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FEYNMAN'S OPERATIONAL CALCULUS VIA FEYNMAN PATH INTEGRALS AND DISENTANGLING ALGEBRAS

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WORKSHOP ON FEYNMAN INTEGRALS ALONG WITH RELATED TOPICS AND APPLICATIONS
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THE FEYNMAN INTEGRAL AND FEYNMAN'S OPERATIONAL CALCULUS

Gerald W. JOHNSON

Michel. L. LAPIDUS

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THE FEYNMAN INTEGRAL AND FEYNMAN'S OPERATIONAL CALCULUS

Gerald W. Johnson, *Department of Mathematics and Statistics, University of Nebraska-Lincoln,* and **Michel L. Lapidus**, *Department of Mathematics, University of California, Riverside*

The aim of this book is to make accessible to mathematicians, physicists and other scientists interested in quantum theory, the beautiful and closely related but mathematically difficult subjects of the Feynman integral and Feynman's operational calculus. Some advantages of the approaches to the Feynman integral which are treated in detail in this book are the following: the existence of the Feynman integral is established for very general potentials in all four cases; under more restrictive but still broad conditions, three of these Feynman integrals agree with one another and with the unitary group from the usual approach to quantum dynamics; these same three Feynman integrals possess pleasant stability properties. Much of the material covered here was previously available only in the research literature, and the book also contains several new results. The background material in mathematics and physics that motivates the study of the Feynman integral and Feynman's operational calculus is discussed, and detailed proofs are provided for the central results. The last chapter of the book includes a discussion of topics in physics and mathematics (including knot theory) where heuristic Feynman integrals have played a significant role. In some cases, perturbation series in the spirit of Feynman's operational calculus are the key objects.

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[Reprinting softcover edition, Jan. 2002]

ORIGINAL PAPER ON THE "FEYNMAN INTEGRAL":

[Fe1] Richard P. FEYNMAN

"Space-time Approach to ~~non~~-relativistic Quantum Mechanics"

Rev. Mod. Phys. **20** (1948), 367-387.

R-1/2

ORIGINAL PAPER ON "FEYNMAN'S OPERATIONAL CALCULUS".

[Fe2] Richard P. FEYNMAN

« An operator calculus having applications in quantum electrodynamics »

Phys. Rev. 84 (1951), 108-128.

MAIN REFERENCE: BOOK BY

for this talk

G. W. JOHNSON
&
M. L. LAPIDUS

« THE FEYNMAN INTEGRAL AND FEYMAN'S OPERATIONAL CALCULUS ».

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(~~to appear~~ 2000)

(Chs. 14–19) F's Op. Calculus & rel. topics

(esp. Chapter 7. The F. $\int$: Heuristic Ideas and Math. Difficulties)

R5

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[dFJL2]

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(↙ ~~Unmea~~ measurability result in: G. W. Johnson, Proc. AMS, [illegible] (1993), 1097-1104,

extended in:

A. Badrikian, G. W. Johnson & Yoo, The composition of operator-valued measurable functions is measurable, Proc. AMS ~~(in press)~~. 123 (1995), 1815-20.)

Related works:

[JeJ] B. Jefferies & G. W. Johnson, Functional calculi for noncommuting operators, ~~in prep~~ series of papers. late 80's.

[R] J. T. Reyes, — extends aspects of [dFJL2] Ph.D. Thesis, in prep., 1995. (U. of Nebraska, Lincoln).

[Ri] T. Riggs, Ph.D. Thesis, 1993 (U. of Nebraska, Lincoln)

{ Poisson processes & Dirac eqt. in 1 dim.)

Also:

G. W. J. & M. L. L. : Feynman's Operational Calculi with Noncommutative Auxiliary Operations, 2002 (in prep.).

Other approaches to Feynman's operational calculus by

MASLOV (1976)

ARAKI (1973)

NELSON (1970)

GILL et al. 1980-90's.

{ FUNCTION SPACE APPROACH: }

(used in our approach but not directly connected with F's OC.)

R. H. CAMERON, A Family of Integrals Serving to Connect the Wiener and Feynman Integrals, J. Math. and Phys. Sci. 39 (1960), 126–140.

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———, The Cameron-Storvick Function Space Integral: an $L(L_p, L_{p'})$ Theory, Nagoya Math. J. 60 (1976), 93–137.

Plan of this Talk:

1. Heuristic Introduction to Feynman's Operational Calculus.
   (with some motivation)

2. A Mathematical Approach to Feynman's Operational Calculus:

A. [JL1] a. via Wiener and/Feynman (analytic) (path) 'integrals'.

- generalized Dyson series (GDSs) and the 'disentangling' process.
- 'disentangling algebras' (brief comments on [Lo-S].

B. [JL2]

- Noncommutative operations * & + on the space of Wiener functionals.
- Links with disentangling algebras and functional integrals.
- Resolution of certain 'paradoxical formulas' in this context.

(if time allows) 3. a. Comments on the more abstract approach: generalized path integral ([dFJL], [L6] [Lo3])

b. Future work: Links with Voiculescu noncommutative probability theory; Links with Connes' noncommutative geometry?

MO

<u>In one sentence:</u>

Feynman's operational calculus consists in treating noncommuting operators as though they were commuting. (!;)

?

$AB = BA$ $\left(= \frac{1}{2}(AB+BA)\right)$

?

$e^{A+B} = e^A e^B$

Feynman's Operator Calculus (Operational)

Hi

Motivations

§ ① Physical Motivations:

1a. Quantum Mechanics

1b. Applications to Quantum Electrodynamics (QED)

§

Schwinger (1949)
Tomonaga (1948)
Feynman (1948-50)

approaches to QED connected by Dyson (1949) by "means of a re-ordering of operators in each term of a perturbation power series" («Dyson series»)

§

"There may be some hope that a thorough understanding of electrodynamics might give a clue as to the possible structure of the more complete theory to which it is an approximation".

"It might be worth [...] expressing [QED] in every possible physical and mathematical way".

(2) Purely Mathematical Reasons

"A second reason is to describe (*) a mathematical method which may be useful in other fields".

"The mathematics is not completely satisfactory. No attempt has been made to maintain mathematical rigor. The excuse is not that it is expected that rigorous demonstra- -tions can be easily supplied. Quite the contrary, it is believed that to put the present methods on a rigorous basis may be quite a difficult task, beyond the abilities of the author."

(*) The quotes are from Feynman's 1951 paper. [Fe2].

Elsewhere: about the "right for a physicist to use poetic license and to be free~~dom~~ from the rules of grammar. ..."

S. Schweber (in "Feynman and the visualization of space-time processes"):

"These papers, on the space-time approach to nonrelativistic quantum mechanics [1948], on quantum electrodynamics [1949-51, including the 'Feynman diagrams'], and on his operator calculus [1951], must surely be placed near the top of any list of the most seminal and influential papers in theoretical physics during the twentieth century".

M_4

Schweber again:

"In any case, Feynman never felt order-by-order was anything but an approximation to the 'thing', and the 'thing' was the path integral"

criticism: "substitute for general path integral"

Dyson's time-ordered perturbation series ?

