Setting: Nonrelativistic quantum mechanics with state space $L^2(\mathbb{R}^d)$ and the standard Hamiltonian

$$H := H_0 + V = -\frac{1}{2}\Delta + V.$$

($V: \mathbb{R}^d \to \mathbb{R}$ is the potential energy function. Standard examples are the harmonic oscillator and the attractive or repulsive Coulomb potentials.)

Within this setting, it is V that varies from problem to problem. Also, it is often the case that $V = \sum_{j=1}^{\ell} V_j$.

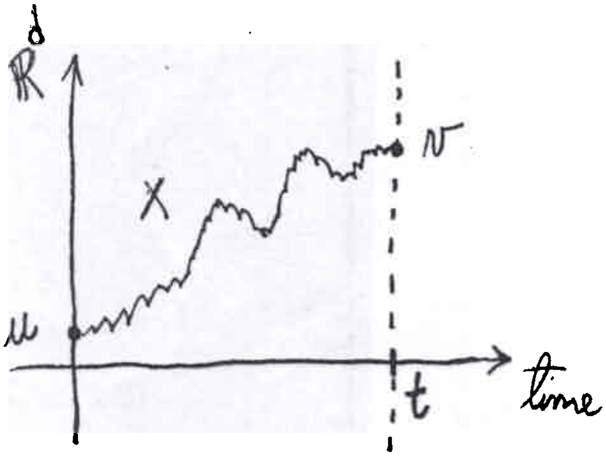
Reference:

G. W. Johnson and M. L. Lapidus, The Feynman Integral and Feynman's Operational Calculus, Oxford U. Press, 2000.
[Paperback: Oxford U. Press, 2002.]

Fey.'s Path "Integral": Amplitude of Transition from u at time 0 to v at time t.

$$\psi(0,t;u,v) = \frac{1}{A} \int_{C^{0,t}_{u,v}} \exp\left\{\frac{i}{\hbar} S(x)\right\} \mathcal{D}(x)$$

where $S(x) = \int_0^t \left\{ \frac{m}{2}\left[\frac{dx}{ds}\right]^2 - V(x(s)) \right\} ds$.



Note: $\psi(t,v) = \int_{\mathbb{R}^d} \psi(0,t;u,v)\,\varphi(u)\,du$

where $\varphi(u) = \psi(0,u)$.

Fey. integrals are used <u>heuristically</u> in several places in quantum physics.

The following ingredients seem typically to be present:

<u>(1.)</u> A set $\mathcal{P}$ of classical objects, the "virtual classical paths."

<u>(2.)</u> An action functional $S = S(p)$, $p \in \mathcal{P}$.

<u>(3.)</u> A "measure" $\mathcal{D}$ (whose existence as a true measure is typically doubtful).

<u>(4.)</u> The Fey. integral

$$(*) \qquad \frac{1}{A} \int_{\mathcal{P}} \exp\{ i k S(p) \} \, \mathcal{D}(p).$$

(The integrand may involve additional functions.)

<u>(5.)</u> The limit as $k \to \infty$ in $(*)$ (stationary phase) remains important.

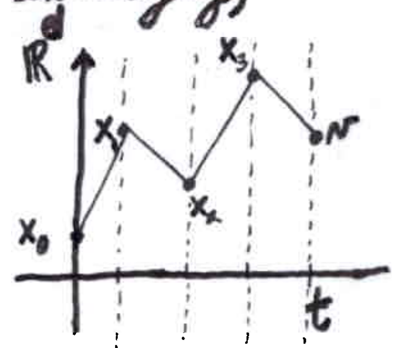
<u>(6.)</u> Perturbation series can be formally calculated from $(*)$ and have been perhaps (??) the most useful result of the heuristics.

