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FEYNMAN'S OPERATIONAL CALCULUS  
VIA FEYNMAN PATH INTEGRALS  
AND DISENTANGLING ALGEBRAS

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WORKSHOP ON FEYNMAN INTEGRALS  
ALONG WITH RELATED TOPICS  
AND APPLICATIONS  
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THE FEYNMAN INTEGRAL AND FEYNMAN'S OPERATIONAL CALCULUS

Gerald W. JOHNSON

Michel. L. LAPIDUS

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# THE FEYNMAN INTEGRAL AND FEYNMAN'S OPERATIONAL CALCULUS



Gerald W. Johnson, *Department of Mathematics and Statistics, University of Nebraska-Lincoln*,  
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The aim of this book is to make accessible to mathematicians, physicists and other scientists interested in quantum theory, the beautiful and closely related but mathematically difficult subjects of the Feynman integral and Feynman's operational calculus. Some advantages of the approaches to the Feynman integral which are treated in detail in this book are the following: the existence of the Feynman integral is established for very general potentials in all four cases; under more restrictive but still broad conditions, three of these Feynman integrals agree with one another and with the unitary group from the usual approach to quantum dynamics; these same three Feynman integrals possess pleasant stability properties. Much of the material covered here was previously available only in the research literature, and the book also contains several new results. The background material in mathematics and physics that motivates the study of the Feynman integral and Feynman's operational calculus is discussed, and detailed proofs are provided for the central results. The last chapter of the book includes a discussion of topics in physics and mathematics (including knot theory) where heuristic Feynman integrals have played a significant role. In some cases, perturbation series in the spirit of Feynman's operational calculus are the key objects.

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[Fe2] Richard P. FEYNMAN

« An operator calculus having applications in quantum electrodynamics »

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(Chen 14-19) F's Op. Calculus & rel. topics  
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with Noncommutative Auxiliary Operations,  
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by

MASLOV  
(1976)

NELSON  
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## FUNCTION SPACE APPROACH:

(used in our approach but not directly connected with F's DG.)

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# Plan of this Talk:

1. Heuristic Introduction to Feynman's Operational Calculus.  
(with some motivations)

2. A Mathematical Approach to Feynman's Operational Calculus:

A. [JL1] via Wiener and Feynman (analytic) (path) integrals.

- generalized Dyson series (GDS) and the 'disentangling process.'
- 'disentangling algebras' (brief comments on [LW-8]).

B. [JL2]

- Noncommutative operations  $\#$  and  $\epsilon$  on the space of Wiener functionals.
- Links with disentangling algebras and resolution of certain integrals.
- 'formulas' in the context of paradoxical

(1) time domain

3. a. Comments on the more abstract approach to generalized path integrals. ([LFT], [16], [12], [13])  
Links with Voiculescu's noncommutative probability theory.

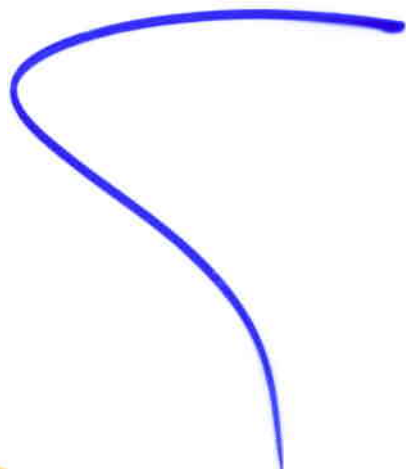
b. Future work: Links with 'curves' in noncommutative geometry.



In one sentence:

Feynman's operational calculus

Consists in treating noncommuting  
operators as though  
they were commuting. (!:)



$AB = BA$

$(= \frac{1}{2}(AB + BA))$



$e^{A+B} = e^A e^B$



# Motivations

## ① Physical Motivations:

1a. Quantum Mechanics

1b. Applications to Quantum Electrodynamics (QED)

- Schwinger (1949)
- Tomonaga (1948)
- Feynman (1948-50)

approaches to QED connected by Dyson's eq. by "means"

of a re-ordering of operators in each term of a perturbation power series"  
("Dyson series")

"There may be some hope that a thorough understanding of electrodynamics might give a clue as to the possible structure of the more complete theory to which it is an approximation."

"It might be worth [...] expressing [QED] in every possible physical and mathematical way."

## ② Purely Mathematical Reasons

"A second reason is to describe a mathematical method which may be useful in other fields!"

"The mathematics is not completely satisfactory. No attempt has been made to maintain mathematical rigor. The excuse is not that it is expected that rigorous demonstrations can be easily supplied.

Quite the contrary, it is believed that to put the present methods on a rigorous basis may be quite a difficult task, beyond the abilities of the author."

⊛ The quotes are from Feynman's 1951 paper. [Fe2].

Elsewhere asks the right for a physicist to use poetic license

and to be freed from the rules of grammar, grammar, ...

S. Schwinger ("Feynman and the visualization of space-time processes"):

"These papers, on the space-time approach to nonrelativistic quantum mechanics [1948], on quantum electrodynamics [1949-51, including the 'Feynman diagrams'] [1951], and on his operator calculus [1951], must surely be placed near the top of any list of the most influential seminal work during the twentieth century."

papers

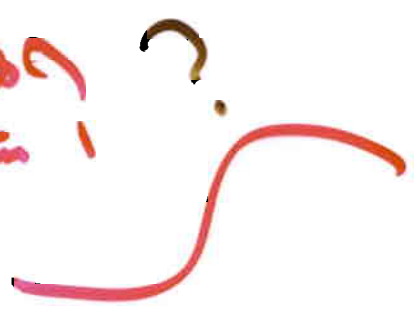
Schwinger again:

"In any case, Feynman never felt order-by-order was anything but an approximation to the 'thing' and the 'thing' was the path integral."