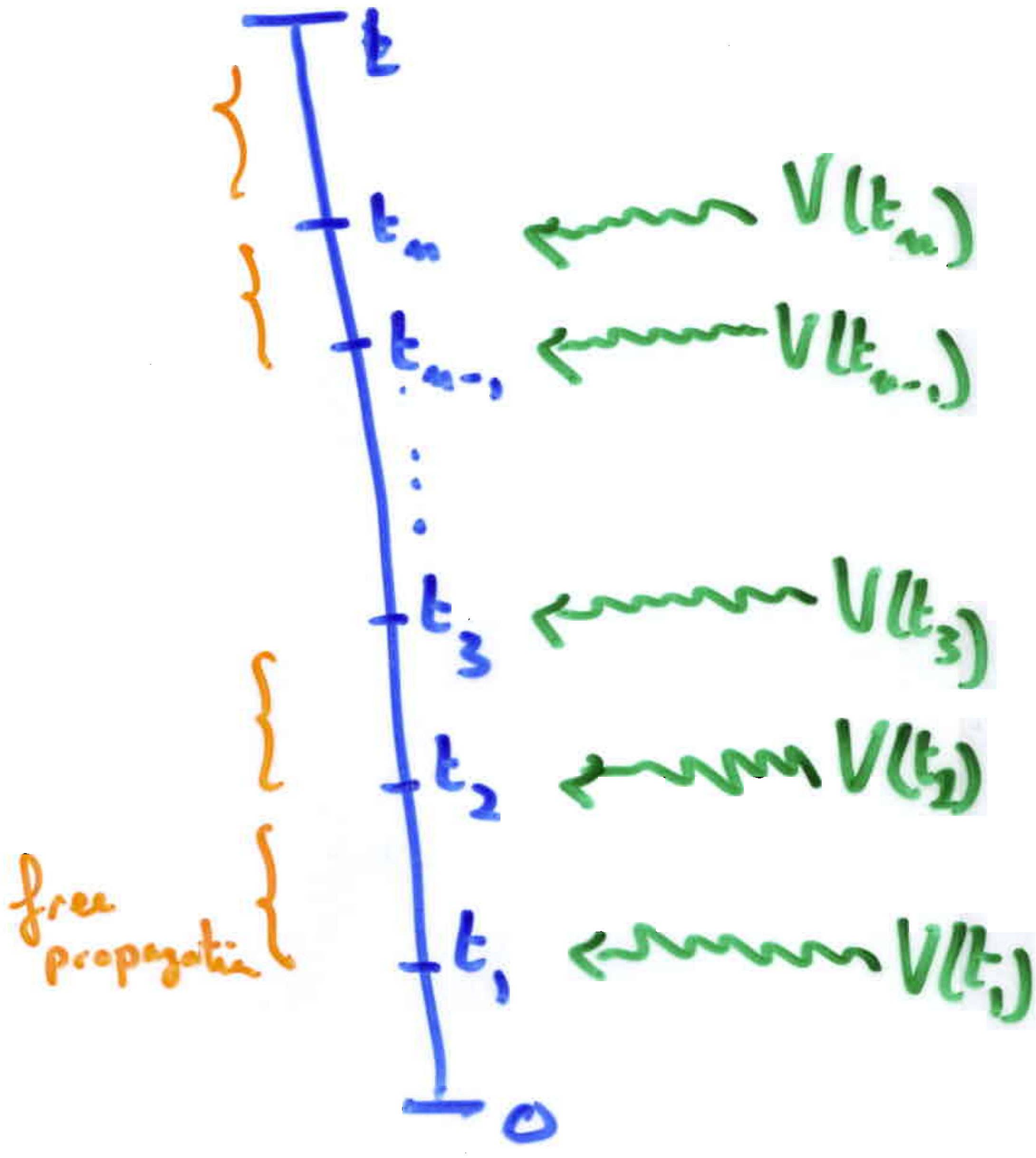
$$\int e^{\frac{i}{\hbar}S(x)}\,\mathcal{D}x$$

Feynman's path integral

$$\sum_{n=0}^{\infty}(-i)^n\int_{\Delta_n} e^{-\frac{i(t-s_n)H_0}{\hbar}}\,V(s_n)\,e^{-\frac{i}{\hbar}(s_n-s_{n-1})H_0}\,V(s_{n-1})\,e^{\dots}\cdots V(s_2)\,e^{-\frac{i(s_2-s_1)H_0}{\hbar}}\,V(s_1)\,e^{-\frac{i s_1 H_0}{\hbar}}\,ds_1\dots ds_n$$

$$\Delta_n=\{(s_1,\dots,s_n)\in\mathbb{R}^n : 0<s_1<\dots<s_n\}.$$

M4 1(2)



(classical) Feynman diagram

(for the nth term of the Dyson series)

Perhaps Feynman's operational calculus could be viewed as a generalized (Wiener or Feynman) path integral, or as a substitute for it.

$$\text{“}\int\text{”} \rightsquigarrow \Sigma$$

Feynman (or functional) integrals

perturbation series

M6

Feynman's Operator Calculus: An Heuristic Introduction

Feynman's rules

[allowing him to calculate (heuristically) time-ordered perturbation series (for instance) without actually having a path integral]:

F's "time-ordering convention"

Rule ① 1. Attach "time" indices to the operators involved in order to specify the order of operations in products.

Rule 2 ② 2. With indices attached, form functions of the operators by treating them as though they commuted.

Rule ③ "" 3. Finally, "disentangle" the resulting expressions, that is, restore the conventional ordering of the operators.

"DISENTANGLING PROCESS": "disentangling process"

III.7) Specifically:

Feynman's "time-ordering convention":

A, B two "operators"! (non commuting)

Step 1.:

$$(*)\quad A(s_1)\,B(s_2) = \begin{cases} BA & \text{if } s_1 < s_2. \\ AB & \text{if } s_2 < s_1. \\ \text{undefined} & \text{if } s_1 = s_2. \end{cases}$$

attach time-indices

Step 2: Now, treat the operators as though they commuted (but respect the rule (*)!).

"disentangling process"

Step 3: Finally, ("disentangle") the resulting expressions; i.e., restore the conventional ordering of the operators.

go back to the real world!

MT 13

Feynman ([Fe 2], 1951) about the 'disentangling process':

"The process is not always easy to perform, and in fact, is the central problem of this operator calculus".

Simple Example:

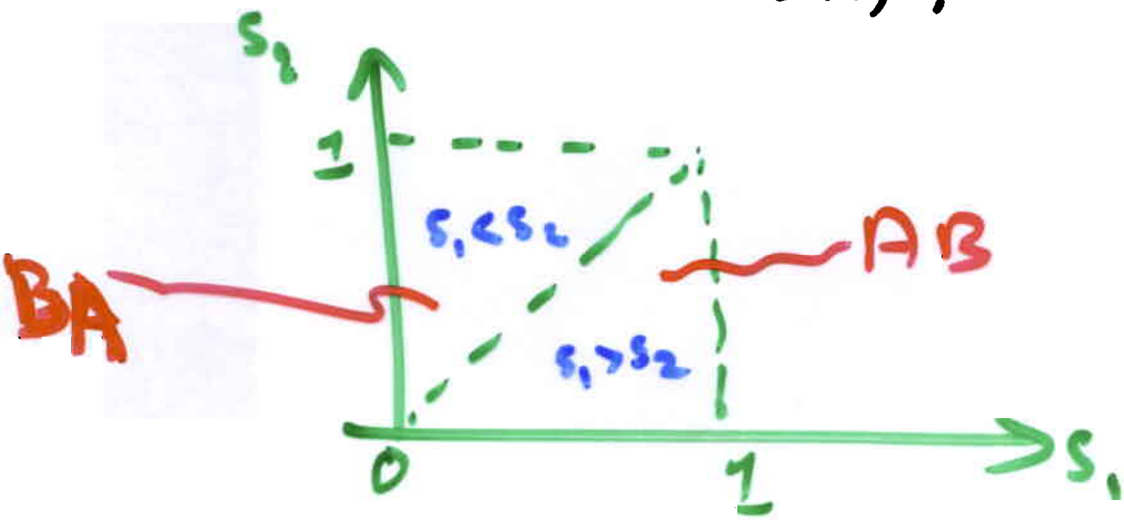
$$\text{"}A.B\text{"} = \left(\int_0^1 A(s_1)\, ds_1\right)\left(\int_0^1 B(s_2)\, ds_2\right)$$

$$= \iint_{[0,1]\times[0,1]} A(s_1)\, B(s_2)\, ds_1\, ds_2$$

$$= \iint_{s_1<s_2} A \cdot B\, d\vec{s} + \iint_{s_1>s_2} B A\, d\vec{s}$$

$$= \frac{1}{2} AB + \frac{1}{2} BA$$

$$= \frac{1}{2}(AB + BA).$$



MgJ

Other examples:

11/4/5

$$\text{"}e^A \cdot B\text{"} = e^{\int_0^1 A(s_1)\,ds_1} \int_0^1 B(s_2)\,ds_2$$

$$= \int^1 e^{\int_{s_2}^1 A(s_1)\,ds_1} B(s_2)\, e^{\int_0^{s_2} A(s_1)\,ds_1} ds_2$$

$$= \int_0^1 e^{(1-s_2)A} B e^{s_2 A}\, ds_2$$

$$\left(= \int_0^1 e^{(1-s)A} B e^{sA}\, ds \right).$$

$$\text{"}e^A \cdot B^2\text{"} = e^{\int_0^1 A(s_3)\,ds_3} \iint_0^1 B(s_1) B(s_2)\,ds_1 ds_2$$

$$= \iint_{s_1 < s_2} e^{\int_{s_2}^1 A(s)\,ds} B(s_2)\, e^{\int_{s_1}^{s_2} A(s)\,ds} B(s_1)\, e^{\int_0^1 A(s)\,ds}\, ds_1 ds_2$$

$$+ \iint_{s_2 < s_1} - - -$$

$$= \iint_{s_1 < s_2} e^{(1-s_2)A} B e^{(s_2 - s_1)A} B\, e^{s_1 A}\, ds_1 ds_2$$

$$+ \iint_{s_2 < s_1} - - - .$$

M1(b)

1/5/6

"<u>Paradoxical</u> formulas".

<u>e.g.</u>,

$$e^{A+B} = e^A \cdot e^B \quad (?!)$$

even if A & B do not commute!

M11)

Time-Ordered Perturbation Series

$$e^{\left\{-tA + \int_0^t B(s)\,ds\right\}} = \exp\left\{-tA + \int_0^t B(s)\,ds\right\}$$

rule (1) "attach time indices"

$$= \exp\left\{-\int_0^t A(s)\,ds + \int_0^t B(s)\,ds\right\}$$

$$= \exp\left\{-\int_0^t A(s)\,ds\right\} \exp\left\{\int_0^t B(s)\,ds\right\}$$

rule (2) now, "treat the operators as though they were commuting"

$$= \exp\left\{-\int_0^t A(s)\,ds\right\} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{\int_0^t B(s)\,ds\right\}^n$$

$$= \sum_{n=0}^{\infty} \exp\left\{-\int_0^t A(s)\,ds\right\} \frac{1}{n!} \left(\int_0^t B(s)\,ds\right)^n.$$

(2) Wait until the "disentangling process" is completed in order to interpret the expressions involved in the standard way. !!

M13

Next, write

$$\left(\int_0^t B(s)\,ds\right)^n = \left(\int_0^t B(s_1)\,ds_1\right)\cdots\left(\int_0^t B(s_n)\,ds_n\right)$$

$$= \int_{[0,t]^n} B(s_n)\cdots B(s_1)\,ds_1\cdots ds_n$$

$$= n!\int_{\Delta_n(t)} B(s_n)\cdots B(s_1)\,ds_1\cdots ds_n,$$

recall: The operators are decreed to be commuting (It's a wonderful world)

where

$$\Delta_n = \Delta_n(t) = \{(s_1,\dots,s_n) : 0 < s_1 < \cdots < s_n < t\}.$$

Further, write

$$"\exp\left\{-\int_0^t A(s)\,ds + \int_0^t B(s)\,ds\right\}"$$

$$= \sum_{n=0}^{\infty}\int_{\Delta_n}\exp\left\{-\int_0^t A(s)\,ds\right\}B(s_n)\cdots B(s_1)\,ds_1\cdots ds_n$$

$$= \sum_{n=0}^{\infty}\int_{\Delta_n}\exp\left\{-\int_{s_n}^{s_{n+1}} A(s)\,ds\right\}B(s_n)$$

$$\exp\left\{-\int_{s_{n-1}}^{s_n} A(s)\,ds\right\}B(s_{n-1})$$

$$\vdots$$

$$\exp\left\{-\int_{s_1}^{s_2} A(s)\,ds\right\}B(s_1)\exp\left\{-\int_0^{s_1} A(s)\,ds\right\}$$

$$ds_1\cdots ds_n$$

rule(?) time-ordering "gives true" and then the conventional ordering

M13) Remember that $A(s) \equiv A$ to conclude:

$$\exp\left\{-tA + \int_0^t B(s)\,ds\right\}$$

$$= \sum_{n=0}^{\infty} \int\!\!\int_{\Delta_n} e^{-(t-s_n)A} B(s_n)\, e^{-(s_n - s_{n-1})A} B(s_{n-1}) \cdots B(s_2) e^{-(s_2-s_1)A} B(s_1) e^{-s_1 A} \, ds_1 \ldots ds_n .$$

(The disentangling is completed.)

time-ordered perturbation expansion

$[A = iH_0, \quad B = -iV$

V: (multiplication operator by a potential)

$\hookrightarrow$ classical Dyson series

$A = H_0, \quad B = -V$

series can also be obtained by functional integration

Functional integrals permit the calculation of perturbation series.

Example:

$$(e^{-t(H_0+V)}\varphi)(\xi) \overset{F\text{-}K}{=} \int_{C_0^t} \exp\left\{-\int_0^t V(x(s)+\xi)\,ds\right\}\varphi(x(t)+\xi)\,dm(x)$$

$$= \int_{C_0^t} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left[\int_0^t V(x(s)+\xi)\,ds\right]^n \varphi(x(t)+\xi)\,dm(x)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{\cancel{n!}}\cdot\cancel{n!}\int_{C_0^t}\left[\int_0^t\int_0^{s_n}\cdots\int_0^{s_2} V(x(s_n)+\xi)\cdots V(x(s_1)+\xi)\,ds_1\cdots ds_n\right] \varphi(x(t)+\xi)\,dm(x)$$

$$\text{'='} \sum_{n=0}^{\infty}(-1)^n\int_{\Delta_n(t)}\left[e^{-(t-s_n)H_0}\,V\,e^{-(s_n-s_{n-1})H_0}\,V\cdots V \cdot e^{-(s_2-s_1)H_0}\,V\,e^{-s_1H_0}\varphi\right](\xi)\,ds_1\cdots ds_n\,.$$

Note: The above calculation is presented rather briefly & the equality '=' is slightly inaccurate.

$$\Delta_n = \Delta_n(t) = \{(s_1,\dots,s_n) : 0 < s_1 < \cdots < s_n < t\}.$$

$$C_0^t = \{x : [0,t] \to \mathbb{R}^m : x \text{ cont.},\ x(0)=0\}.$$

(path space or 'Wiener space')

In general the results are rather involved combinatorially (cf. [JL2 dFJL])

For the (simple) examples:

1. $$e^{\int_0^1 A(s)\,ds + \int_0^1 B(s)\,d\delta_0(s)} = e^A e^B$$

Dirac measure

whereas

2. $$e^{\int_0^1 A(s)\,d\delta_0(s) + \int_0^1 B(s)\,ds} = e^B e^A$$

3. $$e^{\int_0^1 A(s)\,ds + \int_0^1 B(s)\,d\delta_{1/2}(s)} = e^{\frac{1}{2}A} e^B e^{\frac{1}{2}A}.$$