Alternatives to $(*)$:

$$(**) \quad \frac{1}{A} \int_{\mathcal{P}} \exp\{ i k S(p) \} F(p) \, \mathcal{D}(p) \qquad \underline{\text{or}}$$

$$(***) \quad \frac{\frac{1}{A} \int_{\mathcal{P}} \exp\{ i k S(p) \} F(p) \, \mathcal{D}(p)}{\frac{1}{A} \int_{\mathcal{P}} \exp\{ i k S(p) \} \, \mathcal{D}(p)} \quad \Bigg\} \text{ Normalized expectation value of the "observable" } F.$$

By slicing $[0,t]$ into m equal parts and further discretizing, Fey. arrived at an approximation $\psi_m(t,v)$ for his path integral for $\psi(t,v)$:



$$\psi_m(t,v) =$$

$$\left(\frac{-i}{2\pi(t/m)}\right)^{d/2}\int_{\mathbb{R}^d} \exp\left\{\frac{i}{2(t/m)}\|x_m - x_{m-1}\|^2\right\}\exp\left\{\frac{-it}{m}V(x_{m-1})\right\}\times$$

$$\left(\;\;''\;\;\right)^{d/2}\int_{\mathbb{R}^d} \exp\left\{\frac{i}{2(t/m)}\|x_{m-1} - x_{m-2}\|^2\right\}\exp\left\{\frac{-it}{m}V(x_{m-2})\right\}\times\cdots\times$$

$$\left(\;\;''\;\;\right)^{d/2}\int_{\mathbb{R}^d} \exp\left\{\frac{i}{2(t/m)}\|x_2 - x_1\|^2\right\}\exp\left\{-\frac{it}{m}V(x_1)\right\}\times$$

$$\left(\;\;''\;\;\right)^{d/2}\int_{\mathbb{R}^d} \exp\left\{\frac{i}{2(t/m)}\|x_1 - x_0\|^2\right\}\exp\left\{-\frac{it}{m}V(x_0)\right\}\psi(x_0)\,dx_0\,dx_1\cdots dx_{m-1}$$

$$= \left[\left(e^{-i\frac{t}{m}H_0}\, e^{-i\frac{t}{m}V}\right)^m \psi\right](v).$$

in operator notation

mth Trotter product

Fey. then took his path integral according to this approach as

$$\psi(t,v) = \lim_{m\to\infty}\psi_m(t,v).$$

Note: I have taken $\hbar$ and m equal to 1.

OH1

Trotter Product Formula for Unitary Groups

Theorem 3.1. If A and B are self-adjoint operators on a Hilbert space $\mathcal{H}$ and if $A+B$ is essentially self-adjoint (i.e. it has a unique self-adjoint extension, necessarily its closure $\overline{A+B}$) on the intersection of the domain of A with the domain of B, then for all $h \in \mathcal{H}$,

$$(3.1) \quad \left\| e^{-it(\overline{A+B})}h - \left(e^{-i\frac{t}{m}A} e^{-i\frac{t}{m}B}\right)^m h \right\|_{\mathcal{H}} \longrightarrow 0 \text{ as } m \to \infty,$$

uniformly in t on all bounded subsets of $\mathbb{R}$.

Note: (3.1) can also be expressed by writing that

$$e^{-it(\overline{A+B})} = \lim_{m\to\infty} \left(e^{-i\frac{t}{m}A} e^{-i\frac{t}{m}B}\right)^m$$

in the strong operator topology on $\mathcal{L}(\mathcal{H})$.

B is A-operator bounded

Definition 3.2. Let A be self-adjoint and let B be a (densely defined) symmetric operator on a Hilbert space $\mathcal{H}$. We say that B is relatively operator bounded with respect to A (briefly, B is A-operator bounded) with bound less than 1 if and only if the domain of A is a subset of the domain of B; i.e.

$$D(A) \subseteq D(B)$$

and there exist positive constants $a < 1$ and b such that

$$(3.3) \qquad \|B\phi\| \leq a\|A\phi\| + b\|\phi\| \text{ for all } \phi \in D(A).$$

When the concept described in Definition 3.2 is used it is typically the case that both A and B are unbounded. Thus $\|B\phi\|$ is not 'controlled' by $\|\phi\|$ alone as it would be for a bounded operator but is 'controlled' by $\|\phi\|$ and $\|A\phi\|$.

Kato's Theorem

Theorem 3.3. Let V be an $\mathbb{R}$-valued Lebesgue measurable function on $\mathbb{R}^d$ and let $V = V_+ - V_-$ be the decomposition of V into its positive and negative parts. Suppose that $V_+ \in L^2_{loc}(\mathbb{R}^d)$ and V_- is H_0-operator bounded with bound less than 1.