criticism: substitute p. -  
general path integral

Feynman's path integral

Dyson's  
 to the order  
 perturbation  
 series

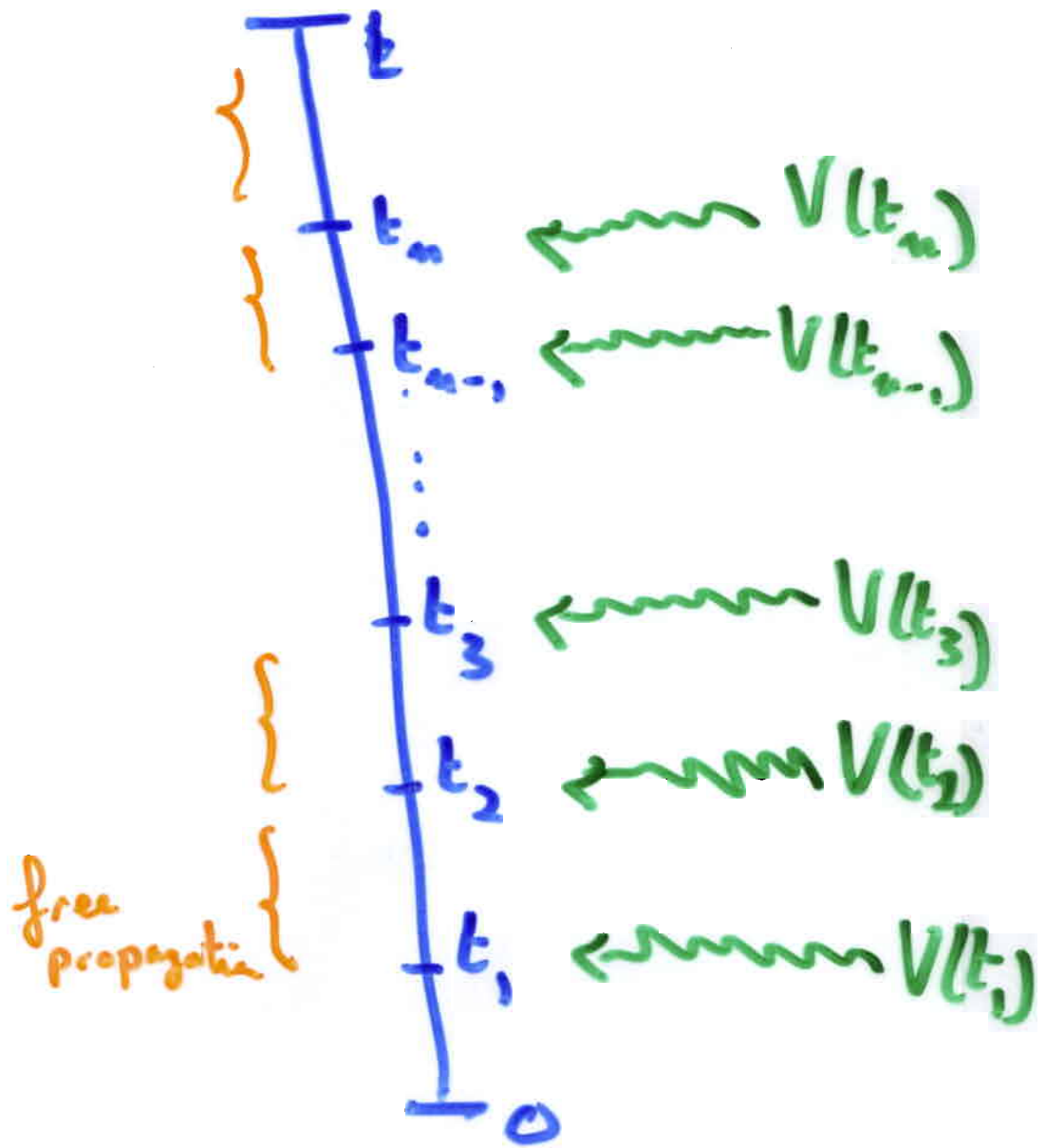


$$\int e^{\frac{i}{\hbar} S(x)} \mathcal{D}x$$

$$\sum_{n=0}^{\infty} \frac{1}{i^n} \int_{\Delta_n} e^{-\frac{i}{\hbar}(t-b_n)H_0} V(s_n) e^{-\frac{i}{\hbar}(s_n-s_{n-1})H_0} V(s_{n-1}) e^{-\frac{i}{\hbar}(s_{n-1}-s_{n-2})H_0} \dots V(s_2) e^{-\frac{i}{\hbar}(s_2-s_1)H_0} V(s_1) e^{-\frac{i}{\hbar}(s_1-s_0)H_0} ds_1 \dots ds_n$$

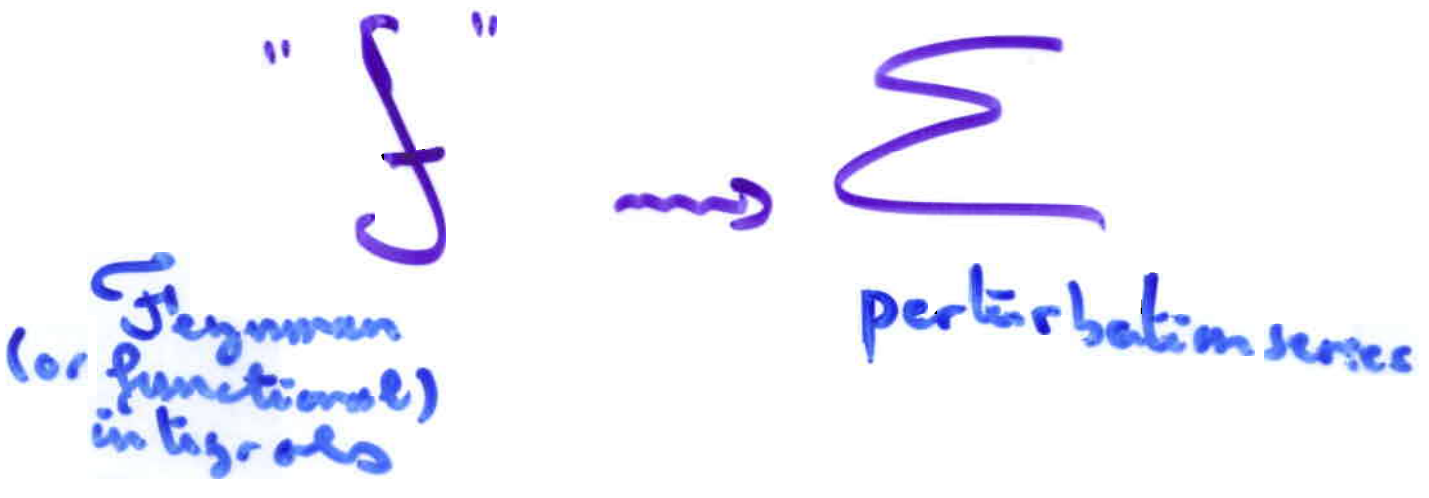
$$\Delta_n = \{(s_1, \dots, s_n) \in \mathbb{R}^n : 0 \leq s_1 \leq \dots \leq s_n\}$$

M41(2)



(classical) Feynman diagram  
(for the  $n$ th term of the  
Dyson series)

Perhaps Feynman's operational  
calculus could be viewed as  
a generalized (Wiener or Feynman)  
path integral, or as a substitute  
for it.



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# Feynman's Operator Calculus: A Heuristic Introduction

## Feynman's rules

Following him to calculate (heuristically) time-ordered perturbation series (for instance) without actually having a path integral):

For "time-ordering convention"

Rule 1 1. Attach "time" indices to the operators involved in order to specify the order of operations in products.

Rule 2 2. With indices attached, form functions of the operators by treating them as though they commuted.

Rule 3 3. Finally, "disentangle" the resulting expressions, that is, restore the conventional ordering of the operators.

"DISENTANGLING PROCESS"

disentangling process"



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Specifically:  
Feynman's "time-ordering convention":

A, B

two "operators",  
(non-commuting)

Step 1:

$$A(s_1) B(s_2) = \begin{cases} BA & \text{if } s_1 < s_2 \\ AB & \text{if } s_2 < s_1 \\ \text{undefined} & \text{if } s_1 = s_2 \end{cases}$$

attach time-indices

Step 2: Now, treat the operators as though they commuted (but respect the rule (\*)!).