4. $$e^{\int_0^t A(s)\,ds + \int_0^t B(s)\,d\nu(s)} = \underbrace{\left(e^{(t/n)A} e^{(t/n)B}\right)^n}_{n\text{-th Trotter product}}$$

where $$\boxed{\nu = \frac{t}{n}\sum_{j=1}^{n} \delta_{j(t/n)}}$$

Some Mathematical Approaches to Feynman's Operational Calculus

I. Approach via Path Integrals

(M.L.L. & G.W.J. Mem. AMS 1989)

$$F \rightsquigarrow K_\lambda^t(F)$$

('functional') on Wiener space, (operator)

setting: Nonrelativistic QM (quantum mechanics): use of Wiener path integrals ($\lambda>0$) and Feynman "path" integrals (λ imaginary) (obtained by analytic continuation)

Ex: Analogy:

$$F \rightsquigarrow K(F)$$

(symbol) ΨDO (pseudodifferential operator)

$$C_0^t = C_0([0,t], \mathbb{R}^N)$$ — Wiener space (2)

$$= \{x : [0,t] \xrightarrow{\text{continuous}} \mathbb{R}^N : x(0) = 0\}.$$

$m = m_t$: Wiener (prob.) measure on C_0^t.

$$F : C^t \longrightarrow \mathbb{C}$$ "functional" (suitable meas. function on Wiener space)

$$C^t = \{x : [0,t] \xrightarrow{\text{continuous}} \mathbb{R}^N\}$$

$$F \longmapsto K_\lambda^t(F) \in \mathcal{L}(L^2(\mathbb{R}^N)).$$

$\lambda > 0.$ ($\xi \in \mathbb{R}^N$, $t > 0$, $\psi \in L^2(\mathbb{R}^N)$)

$$(K_\lambda^t(F)\psi)(\xi) =$$

$$\int_{C_0^t} F(\lambda^{-1/2}x + \xi)\,\psi(\lambda^{-1/2}x(t) + \xi)\,dm_t(x)$$

$$\left(= \underbrace{\mathbb{E}_\xi}_{\text{conditional expectation}}\{F(\lambda^{-1/2}x)\,\psi(\lambda^{-1/2}x(t))\}\right).$$

G 2.8

Let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0, \lambda \neq 0\}$.

$\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$.

Then if $K_\lambda^t(F)$ exists for all $\lambda \in \tilde{\mathbb{C}}_+$,

$K_\lambda^t(F)$ is strongly continuous for $\lambda \in \tilde{\mathbb{C}}_+$

and is analytic for $\lambda \in \mathbb{C}_+$.

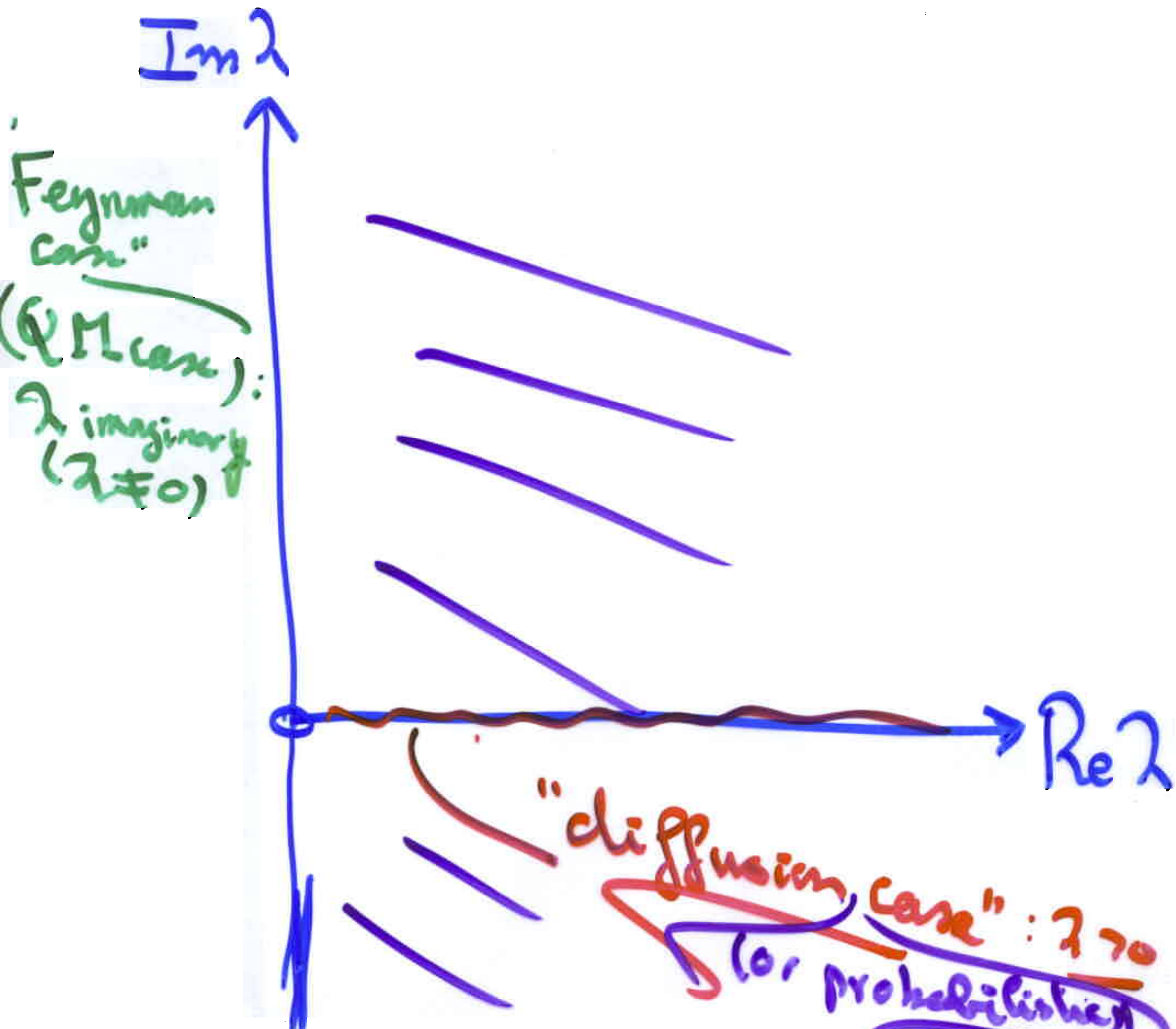
(2)

- $K_\lambda^t(F)$ ($\lambda > 0$) is a functional integral given by a Wiener path integral.
- For $\lambda \in \mathbb{C}_+$ (and a fortiori for λ purely imaginary), $K_\lambda^t(F)$ is no longer given by a path integral. (cf. CAMERON)

For λ purely imaginary, $K_\lambda^t(F)$ is called the (operator-valued) analytic Feynman integral of F.

("in mass")

(cf. CAMERON; CAMERON–STORVICK, NELSON, JOHNSON–SKOUG, JOHNSON–LAPIDUS ...).



Def.: $K_\lambda^t(F)$ $(\mathrm{Re}\,\lambda \geq 0, \lambda \neq 0)$ if:

(a) The path integral $K_\lambda^t(F)$ exists for all $\lambda > 0$ and defines an operator in $\mathcal{L}(L^2(\mathbb{R}^n))$

(b) $\lambda \mapsto K_\lambda^t(F)$ can be analytically continued to the half-plane $\mathrm{Re}\,\lambda > 0$.

(c) $\lambda \mapsto K_\lambda^t(F)$ admits a strong limit along the imaginary axis (while $\mathrm{Re}\,\lambda > 0$).