Then $H_0 + V$ is essentially self-adjoint on $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ where $\mathcal{D}$ is the space of compactly supported, infinitely differentiable functions on $\mathbb{R}^d$.

Definition 3.4. (Brief version)

$$\mathcal{F}^{t}_{TP}(V) := \lim_{n\to\infty} \left(e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}V}\right)^n \tag{3.7}$$

in the strong operator topology on $\mathcal{L}(L^2(\mathbb{R}^d))$ whenever the limit exists.

Existence Theorem for $\mathcal{F}_{TP}^{t}(V)$ (abbreviated slightly)

Theorem 3.5. Let $V: \mathbb{R}^d \to \mathbb{R}$ be Lebesgue measurable and such that $V_+ \in L^2_{loc}(\mathbb{R}^d)$ and V_- is relatively operator bounded with respect to H_0 with relative bound less than 1.

Then $\mathcal{F}_{TP}^{t}(V)$ exists for all $t \in \mathbb{R}$. Further, for all $t \in \mathbb{R}$,

(3.9) $$\mathcal{F}_{TP}^{t}(V) = e^{-itH},$$

where H is the unique self-adjoint extension of the operator $(H_0 + V)|_D$; that is, $\mathcal{F}_{TP}^{t}(V)$ agree with the unitary group which specifies the dynamics in the standard approach to quantum mechanics. Finally, for $\phi \in D(H)$, $\mathcal{F}_{TP}^{t}(V)\phi$ is the unique solution (in the semigroup sense) of the Schrödinger equation with initial state ϕ:

(3.10) $$\frac{\partial \psi}{\partial t} = -iH\psi, \quad \psi(0,\cdot) = \phi.$$

In the <u>Modified Feynman Integral</u> of Lapidus, the approximators $\left(e^{-i(t/n)H_0}\, e^{-i(t/n)V}\right)^n$ are replaced by

$$\left([I+i(t/n)H_0]^{-1}[I+i(t/n)V]^{-1}\right)^n.$$

In the <u>Analytic in Time Operator-valued Feynman Integral</u>, one begins with the Wiener integral

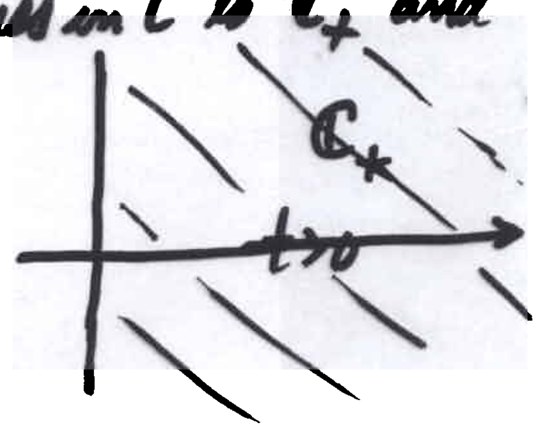
$$\int_{C_0([0,t],\mathbb{R}^d)} \exp\left[-\int_0^t V(x(s)+\xi)\,ds\right]\varphi(x(s)+\xi)\, m(dx)$$

$$= E_\xi\left\{\exp\left[-\int_0^t V(\omega(s))\,ds\right]\varphi(\omega(t))\right\}$$

$$= \left[e^{-t(H_0+V)}\varphi\right](\xi).$$

(Feynman-Kac Formula (True under certain conditions))

One then analytically continues in t to $\mathbb{C}_+$ and by (strong) continuity to $\overline{\mathbb{C}_+}$.



Some Properties Which One Would Like a Theory of the Feynman Integral to Have:

(1.) The FI ought to exist for a wide class of potentials V and, in particular, ought to exist for the standard examples.

(2.) The relationship between the FI and the unitary group from the usual approach to quantum dynamics ought to be clear. Further, if more than one approach to the FI is being discussed, one would like to know how they are related to one another.