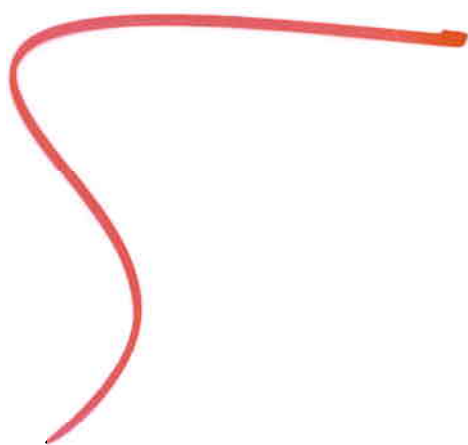
"disentangling process"

Step 3: Finally, "disentangle" the resulting expressions; i.e., restore the conventional ordering of the operators. I go back to the real world!

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Feynman ([Fe 2], 1951) about the  
'disentangling process':

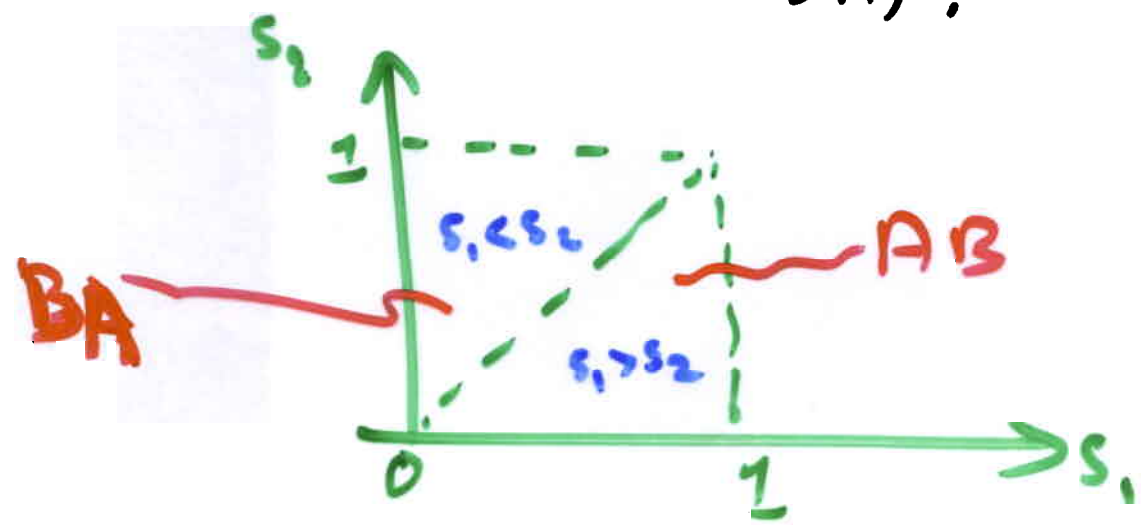
" The process is not always  
easy to perform, and in fact, is  
the central problem of  
this operator calculus".



# Simple Example:



$$\begin{aligned}
 "A \cdot B" &= \left( \int_0^1 A(s_1) ds_1 \right) \left( \int_0^1 B(s_2) ds_2 \right) \\
 &= \iint_{[0,1] \times [0,1]} A(s_1) B(s_2) ds_1 ds_2 \\
 &= \iint_{s_1 < s_2} A \cdot B + \iint_{s_1 > s_2} B A ds' \\
 &= \frac{1}{2} AB + \frac{1}{2} BA \\
 &= \frac{1}{2} (AB + BA)
 \end{aligned}$$



Other examples:

$$\begin{aligned}
 e^A \cdot B &= e^{\int_0^1 A(s_1) ds_1} \int_0^1 B(s_2) ds_2 \\
 &= \int_0^1 e^{\int_{s_2}^1 A(s_1) ds_1} B(s_2) e^{\int_0^{s_2} A(s_1) ds_1} ds_2 \\
 &= \int_0^1 e^{(1-s_2)A} B e^{s_2 A} ds_2 \\
 &= \int_0^1 e^{(1-s)A} B e^{sA} ds.
 \end{aligned}$$

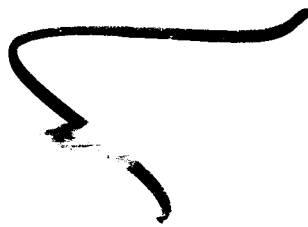
$$\begin{aligned}
 e^A \cdot B^2 &= e^{\int_0^1 A(s_1) ds_1} \iint_0^1 B(s_1) B(s_2) ds_1 ds_2 \\
 &= \iint_{s_1 < s_2} e^{\int_{s_2}^1 A(s_1) ds_1} B(s_2) e^{\int_{s_1}^{s_2} A(s_1) ds_1} B(s_1) \\
 &\quad + \iint_{s_2 < s_1} e^{\int_0^{s_2} A(s_1) ds_1} B(s_1) e^{\int_{s_2}^{s_1} A(s_1) ds_1} B(s_2) \\
 &= \iint_{s_1 < s_2} e^{(1-s_2)A} B e^{(s_2-s_1)A} B e^{s_1 A} ds_1 ds_2 \\
 &\quad + \iint_{s_2 < s_1} \dots
 \end{aligned}$$

M10)

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"  
Paradoxical

formulas:



e.g.,

$$e^{A+B} = e^A \cdot e^B$$

even if  $A$  &  $B$  do not commute! (???)

# Time-Ordered Perturbation Series

$$e^{\left\{ -t A + \int_0^t B(s) ds \right\}} = \exp \left\{ -t A + \int_0^t B(s) ds \right\}$$

rule (1) "attach time indices"

$$= \exp \left\{ - \int_0^t A(s) ds \right\} \int_0^t B(s) ds$$

rule (2) "treat the operators as though they were commuting"

$$= \exp \left\{ - \int_0^t A(s) ds \right\} \exp \left\{ \int_0^t B(s) ds \right\}$$

$$= \exp \left\{ - \int_0^t A(s) ds \right\} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \int_0^t B(s) ds \right\}^n$$

$$= \sum_{n=0}^{\infty} \exp \left\{ - \int_0^t A(s) ds \right\} \frac{1}{n!} \left( \int_0^t B(s) ds \right)^n$$

⊕ Wait until the "disentangling process" is completed in order to interpret the expressions involved in the standard way.!!