G $3\frac{1}{2}$

Each operator $K_\lambda^t(F)$ is "disentangled" via a time-ordered perturbation series, called a '<u>generalized Dyson series</u>' (GDS).

In fact, we construct a family of <u>algebras</u> $\{\mathcal{A}_t\}_{t>0}$ (called the '<u>disentangling algebras</u>'), such that for each $F \in \mathcal{A}_t$, $K_\lambda^t(F)$ exists $\forall\, \mathrm{Re}\,\lambda \geq 0$, $\lambda \neq 0$ and can be disentangled via a GDS.

<u>Note</u>: Each $\mathcal{A}_t$ is a <u>commutative</u> Banach algebra.

F4

Examples of functionals allowed:

$$F(x) = \exp\left\{\int_0^t \Theta(s, x(s))\, \eta(ds)\right\}$$

$\left(\Theta : [0,t] \times \mathbb{R}^N \to \mathbb{C},\ \int_0^t \|\Theta(s,\cdot)\|_\infty |\eta|(ds) < \infty \right)$

$\int_0^t \|\Theta(s,\cdot)\|_\infty |\eta|(ds) =: \|\Theta\|_{\infty,1;\eta}$

η : Borel measure on $[0,t]$.

'Feynman-Kac functional with Lebesgue-Stieltjes measure η'.

$$\eta = \mu + \nu$$

μ: continuous measure (may have a singular part)

ν: discrete measure

The use of Lebesgue-Stieltjes measures (like $\mathfrak{I}$) enables us to blend continuous and discrete structures.

→ more complicated (and richer) combinatorics for the perturbation series (GNS).

→ 'generalized Feynman diagrams'.

G6

Example: ($\mu(\{s\})=0, \forall s$) continuous measure

Special case: $\eta = \mu + \omega\,\delta_\tau$ (Dirac mass)

($\omega \in \mathbb{C}$, $0<\tau<t$).

Then:

$$K_\lambda^t(F) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\omega^{n-k}}{(n-k)!} \sum_{j=0}^{k}$$

$$\Big[\int_{\Delta_{k,j}} \Big\{ e^{-(t-s_k)(H_0/\lambda)}\, \theta(s_k)\, e^{-(s_k-s_{k-1})H_0/\lambda}$$

$$\cdots \theta(s_{j+1})\, e^{-(s_{j+1}-\tau)(H_0/\lambda)}$$

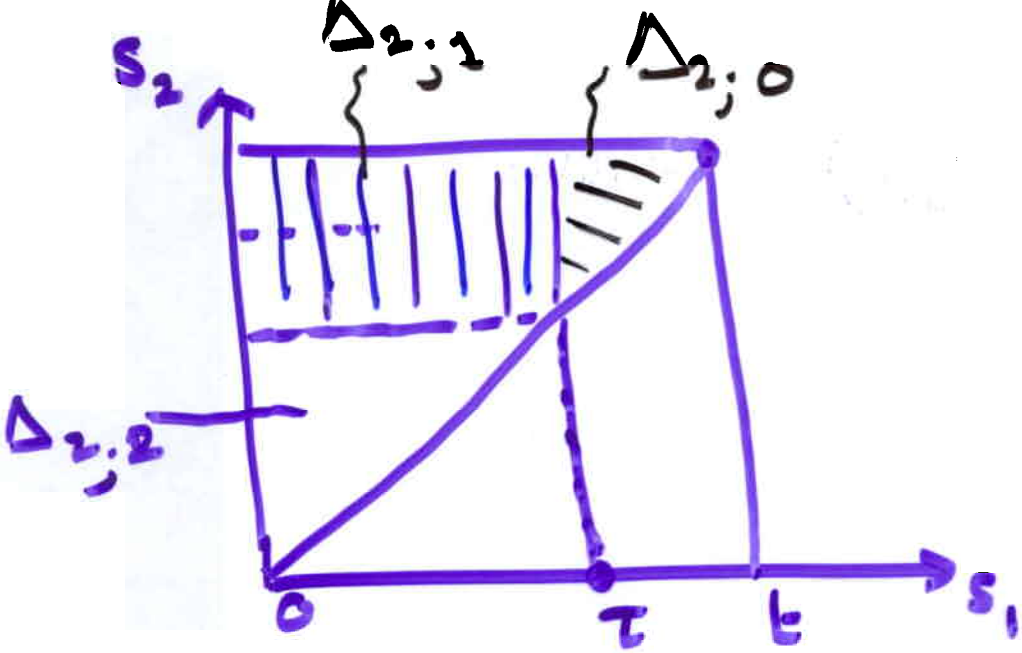
$$[\theta(\tau)]^{n-k}\, e^{-(\tau-s_j)(H_0/\lambda)}$$

$$\theta(s_2)\, e^{-(s_2-s_1)(H_0/\lambda)}\, \theta(s_1)\, e^{-s_1(H_0/\lambda)} \Big\}$$

