of order $O(\frac{1}{n})$. Define $\rho^2 = \frac{1}{n}x_i x_i$. The norm of the ground state

$$||\psi||^2 = \int |\psi(x)|^2 d^n x = K_n \int_0^\infty \rho^{n-1}|\psi(\rho)|^2 d\rho \tag{2}$$

can be written more neatly in terms of the radial wave-function $\Psi(\rho) = \sqrt{K_n}\rho^{\frac{n-1}{2}}\psi(\rho)$ so that

$$||\Psi||^2 = \int_0^\infty |\Psi(\rho)|^2 d\rho, H_{\text{eff}}\Psi = \frac{E}{n}\Psi \tag{3}$$

where

$$H_{\text{eff}} = \frac{\hbar^2}{2m}\pi_\rho^2 + \frac{\hbar^2}{8m\rho^2}\frac{(n-1)(n-3)}{n^2} - \frac{Z\alpha}{\rho}, \quad \pi_\rho = -\frac{i}{n}\frac{\partial}{\partial\rho}. \tag{4}$$

The Large n-limit for the Hydrogen Atom

If we let $n \rightarrow \infty$ keeping $\hbar$ and $\alpha = \frac{e^2}{n^{\frac{3}{2}}}$ fixed, we get a classical limit , since $[\pi_\rho, \rho] = -\frac{i}{n}$.

Unlike the usual classical limit this 'neoclassical limit' has a ground state. It is not a good approximation (off by about 30%) but is qualitatively correct.

Expansion in powers of $\frac{1}{n}$ and use of Pade approximants can get quantitatively correct answers. This method is not any more complicated for multi-electron atoms and gives a remarkably effective method for understanding all atoms. (Hershbach).This might give a useful approach to understanding the shapes of molecules: a notoriously hard problem.

The competition between the centrifugal repulsion and the attractive potential leads to the existence of the ground state. This repulsion can be thought of as a sort of entropy (more precisely 'information') due to the averaging over the angular variables.

Static Matrix Models

Let $A_i, i = 1, \cdots r$ be a collection of hermitean $N \times N$ matrices. Also let $S(A)$ be a real-valued polynomial invariant under the action of the unitary group, $A_i \mapsto gA_ig^\dagger$, specifically

$$S(A) = S^{i_1 \cdots i_k} \text{ tr } A_{i_1} \cdots A_{i_k}. \quad (5)$$

We assume that the integral $\int e^{NS(A)}dA$ converges; a constant can then be added to $S(A)$ so that the integral is then equal to one. Such an 'action' defines a probability distribution on the space of multi-matrices.

The basic problem is, given $S(A)$, to find the expectation value of an arbitrary polynomial w.r.t. this distribution.

$$< f(A) >= \int e^{NS(A)} f(A) dA. \quad (6)$$

(We will for now assume that it exists.)

Large N-limit of Static Matrix Models

The basic simplification of the large N limit is that the expectation values of invariant polynomials factorize:

$$\frac{< f(A)h(A) > - < f(A) >< h(A) >}{< f(A) >< h(A) >} = \mathrm{O}(\frac{1}{N^2}). \quad (7)$$

Thus it is enough to know the 'moments'

$$G_{i_1 \cdots i_k} = \frac{1}{N} \operatorname{tr} A_{i_1} \cdots A_{i_k}. \quad (8)$$

These tensors are *not* symmetric: only cyclically symmetric. A 'generating function'

$$Z(J) = \sum_{k=0}^{\infty} G_{i_1 \cdots i_k} J^{i_1} \cdots J^{i_k} \quad (9)$$

captures all this data if it is regarded as a formal power series in non-commuting variables $J^1, \cdots J^r$. It satisfies the 'factorized Schwinger-Dyson' equations:

$$\mathcal{S}^i(D)Z(J) + Z(J)J^i Z(J) = 0, \mathcal{S}^i = \sum_{k=0} S^{ii_1 \cdots i_k} D_{i_1} \cdots D_{i_k} \quad (10)$$

and the operator D_i is defined by

$$D_i \left[J^{i_1} \cdots J^{i_k} \right] = \delta_i^{i_1} J^{i_2} \cdots J^{i_k}. \quad (11)$$

The Wigner Distribution

The simplest case is when $S(A)$ is quadratic:

$$S(A) = -\frac{1}{2}\,\mathrm{tr}\, A_i A_j K^{ij}, \quad K > 0. \tag{12}$$

This example (due to Wigner) is the analogue of a Gaussian; all odd moments are zero and even moments are determined by products of the covariance matrix:

$$G_{i_1 \cdots i_{2k}} = \sum_{\mathrm{non-crossing partitions}} \prod_{\mathrm{pairs} ab} K_{i_a i_b}, \quad K_{ij} K^{jl} = \delta_i^l. \tag{13}$$

The number of terms in this sum is the Catalan number: $\left(\frac{2k+1}{k}\right) \sim 4^k$ as $k \to \infty$. This solution can also be expressed as a continued fraction, generalizing to the non-commutative case the usual method of solving a quadratic equation: $Z_{s+1}(J) = [1 - K_{ij} J^i Z_s(J) J^j Z_s(J)]^{-1}, \quad Z_0(J) = 1$.