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Next, write

$$\begin{aligned} \left( \int_0^t B(s) ds \right)^n &= \left( \int_0^t B(s_1) ds_1 \right) \cdots \left( \int_0^t B(s_n) ds_n \right) \\ &= \int_{[0,t]^n} B(s_n) \cdots B(s_1) ds_1 \cdots ds_n \\ &= n! \int_{\Delta_n(t)} B(s_n) \cdots B(s_1) ds_1 \cdots ds_n \end{aligned}$$

recall:  
 The operators  
 are described  
 to be commuting  
 (It's a wonderful  
 world)

where

$$\Delta_n = \Delta_n(t) = \{ (s_1, \dots, s_n) : 0 < s_1 < \dots < s_n \leq t \}$$

Further, write

$$\begin{aligned} & \exp \left\{ - \int_0^t A(s) ds + \int_0^t B(s) ds \right\} \\ &= \sum_{n=0}^{\infty} \int_{\Delta_n} \exp \left\{ - \int_0^t A(s) ds \right\} B(s_n) \cdots B(s_1) ds_1 \cdots ds_n \\ &= \sum_{n=0}^{\infty} \int_{\Delta_n} \exp \left\{ - \int_{s_n}^{s_{n+1}} A(s) ds \right\} B(s_n) \\ & \quad \exp \left\{ - \int_{s_{n-1}}^{s_n} A(s) ds \right\} B(s_{n-1}) \\ & \quad \vdots \\ & \quad \exp \left\{ - \int_{s_1}^{s_2} A(s) ds \right\} B(s_1) \exp \left\{ - \int_0^{s_1} A(s) ds \right\} \\ & \quad ds_1 \cdots ds_n \end{aligned}$$

rule of  
 ordering  
 "reverse" with  
 the conventional  
 ordering

M13) Remember that  $A(s) \equiv A$  to conclude:

$$\exp\left\{-tA + \int_0^t B(s) ds\right\}$$

$$= \sum_{n=0}^{\infty} \int \int \int \dots \int \left[ e^{-(t-s_n)A} B(s_n) e^{-(s_n-s_{n-1})A} B(s_{n-1}) \dots B(s_2) e^{-(s_2-s_1)A} B(s_1) e^{-s_1 A} \right]$$

$$ds_1 \dots ds_n.$$

(The disentangling is complete.)

time-ordered perturbation expansion

$$[A = iH_0, B = -iV$$

↪ classical Dyson series

$$A = H_0, B = -V$$

$V$ :  
(multiplication operator by a potential)

series can also be obtained by functional integration



Functional integrals permit the calculation of perturbation series. Example:

$$\begin{aligned}
(e^{-t(H_0+V)} \psi)(z) &= \int_{C_0^t}^{F-K} \exp\left\{-\int_0^t V(x(s)+z) ds\right\} \psi(x(t)+z) d\mu(x) \\
&= \int_{C_0^t} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left[ \int_0^t V(x(s)+z) ds \right]^m \psi(x(t)+z) d\mu(x) \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{C_0^t} \left[ \int_0^t \int_0^{A_m} \dots \int_0^{A_2} V(x(A_m)+z) \dots V(x(A_1)+z) ds_1 \dots ds_m \right] \\
&\quad \psi(x(t)+z) d\mu(x)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{\Delta_m(t)} \left[ e^{-(t-s_m)H_0} V e^{-(s_m-s_{m-1})H_0} V \dots V \right. \\
&\quad \left. \cdot e^{-(s_2-s_1)H_0} V e^{-s_1 H_0} \psi \right](z) ds_1 \dots ds_m.
\end{aligned}$$

Note: The above calculation is presented rather briefly & the equality '=' is slightly inaccurate.

$\Delta_m = \Delta_m(t) = \{(s_1, \dots, s_m) : 0 \leq s_1 < \dots < s_m \leq t\}.$   
 $C_0^t = \{x : [0, t] \rightarrow \mathbb{R}^n; x \text{ c.b.}, x(0) = 0\}.$   
 (path space or 'Wiener space')

In general the results are rather involved combinatorially (cf. [JL2, dfJL])

For the simple examples:

$$1. e^{\int_0^1 A(s) ds} + \int_0^1 B(s) d\delta_0(s) = e^A e^B$$

Dirac measure

whereas

$$2. e^{\int_0^1 A(s) d\delta_0(s) + \int_0^1 B(s) ds} = e^B e^A$$

$$3. e^{\int_0^1 A(s) ds + \int_0^1 B(s) d\delta_{1/2}(s)} = e^{\frac{1}{2}A} e^B e^{\frac{1}{2}A}$$

$$4. e^{\int_0^t A(s) ds + \int_0^t B(s) d\nu(s)} = \left( e^{(t/m)A} e^{(t/m)B} \right)^n$$

where

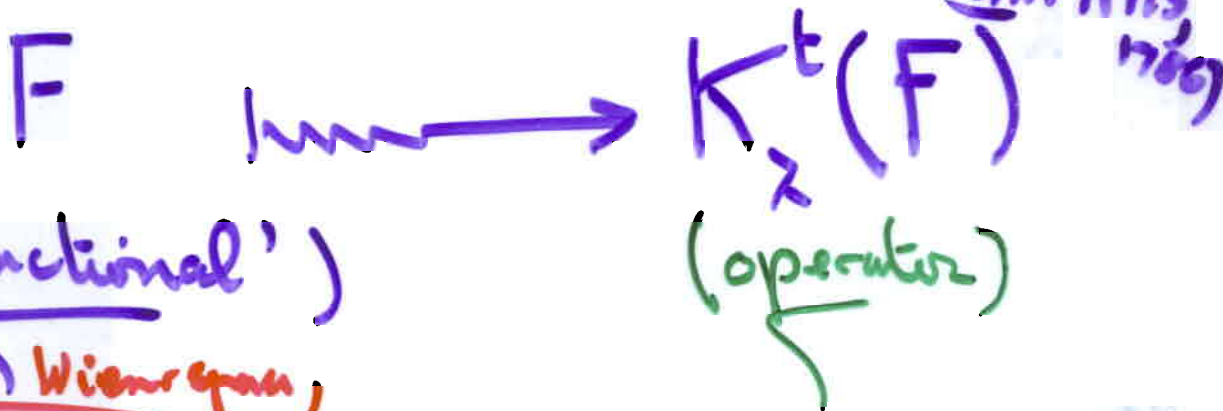
$$\nu = \frac{t}{n} \sum_{j=1}^n \delta_{j(t/m)}$$

$n$ -th Trotter product

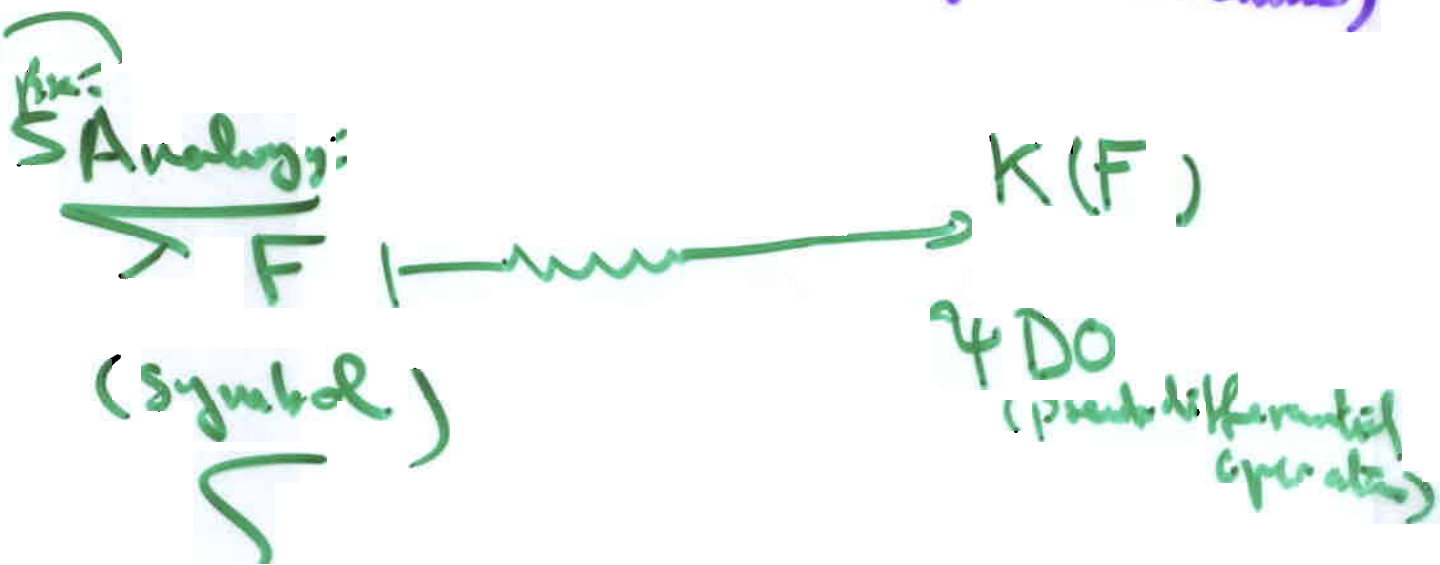
# Some Mathematical Approaches to Feynman's Operational Calculus