$$\mu(ds_1)\cdots\mu(ds_k) \Big],$$

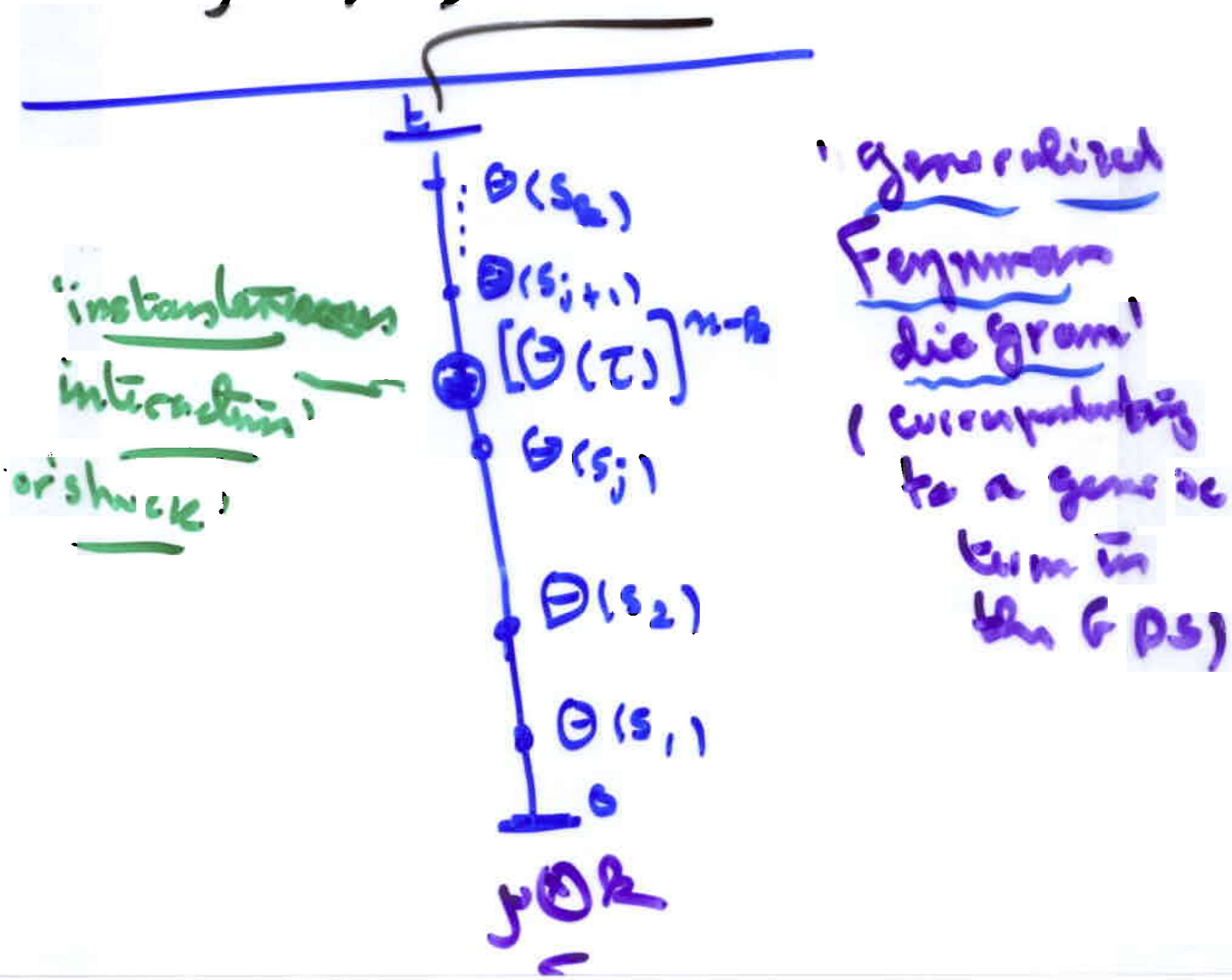
where (for $0 \le j \le k \le n$),

$$\Delta_{k,j} = \{(s_1,\dots,s_k) \in (0,t)^k : 0 < s_1 < \cdots s_j < \tau < s_{j+1} < \cdots < s_k < t\}.$$



The sets $\Delta_{k;j}$ for $k=2$.

$$\Delta_{2;0} = \{(s_1,s_2) : 0<\tau<s_1<s_2<t\}$$
$$\Delta_{2;1} = \{(s_1,s_2) : 0<s_1<\tau<s_2<t\}$$
$$\Delta_{2;2} = \{(s_1,s_2) : 0<s_1<s_2<\tau<t\}.$$



G7.5

Rks. 1) Time-ordering convention
(physical ordering and time-reversal)

2)

3) If $\omega = 0$ ($\&\ \mu = \ell =$ Lebesgue measure on $[0,t]$ $\&\ \lambda = -i$, $\Theta = -iV$)

recover the 'classical Dyson series':

$$\sum_{n=0}^{\infty} (-i)^n \int_{\Delta_n} e^{-i(t-s_n)H_0} V(s_n) \cdots \cdots V(s_2) e^{-i(s_2-s_1)} V(s_1) \, \mu(ds_1, \dots, ds_n)$$

$\mu = \ell$

where $\Delta_n = \{(s_1, \dots, s_n) : 0 < s_1 < \dots < s_n < t\}$.

Feynman Graph

Scatterings at times $s_1, \dots, s_n$

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Rk: In [L-4], a 'Feynman-Kac formula with Lebesgue-Stieltjes measure' is obtained:

1) for $\eta = \mu + \nu$ - finitely supported

($\nu = \sum_{p=1}^{k} \omega_p \delta_{\tau_p}$, $0 < \tau_1 < \dots < \tau_k < b$, $\omega_p \in \mathbb{C}$)

$u(t) = K_x^{t}(F)$ viewed as a function of time t

2) for $\eta = \mu + \nu$ - (countable support)

General Borel measure

$\nu = \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}$ ($\sum |\omega_p| < \infty$)

Integral equation

Differential equation.

(Distributional differential equation)

Product integral representation

$F=1$, $u(t) = K_x^{t}(F)$

that make very concrete the notion of 'shocks' or 'instantaneous interaction'.

N1

Back to the algebra $\mathcal{A}_t$:

"DISENTANGLING ALGEBRA"

Examples of functionals:

$$f\left(\int_0^t \Theta_1(s,x(s))\,\eta_1(ds), \ldots, \int_0^t \Theta_k(s,x(s))\,\eta_k(ds), \ldots\right)$$

analytic (possibly a fct. of ∞-many variables)

$C^1([0,t],\mathbb{R}^N)$

$F: \mathcal{C}^t \to \mathbb{C}$

General form: (of $F \in \mathcal{A}_t$)

$F: \mathcal{C}^t \to \mathbb{C}$

$$F(x) = \sum_{m=0}^{\infty} F_m(x)$$

each F_m of the form:

Borel measures $[0,t]$

$$F_m(x) = \prod_{u=1}^{m_n} \int_0^t \Theta_{m,u}(s,x(s))\,\eta_{m,u}(ds)$$

$$\left(\text{with } \sum_{m=0}^{\infty} \prod_{u=1}^{m_n} \int_0^t \|\Theta_{m,u}(s,\cdot)\|_\infty\,|\eta_{m,u}|(ds) < \infty\right).$$

norm in $\mathcal{A}_t$:

$\|F\|_t = \inf\{\text{expressions}\}$ over all representations of F of the above form

Rk.: Maeno - kinetic technique

$\to$ equivalence classes of functionals

Rm. (cont.) equivalence relation: $F \sim G$ if $\forall \lambda > 0$ N2

Transl.:

$$F(\lambda^{-1/2} x + \xi) = G(\lambda^{-1/2} x + \xi)$$

for m×Leb.-a.e. (x, ξ) in $\mathcal{C}_0^t \times \mathbb{R}^N$.

Rm 3: Use pointwise operations only.

Thm.: (1) $\mathcal{A}_t$ is a commutative $*$-Banach algebra.

'Disentangling algebras'

Thm.: (2) Given any $F \in \mathcal{A}_t$, the analytic (operator-valued) Feynman integral $K_\lambda^t(F)$ exists and can be 'disentangled' via a 'generalized Dyson series' (GDS) (or time-ordered perturbation series).

Moreover, $\forall \lambda$, $\|K_\lambda^t(F)\| \leq |||F|||_t$.

Rm: 1) $*$-operation on $\mathcal{A}_t$. ($\eta \mapsto \bar{\eta}$, $\lambda \mapsto \bar{\lambda}$, $s \mapsto t-s$)

2) Non-uniqueness of the 'disentangling process'.

Rm: Recall:

Feynman (1951, (about the 'disentangling process'):

"The process is not always easy to ~~define~~ perform, and, in fact, is the central problem of this operator calculus".

Here, we use path integration (followed by analytic continuation & passage to a limit) to justify the disentangling process.

Rm: Earlier (related) works: Cameron & Storvick, Johnson & Skoug (smaller algebra of fcts., no direct link with F's op. calculus).

$$F \in \underset{\text{commutative}}{\mathcal{A}_t} \xmapsto{K_\lambda^t} K_\lambda^t(F) \in \underset{\text{noncommutative}}{\mathcal{L}(L^2(\mathbb{R}^n))}$$

$$K_\lambda^t : \mathcal{A}_t \longrightarrow \mathcal{L}(\mathcal{H}) \;;\; \mathcal{H} := L^2(\mathbb{R}^N)$$

$$F \longmapsto K_\lambda^t(F)$$

commutative algebra of functions — noncommutative algebra.

K_λ^t is a bounded linear operator, but is not an algebra homomorphism

($K_\lambda^t(F \cdot G) \neq K_\lambda^t(F) \cdot K_\lambda^t(G)$, in general). (pointwise product of functions)

Example: $K_\lambda^t(1) = e^{-t(H_0/\lambda)}$

$$K_\lambda^t(1) = K_\lambda^t(1 \cdot 1) \qquad K_\lambda^t(1) \cdot K_\lambda^t(1)$$

$$e^{-t(H_0/\lambda)} \neq e^{-2t(H_0/\lambda)} = e^{-t(H_0/\lambda)}\, e^{-t(H_0/\lambda)}$$

Goal: Introduce a noncommutative multiplication $*$ (on $\{\mathcal{A}_t\}_{t\geq 0}$) so that the above defect is remedied.