The Schwinger-Dyson equation of a some static matrix models can be solved by an algorithm that can be implemented on a push down automaton. (S. G. Rajeev, paper in preparation.)

Variational Principle for the Schwinger-Dyson Equations

Beyond the Wigner (Gaussian) models, there are very few cases for which the Schwinger-Dyson equations can be solved exactly. If we could think of the SD equations as the conditions for the minimization of a function the space of probability distributions, then we could find variational approximations.

We found such a variational principle (also found independently by D. V. Voiculescu). The (*enthalpy*)

$$\Omega(G) = \sum_{k=0}^{\infty} S^{i_1\cdots i_k} G_{i_1\cdots i_k} + \chi(G) \tag{14}$$

which is the sum of the *action* $S(G)$ and the *free entropy* of Voiculescu $\chi(G)$ of the distribution is an extremum at a solution of the SD equations. This entropy arises in previous studies of matrix models by L. Yaffe and A. Jevicki-B. Sakita as a *collective potential.*

Variational approximations

We showed that it cannot be written as a formal power series of the moments (even one that doesnt converge): it is a non-trivial element of the cohomology of variations of the moments. But we do have an explicit expression for it as a formal power series in a larger space of variables.

In the special case of a Wigner (noncommutative Gaussian) of covariance there is an explicit formula (upto some uninteresting constants)

$$\chi(G) = \log\det K. \tag{15}$$

Even when $S(A)$ is not a quadratic function, we can use a Wigner(Gaussian) ansatz to get an approximate solution: find the minimum of Ω restricted to this subspace parametrized by K. MOre accurate answers can be obtained by expanding around this point. This way have solved approximately several previously intractable models. This method is analogous to mean field theory.

Path Integrals over Non-Commutative Variables

A *dynamical matrix model* is defined by the path integral in the space of hermitean matrices

$$< f(A) >= \int e^{N\int\left[-\frac{1}{2}\dot{A}_i\dot{A}_k K^{ij}+V(A)\right]dt} f(A)\mathcal{D}[A]. \quad (16)$$

Again, $f(A)$ is an invariant polynomial of matrix elements such as $\frac{1}{N}$ tr $A_{i_1}(t_1)A_{i_2}(t_2)$. There is again a factorization of amplitudes so that it is sufficient to find the Green's functions

$$G_{i_1\cdots i_k}(t_1,\cdots t_k) =< \frac{1}{N} \text{ tr } A_{i_1}(t_1)\cdots A_{i_k}(t_k) > . \quad (17)$$

The expectation value at equal time are of interest. They are also determined by a variational principle: they are an extremum of a *effective potential*

$$\mathcal{V}(G) = V(G) + \mathcal{I}^{ij}(G)K_{ij} \quad (18)$$

where $\mathcal{I}$ is the free Fisher Information of Voiculescu. (We thus relate the ideas of Voiculescu to those of Yaffe , Jevicki and Sakita.)

Free Fischer Information

In commutative probability theory, the Fisher Information is the change in entropy when a random variable ξ is pertrurbed by a Gaussian random variable $\eta$of small covariance. This change is proportional to the covariance matrix K_{ij} :

$$\mathcal{I}^{ij}(\xi)K_{ij} = \lim_{\epsilon\to 0}\frac{\chi(\xi+\epsilon\eta)-\chi(\xi)}{\epsilon}. \qquad (19)$$

The very same idea applies to noncommutative random variables, except that instead of the Gaussian variable we use a Wigner variable. This is the non-commutative analogue of the Fischer information was discovered by Voiculescu. We find it remarkable that it appears so naturally in the solution of dynamical matrix models.

Our derivation of this result uses the hamiltonian picture, following earlier ideas of L. Yaffe and A. Jevicki-B. Sakita. A less rigorous but heuristic ('physicists') derivation using path integrals over non-commutative variable can also be given. It would be interesting to derive this result rigorously using the techniques of A. Nica and R. Speicher on such path integrals.

Matrix Field Theories

The next step is to understand the ground state of matrix field theories defined by an integral over matrix valued fields with action

$$\int \left[-\frac{1}{2} \partial_\mu A_i \partial_\nu A_j K^{ij} + V(A) \right] d^2x. \qquad (20)$$

There should be a variational principle that determines the low momentum limit of the Green's functions. What is the quantity that generalizes entropy and information?

$$\text{Entropy} \rightarrow \text{Fisher Information} \rightarrow ??? \quad .$$