## I. Approach via Path Integrals

(M.L.L. & C.W.J. Mem. AHS 1966)



setting: Nonrelativistic QM (quantum mechanics):  
 use of Wiener <sup>path</sup> integrals (270)  
 and Feynman "path" integrals (2 imaginary)  
 (obtained by analytic continuation)



$$C_0^t = C_0([0, t], \mathbb{R}^N)$$

— Wiener space (2)

$m: m_t: \text{Wiener (prob.) measure on } C_0^t = \{x: [0, t] \xrightarrow{\text{continuous}} \mathbb{R}^N : x(0) = 0\}.$

$F: C^t \rightarrow \mathbb{C}$  "functional"  
 (suitable meas. function on Wiener space)

$$C^t = \{x: [0, t] \xrightarrow{\text{continuous}} \mathbb{R}^N\}$$

$$F \longmapsto K_\lambda^t(F) \in \mathcal{L}(L^2(\mathbb{R}^N))$$

$\lambda > 0.$

$(\xi \in \mathbb{R}^N, t > 0, \psi \in L^2(\mathbb{R}^N))$

$$(K_\lambda^t(F) \psi)(\xi) =$$

$$\int_{C_0^t} F(\lambda^{-1/2} x + \xi) \psi(\lambda^{-1/2} x(t) + \xi) dm_t(x)$$

$$= \mathbb{E}_\xi \left\{ F(\lambda^{-1/2} x) \psi(\lambda^{-1/2} x(t)) \right\}.$$

conditional expectation

Let  $\mathbb{C}_+^> = \{ \lambda \in \mathbb{C} : \text{Re } \lambda > 0, \lambda \neq 0 \}$ .

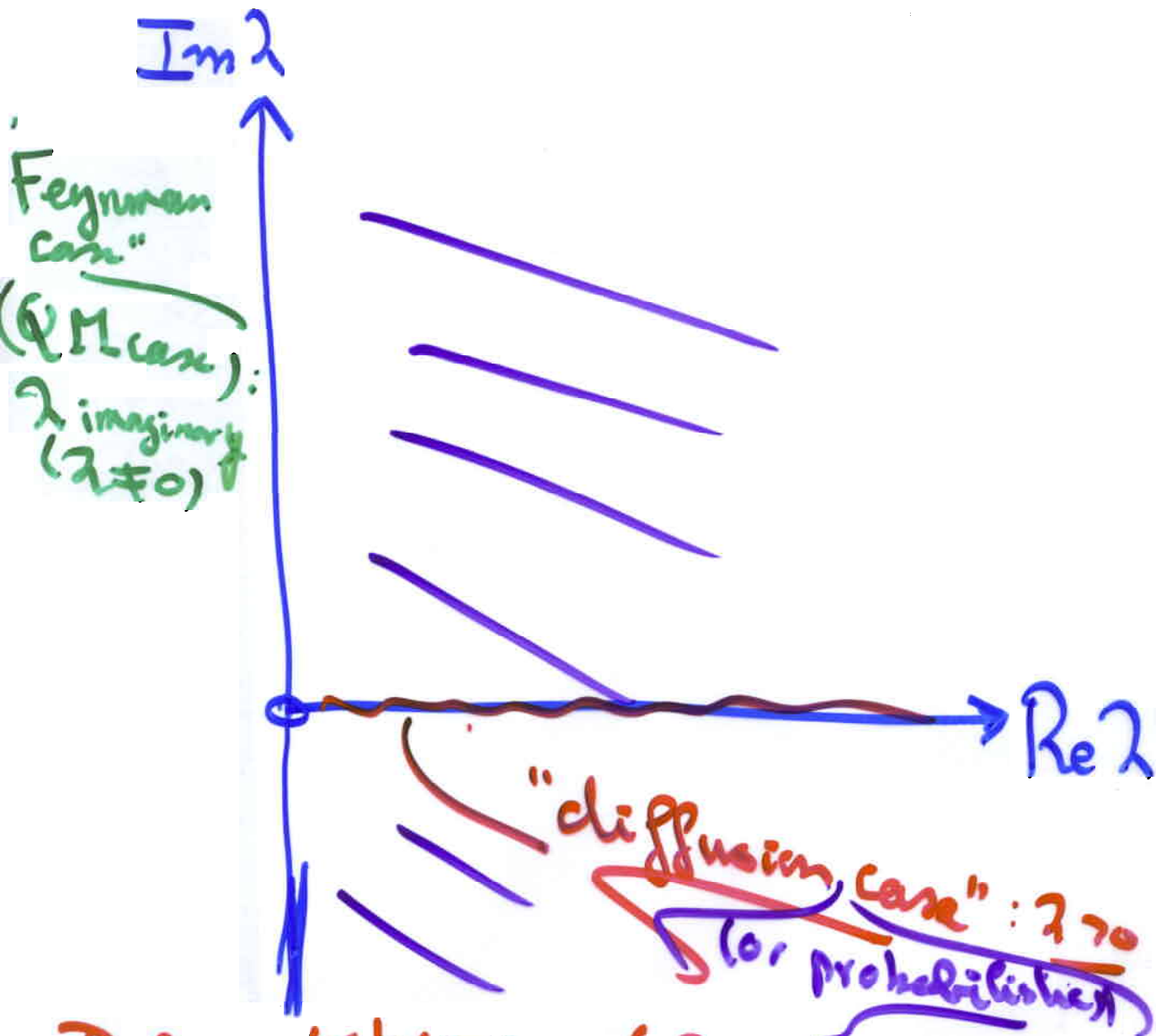
Then if  $K_\lambda^t(F)$  exists for all  $\lambda \in \mathbb{C}_+^>$ ,

$K_\lambda^t(F)$  is strongly continuous for  $\lambda \in \mathbb{C}_+^>$  and is analytic for  $\lambda \in \mathbb{C}_+$ .