([JL2], G.W. Johnson & MLL, J. Funct. Anal. 1988)

In fact, we introduce two (on Wiener functionals) noncommutative operations $*$ ('noncommutative multiplication') and $\dot{+}$ ('noncommutative addition'!) so that, in particular, if $F \in \mathcal{A}_{t_1}$ and $G \in \mathcal{A}_{t_2}$, then

$$F * G \in \mathcal{A}_{t_1+t_2} \quad (\& \ F \dot{+} G \in \mathcal{A}_{t_1+t_2})$$

and

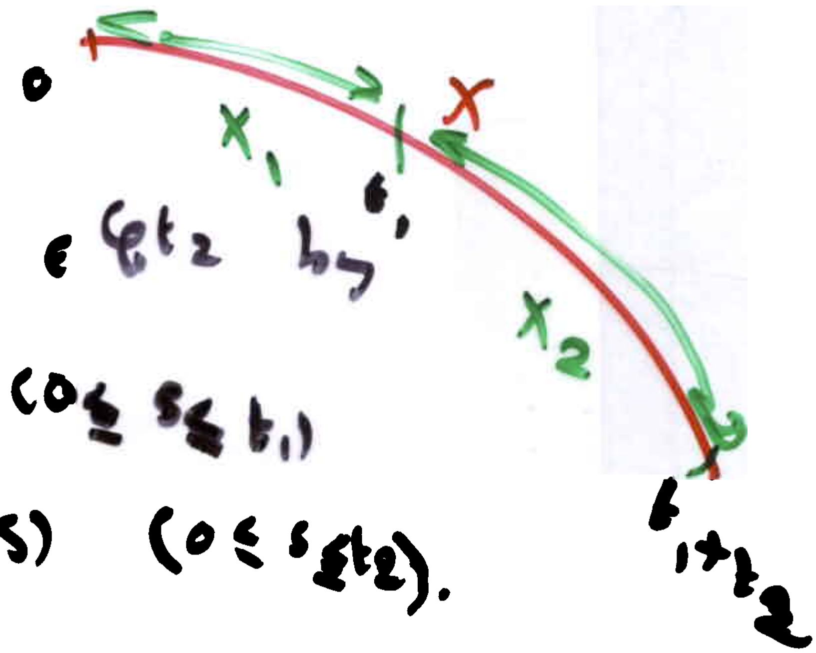
$$K_\lambda^{t_1+t_2}(F * G) = K_\lambda^{t_1}(F)\, K_\lambda^{t_2}(G).$$

Recall that $\mathcal{C}^t = C([0,t], \mathbb{R}^N)$.

Noncommutative operations [JL2]

$$F: \mathcal{C}^{t_1} \to \mathbb{C}, \qquad G: \mathcal{C}^{t_2} \to \mathbb{C}.$$

Let $x \in \mathcal{C}^{t_1+t_2}$.



Define $x_1 \in \mathcal{C}^{t_1}$ and $x_2 \in \mathcal{C}^{t_2}$ by

$$x_1(s) = x(s) \quad (0 \le s \le t_1)$$

& $$x_2(s) = x(t_1+s) \quad (0 \le s \le t_2).$$

Then

$$F * G : \mathcal{C}^{t_1+t_2} \longrightarrow \mathbb{C}$$

is defined by

$$\boxed{(F * G)(x) \stackrel{\text{def.}}{=} F(x_1)\, G(x_2)}$$

(noncommutative multiplication)

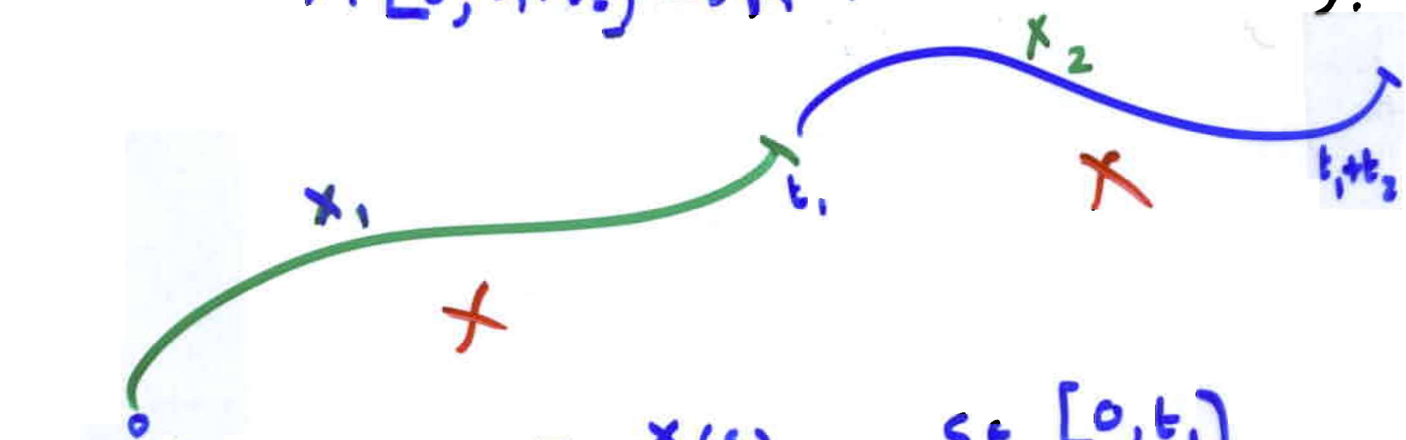
Similarly, $F \dot{+} G : \mathcal{C}^{t_1+t_2} \longrightarrow \mathbb{C}$

$$\boxed{(F \dot{+} G)(x) \stackrel{\text{def.}}{=} F(x_1) + G(x_2)}$$

(noncommutative addition)

$t_1, t_2 > 0$. **In summary:** (N7)

$x : [0, t_1+t_2] \to \mathbb{R}^N$. $(x \in \mathcal{C}^{t_1+t_2})$.



$x_1 \in \mathcal{C}^{t_1}$; $\quad x_1(s) = x(s)$, $s \in [0, t_1]$.

$x_2 \in \mathcal{C}^{t_2}$; $\quad x_2(s) = x(t_1+s)$, $s \in [t_1, t_1+t_2]$.

$F : \mathcal{C}^{t_1} \to \mathbb{C}$, $G : \mathcal{C}^{t_2} \to \mathbb{C}$.

pointwise product

$$(F * G)(x) = F(x_1) \cdot G(x_2).$$

$$\& \quad (F \dotplus G)(x) = F(x_1) + G(x_2).$$

pointwise addition

Rk.: Actually, due to measure-theoretic technicalities, we work with equivalence classes of Wiener functionals. Of course, the above operations are compatible with this eq. rel.

$$\left(\begin{array}{l} [F] * [G] = [F * G] \\ [F] \dotplus [G] = [F \dotplus G] \end{array} \right)$$

As before:

$F \sim G$ iff $\forall \lambda > 0$, $F(\lambda^{-1/2} x + \xi) = G(\lambda^{-1/2} x + \xi)$ f.a. $m \times$ leb. $(x, \xi) \in \mathcal{C}_0^t \times \mathbb{R}^N$

The Disentangling Algebras $\{\mathcal{A}_t\}_{t>0}$ and the Noncommutative Operations $*$ and $\dot{+}$.

$(\mathcal{A}_t, |||\cdot|||_t)$, commutative Banach algebra

These ops. are compatible with the Banach algebra structure of the $\mathcal{A}_t$'s.:

Thm. 1: If $F \in \mathcal{A}_{t_1}$ and $G \in \mathcal{A}_{t_2}$, then $F * G$ and $F \dot{+} G$ are in $\mathcal{A}_{t_1+t_2}$; further,

$$|||F * G|||_{t_1+t_2} \leq |||F|||_{t_1} \, |||G|||_{t_2}$$

(norm on $\mathcal{A}_t$)

and

$$|||F \dot{+} G|||_{t_1+t_2} \leq |||F|||_{t_1} + |||G|||_{t_2}.$$

Rk.: Also: If $F_j \in \mathcal{A}_{t_j}$ $(j = 1, \dots, n)$, then $F_1 * \cdots * F_n$ & $F_1 \dot{+} \cdots \dot{+} F_n$ are in $\mathcal{A}_{t_1+\cdots+t_n}$ and the 'associative law' holds.