(3.) One would like to know that the FI is "stable" under small changes in the potential V and the initial state ϕ in $L^2(\mathbb{R}^d)$. Related: If $V_n \to V$ and $\phi_n \to \phi$, one would like to know that

$$\mathcal{F}_{(\cdot)}^{t}(V_n)\phi_n \longrightarrow \mathcal{F}_{(\cdot)}^{t}(V)\phi.$$

Corollary. If $V_+ \in L^1_{loc}(\mathbb{R}^d)$ and V_- is H_0-form bounded with bound less than 1, then $e^{-it(H_0 \dot{+} V)}$, $\mathcal{F}_M^{\,t}(V)$ and $J^{it}(F_V)$ all exist and we have

$$J^{it}(F_V) = \mathcal{F}_M^{\,t}(V) = e^{-it(H_0 \dot{+} V)} \quad \forall t \in \mathbb{R}.$$

Corollary. If $V_+ \in L^2_{loc}(\mathbb{R}^d)$ and V_- is H_0-bounded with bound less than 1, then all of $e^{-it(H_0+V)}$, $\mathcal{F}_{TP}^{\,t}(V)$, $\mathcal{F}_M^{\,t}(V)$, $J^{it}(F_V)$ exist $\forall t \in \mathbb{R}$ and

$$J^{it}(F_V) = \mathcal{F}_M^{\,t}(V) = \mathcal{F}_{TP}^{\,t}(V) = e^{-it(H_0+V)}.$$

Proposition. (a) If $V_- \in S_d$, then V_- satisfies the hypothesis of the last Corollary.

(b) Let $p = 2$ for $d \le 3$ and let $p > d/2$ for $d \ge 4$. Then $L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \subseteq L^p_{loc}(\mathbb{R}^d)_u \subseteq S_d$.

Proposition (a) If $V_- \in K_d$, then V_- satisfies the hypotheses of the 1st Corollary above.

(b) Let $p = 1$ for $d = 1$ and $p > d/2$ for $d \ge 2$. Then $L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \subseteq L^p_{loc}(\mathbb{R}^d)_u \subseteq K^d$.

"Dominated-type" Convergence Theorems for the Modified Feynman Integral and for the Analytic in Time Operator-valued Feynman Integral

Assumptions Let V, V_m, $m = 1, 2, \dots$, be Lebesgue measurable $\mathbb{R}$-valued functions on $\mathbb{R}^d$. Assume that "V_m converges to V dominatedly" in the following sense:

(a.) $V_m \longrightarrow V$ Leb.-a.e. in $\mathbb{R}^d$,

(b.) $V_{m,+} \leq U$ for some $U \in L^1_{loc}(\mathbb{R}^d)$,

(c.) $V_{m,-} \leq W$ for some $W \in L^p_{loc}(\mathbb{R}^d)_{\tilde{u}}$, where $p = 1$ if $d = 1$ and $p \in (d/2, \infty)$ for $d \geq 2$.

Then $\mathcal{F}_M^t(V_m), \mathcal{F}_M^t(V), J^{it}(F_{V_m}), J^{it}(F_V)$ all exist and

1. $\mathcal{F}_M^t(V_m) \longrightarrow \mathcal{F}_M^t(V)$ [Modified Feynman Integral]

2. $J^{it}(F_{V_m}) \longrightarrow J^{it}(F_V)$ [Analytic Op.-Valued Fey. Integral]

in the strong operator topology, uniformly in t on compact subsets of $\mathbb{R}$.

"Dominated-type" Convergence Theorem for the Feynman Integral defined via the Trotter Product Formula

Assumptions Let V, V_m, $m = 1, 2, \ldots$, be Lebesgue measurable $\mathbb{R}$-valued functions on $\mathbb{R}^d$. Assume that "V_m converges to V dominatedly" in the following sense:

(a.) $V_m \to V$ Leb.–a.e. in $\mathbb{R}^d$,

(b.) $V_{m,+} \leq U$ for some $U \in L^2_{loc}(\mathbb{R}^d)$,

(c.) $V_{m,-} \leq W$ for some $W \in L^p_{loc}(\mathbb{R}^d)_u$, where $p = 2$ if $d = 1, 2, 3$ and $p \in (d/2, \infty)$ for $d \geq 4$.

Then $\mathcal{F}^t_{TP}(V_m)$ and $\mathcal{F}^t_{TP}(V)$ all exist and

$$\mathcal{F}^t_{TP}(V_m) \to \mathcal{F}^t_{TP}(V)$$

in the strong operator topology, uniformly in t on compact subsets of $\mathbb{R}$.