- ②  $K_\lambda^t(F)$  ( $\lambda > 0$ ) is a functional integral given by a Wiener path integral.
- For  $\lambda \in \mathbb{C}_+$  (and a factor for  $\lambda$  purely imaginary),  $K_\lambda^t(F)$  is no longer given by a path integral. (cf. CAMERON)

For  $\lambda$  purely imaginary,  $K_\lambda^t(F)$  is called the (operator-valued) analytic Feynman integral of  $F$ . ("in mass")

(cf. CAMERON, NELSON, CAMERON-STORVICK, JOHNSON-SKOUGA, JOHNSON-LAFLORE, ...)



Def.:  $K_\lambda^t(F)$  ( $\text{Re } \lambda \geq 0, \lambda \neq 0$ ):

(a) The path integral  $K_\lambda^t(F)$  exists for all  $\lambda > 0$  and defines an operator in  $\mathcal{L}(L^2(\mathbb{R}^n))$

(b)  $\lambda \mapsto K_\lambda^t(F)$  can be analytically continued to the half-plane  $\text{Re } \lambda > 0$ .

(c)  $\lambda \mapsto K_\lambda^t(F)$  admits a strong limit along the imaginary axis (while  $\text{Re } \lambda > 0$ ).

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Each operator  $K_2^t(F)$  is  
 "disentangled" via a  
time-ordered perturbation series,  
 called a 'generalized Dyson series'  
 (GDS).

In fact, we construct a  
 family of algebras  $\{A_t\}_{t>0}$   
 (called the 'disentangling algebras'),  
 such that for each  $F \in \mathcal{F}_t$ ,  $K_2^t(F)$  exists  
 and can be disentangled via a GDS.

Note: Each  $A_t$  is a commutative Banach  
algebra.

# Examples of functionals allowed:

(54)

$$F(x) = \exp \left\{ \int_0^t \Theta(s, x(s)) \eta(ds) \right\}$$

$$(\Theta: [0, t] \times \mathbb{R}^N \rightarrow \mathbb{C}, \int_0^t \|\Theta(s, \cdot)\|_\infty |\eta|(ds) < \infty)$$

$\eta$ : Borel measure on  $[0, t]$  with  $\|\Theta\|_\infty < \infty$

Feynman-Kac functional with Lebesgue-Stieltjes measure  $\eta$ .

$$\eta = \nu + \nu'$$

continuous measure

discrete measure

(may have a singular part)



The use of Lebesgue-Stieltjes  
measures (like  $\gamma$ ) enables us  
to blend continuous and discrete  
structures.

- more complicated combinatorics <sup>and rules</sup> for the perturbation series (GNS).
- 'generalized Feynman diagrams'.

Example:  $(\nu(f(s))=0, \forall s)$  Continuous measure Dirac mass (6)

Special case:  $\gamma = \nu + \omega \delta_\tau$

$(\omega \in \mathbb{C}, 0 < \tau < t)$ .

Then:

$$K_2^t(F) = \sum_{n=0}^{\infty} \sum_{R=0}^n \frac{\omega^{n-R}}{(n-R)!} \sum_{j=0}^R$$

$$\left[ \int_{\Delta_{R,j}} \left\{ e^{-(t-s_R)(H_0/2)} \Theta(s_R) e^{-(s_2-s_{R-1})(H_0/2)} \right. \right.$$

$$\dots \Theta(s_{j+1}) e^{-(s_{j+1}-\tau)(H_0/2)}$$

$$\left. \left[ \Theta(\tau) \right]^{n-R} e^{-(\tau-s_j)(H_0/2)} \right.$$

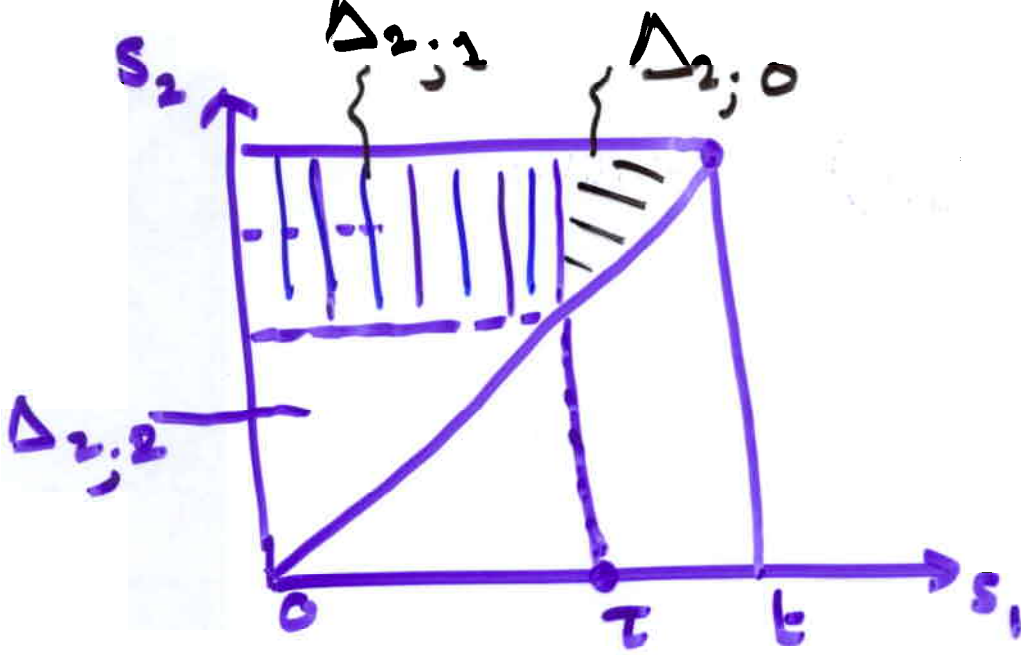
$$\left. \Theta(s_2) e^{-(s_2-s_1)(H_0/2)} \Theta(s_1) e^{-s_1(H_0/2)} \right\}$$

$$\nu(ds_1) \cdots \nu(ds_R) \Big],$$

where (for  $0 \leq j \leq R \leq n$ ),

$$\Delta_{R,j} = \left\{ (s_1, \dots, s_R) \in (0, t)^R : 0 < s_1 < \dots \right.$$