(Nb)

Main Theorem:

Thm. 2 [JL2]. If $F \in \mathcal{A}_{t_1}$ and $G \in \mathcal{A}_{t_2}$, then for all $\lambda \in \tilde{\mathbb{C}}_+$, $K_\lambda^{t_1}(F)$, $K_\lambda^{t_2}(G)$ and $K_\lambda^{t_1+t_2}(F*G)$ exist and

$$\boxed{K_\lambda^{t_1+t_2}(F*G) = K_\lambda^{t_1}(F)K_\lambda^{t_2}(G).}$$

Cor. a. If $F \in \mathcal{A}_{t_1}$ & $G \in \mathcal{A}_{t_2}$, then for all $\lambda \in \tilde{\mathbb{C}}_+$, $K_\lambda^{t_1}(F)K_\lambda^{t_2}(G)$ can be disentangled via a generalized Dyson series (G-DS) for $K_\lambda^{t_1+t_2}(F*G)$.

Cor. 6. Let $F \in \mathcal{A}_{t_1}$ & $G \in \mathcal{A}_{t_2}$.

Then $[F,G] := F * G - G * F$ (commutator of F&G)

is in $\mathcal{A}_{t_1+t_2}$ and, for all $\lambda \in \mathbb{C}_+^2$,

$$K_\lambda^{t_1+t_2}([F,G]) = [K_\lambda^{t_1}(F), K_\lambda^{t_2}(G)]$$

commutator of bd. operators $\left(:= K_\lambda^{t_1}(F) K_\lambda^{t_2}(G) - K_\lambda^{t_2}(G) K_\lambda^{t_1}(F) \in \mathcal{L}(\mathcal{H}) \right)$.

Roles of the operations $\dotplus$ and $*$ combined:

Thm. 3. Let $F \in \mathcal{A}_{t_1}$ & $G \in \mathcal{A}_{t_2}$.

Then the following equality holds in $\mathcal{A}_{t_1+t_2}$:

(1) $\boxed{\exp(F \dotplus G) = \exp(F) * \exp(G)}$;

further, for all $\lambda \in \mathbb{C}_+^2$:

(2) $K_\lambda^{t_1+t_2}(\exp(F \dotplus G)) = K_\lambda^{t_1}(\exp(F))\, K_\lambda^{t_2}(\exp(G))$

Rk.: Compare with Feynman's 'paradoxical formula' " $e^{A+B} = e^A . e^B$ ".

Here, (in (1),) we work with functionals ("symbols") and not operators.
We then deduce a corresponding formula for operators; see (2).

Note then in (1) we use the noncommutative addition $\dot{+}$ (and multiplication $*$).

exponential defined via the functional calculus in the Banach algebra $\mathcal{A}_{t_1+t_2}$.

$$e^{F \dot{+} G} = e^F * e^G.$$

Cons. For $F \in \mathcal{A}_{t_1}$ and $G \in \mathcal{A}_{t_2}$,

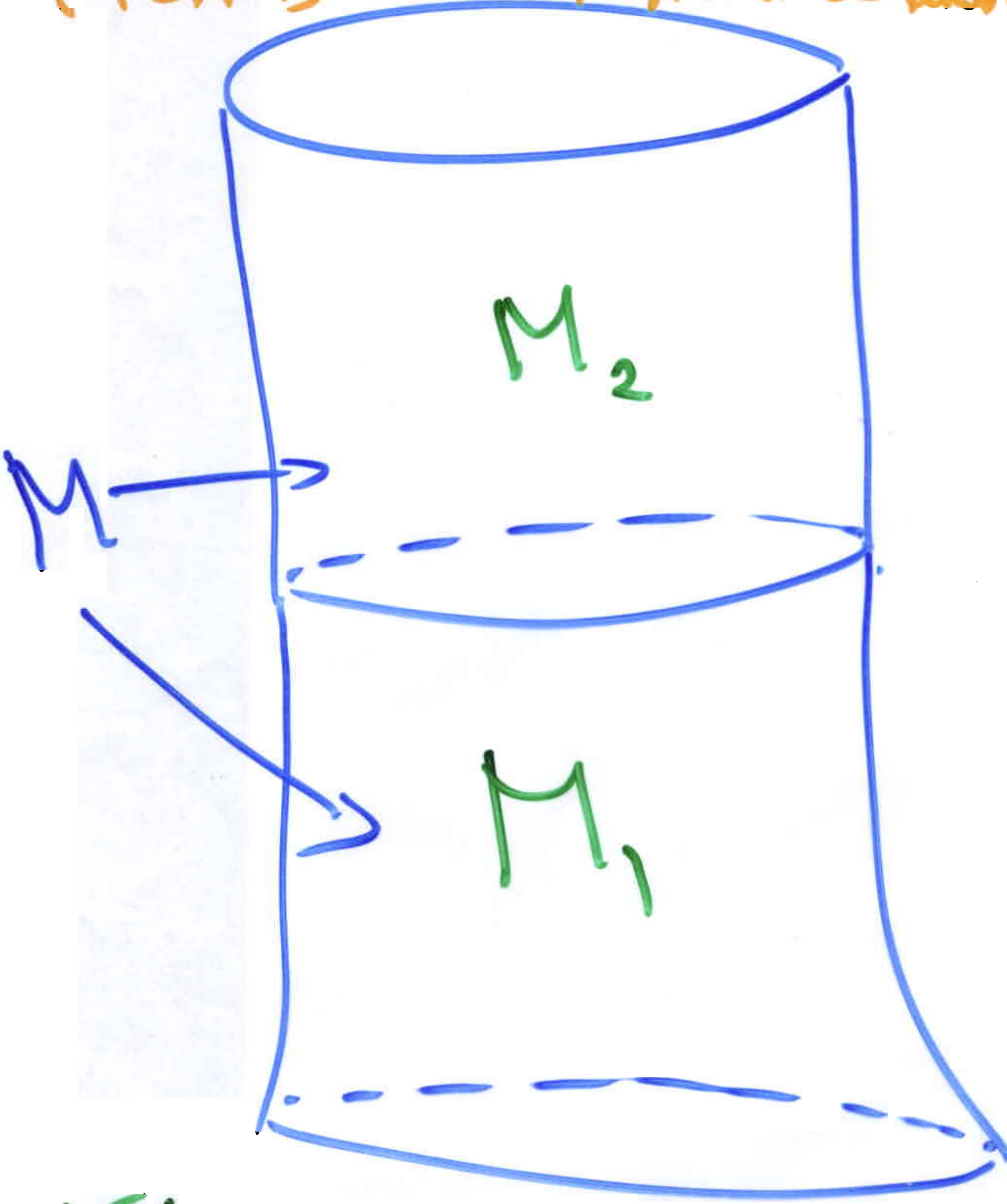
$$[\exp(F), \exp(G)] = \exp(F \dot{+} G) - \exp(G \dot{+} F)$$

and

$$[K_\lambda^{t_1}(\exp(F)), K_\lambda^{t_2}(\exp(G))] = K_\lambda^{t_1+t_2}(\exp(F \dot{+} G)) - K_\lambda^{t_1+t_2}(\exp(G \dot{+} F)).$$

Analogy:

(TQFT, Jones polynomial – knot invariant)



$$K(M) = K(M_1) \cdot K(M_2)$$