$$\left. s_j < \tau < s_{j+1} < \dots < s_R < t \right\}.$$

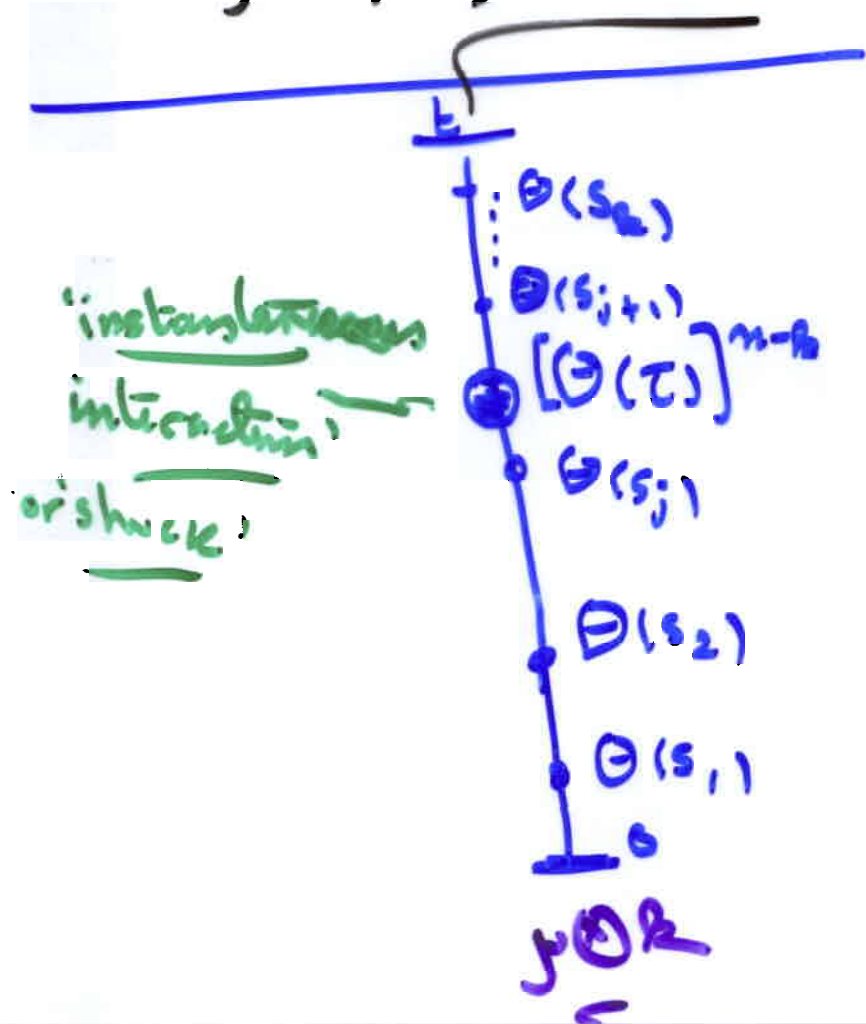


That's  $\Delta_{2,j}$  for  $k=2$ .

$$\Delta_{2,0} = \{(s_1, s_2) : 0 < \tau < s_1 < s_2 < t\}$$

$$\Delta_{2,1} = \{(s_1, s_2) : 0 < s_1 < \tau < s_2 < t\}$$

$$\Delta_{2,2} = \{(s_1, s_2) : 0 < s_1 < s_2 < \tau < t\}$$



'generalized Feynman diagrams' (circumventing to a general term in the G.P.S.)

RKS. , Time-ordering convention  
 (physical ordering and time-reversal)

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2)

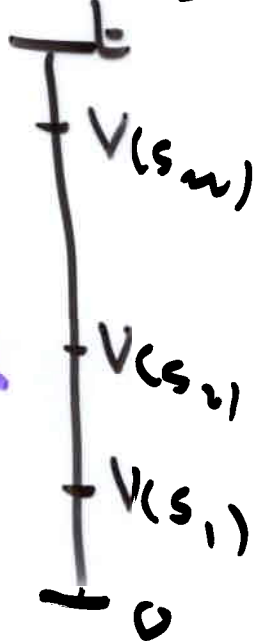
1) If  $\omega = 0$  ( $\alpha \nu = \ell = \text{Lebesgue measure on } [0, t]$ )

recover the 'classical Dyson series':

$$\sum_{n=0}^{\infty} (-i)^n \int_{\Delta_n} e^{-i(t-s_n)H_0} V(s_n) \dots V(s_2) e^{-i(s_2-s_1)H_0} V(s_1) \dots$$

where  $\Delta_n = \{(s_1, \dots, s_n) : 0 < s_1 < \dots < s_n < t\}$ .

Feynman graph



scatterings at times  $s_1, \dots, s_n$

Prk: In  $[L-4]$ , a Feynman-Kac formula with Lebesgue-Stieltjes measure is obtained: (68)

1) for  $\eta = \delta + \nu$  - locally supported

$$u(t) = K_x^{\eta}(F)$$

viensianc  
time t, substituted

( $\nu = \sum_{j=1}^n \omega_j \delta_{\tau_j}$ )  
 $\nu = \sum_{j=1}^n \omega_j \delta_{\tau_j}$   
 $0 < \tau_1 < \dots < \tau_n < b$   
 $\omega_j \in \mathbb{C}$

2) general  $\eta = \delta + \nu$   
Borel measure

(countable support)  
 $\nu = \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}$   
 $(\sum_{p=1}^{\infty} |\omega_p| < \infty)$

Integral equation

Differential equation

Distributional differential equation

Product integral representation

that make very easy the notation of 'stochastic' or 'instantaneous' integration!

$$\left\{ \begin{array}{l} F_x \\ u(t) \\ K_x^{\eta}(F) \end{array} \right.$$

# Back to the algebra $\mathcal{A}_t$ :

## "DISENTANGLING ALGEBRA"

Ex. ple of functions:

$$f\left(\int_0^t \Theta_1(s, x(s)) \gamma_1(ds), \dots, \int_0^t \Theta_k(s, x(s)) \gamma_k(ds)\right)$$

analytic  
(possibly a set of  $\infty$ -many variables)

$$F: \mathcal{C}^t \rightarrow \mathbb{C}$$

General form:  $F(x) = \sum_{n=0}^{\infty} F_n(x)$   
(of  $F \in \mathcal{A}_t$ )

each  $F_n$  of the form:

$$F_n(x) = \prod_{u=1}^{m_n} \int_0^t \Theta_{n,u}(s, x(s)) \gamma_{n,u}(ds)$$

(with  $\sum_{n=0}^{\infty} \int_0^t \prod_{k=1}^{m_n} \|\Theta_{n,k}(s, \cdot)\|_{\infty} |\gamma_{n,k}|(ds) < \infty$ )

Borel measurable  $(0, \theta)$

norm in  $\mathcal{A}_t$ :

$$\|F\|_t = \inf \{ \text{expression} \}$$

over all representations of  $F$  of the above form

Rx.: Meaner - theoretic techniques  
 $\equiv \rightarrow$  equivalence classes of functions

Pr. (cont.)  $\leftarrow$  equivalence relation:  $F \sim G$  if  $\forall \lambda > 0$  (N2)

~~Thm~~

$$F(\lambda^{-1/2} x + \xi) = G(\lambda^{-1/2} x + \xi)$$

for all  $x, \xi$  in  $\mathbb{C}_0^k \times \mathbb{R}^N$ .

Pr. 3: Use pointwise operation on  $\mathcal{F}$ .

Thm: (1)  $\mathcal{A}_\epsilon$  is a commutative  $\ast$ -Banach algebra.

Disentangling  
algebra

Thm: (2) Given any  $F \in \mathcal{A}_\epsilon$  (operator-valued), the analytic Feynman integral  $K_\epsilon^F(F)$  exists and can be

'disentangled' via a 'generalized

Dyson series (GOS) (or time-ordered perturbation series).

Moreover,  $\forall \lambda, \|K_\epsilon^F(F)\| \leq \|F\|_\epsilon$ .

Pr: (1)  $\ast$ -operation on  $\mathcal{A}_\epsilon$  ( $a \mapsto \bar{a}, \lambda \mapsto \bar{\lambda}$ )

(2) Non-uniqueness of the 'disentangling process'.

Rev:

Recall:

Feynman (1951), (about the 'disentangling process'):

"The process is not <sup>always</sup> easy to ~~define~~ perform, and, in fact, is the central problem of this operator calculus".



Here, we use path integration (followed by analytic continuation & passage to a limit) to justify the disentangling process.

Rev:

Earlier <sup>notes</sup> notes:

Cameron & Glöckle  
Johnson & Skov

(smaller algebra of paths, no direct link with F's op. calculus)





$$K_2^t: \mathcal{A}_t \longrightarrow \mathcal{L}(\mathcal{A}_t); \quad \mathcal{A}_t = L^2(\mathbb{R}^n)$$

$$F \longmapsto K_2^t(F)$$

commutative  
algebra of functions

noncommutative  
algebra.

$K_2^t$  is a bounded linear operator, but is not an algebra isomorphism

(  $K_2^t(F \cdot G) \neq K_2^t(F) \cdot K_2^t(G)$ ,  
in general )

*pointwise product of functions*

Example:  $K_2^t(1) = e^{-t(H_0/2)}$

$$K_2^t(1) = K_2^t(1 \cdot 1)$$

$$e^{-t(H_0/2)} \neq e^{-2t(H_0/2)} = e^{-t(H_0/2)} \cdot e^{-t(H_0/2)}$$

Goal: Introduce a noncommutative multiplication  $*$  (on  $\{\mathcal{A}_t\}_{t \geq 0}$ )  
so that the above defect is remedied

([JL2], G.W. Johnson & N.L.L. J. Funct. Anal. 1988)

In fact, we introduce two on Wiener functionals noncommutative operations  $*$

(‘noncommutative multiplication’) and  
+ (‘noncommutative addition’) so that,

in particular, if  $F \in \mathcal{A}_{t_1}$  and  $G \in \mathcal{A}_{t_2}$ ,  
then

$$F * G \in \mathcal{A}_{t_1 + t_2} \quad (\& F + G \in \mathcal{A}_{t_1 + t_2})$$

and

$$\kappa_{\lambda}^{t_1 + t_2}(F * G) = \kappa_{\lambda}^{t_1}(F) \kappa_{\lambda}^{t_2}(G).$$

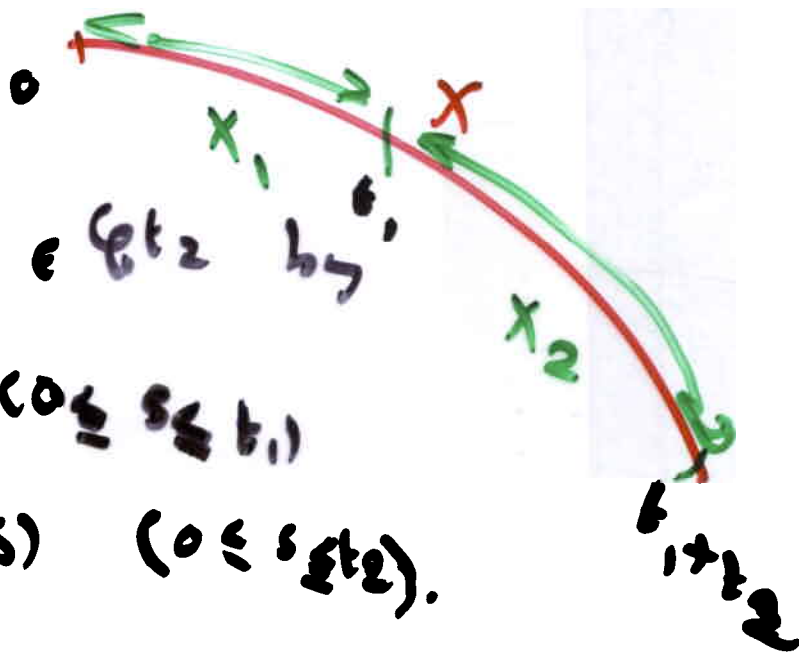
Recall that  $\mathcal{G}^t = C([0, t], \mathbb{R}^N)$ .

Non commutative operations

[JL2]

$$F: \mathcal{C}^{t_1} \rightarrow \mathbb{C}, \quad G: \mathcal{C}^{t_2} \rightarrow \mathbb{C}$$

Let  $x \in \mathcal{C}^{t_1+t_2}$ .



Define  $x_1 \in \mathcal{C}^{t_1}$  and  $x_2 \in \mathcal{C}^{t_2}$  by

$$x_1(s) = x(s) \quad (0 \leq s \leq t_1)$$

$$\text{and } x_2(s) = x(t_1+s) \quad (0 \leq s \leq t_2).$$

Then

$$F * G: \mathcal{C}^{t_1+t_2} \rightarrow \mathbb{C}$$

is defined by

$$(F * G)(x) \stackrel{\text{def.}}{=} F(x_1) G(x_2)$$

Similarly, non commutative multiplication

$$F + G: \mathcal{C}^{t_1+t_2} \rightarrow \mathbb{C}$$

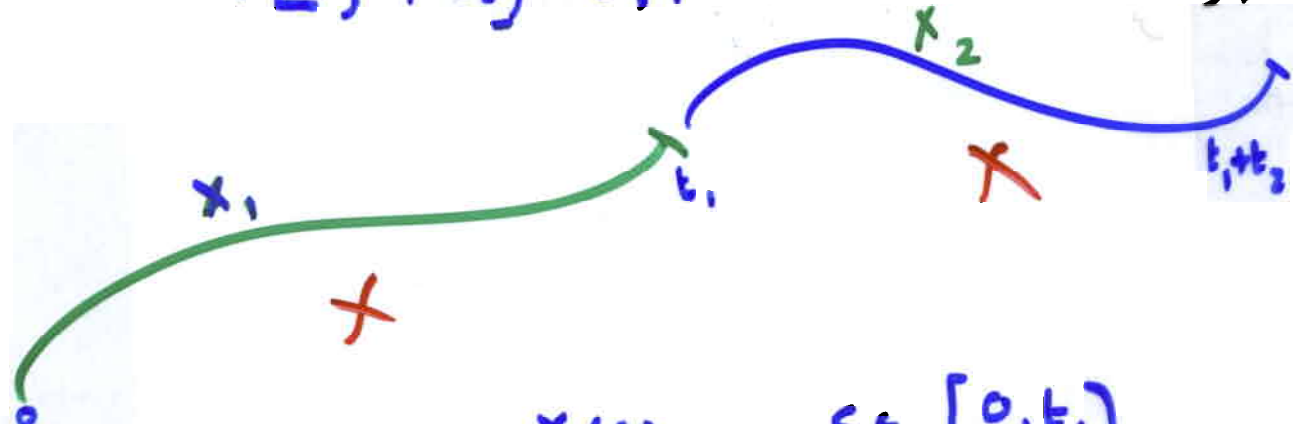
$$(F + G)(x) \stackrel{\text{def.}}{=} F(x_1) + G(x_2)$$

(non commutative addition)

$t_1, t_2 > 0$ .

In summary:

$x: [0, t_1+t_2] \rightarrow \mathbb{R}^N$ . ( $x \in \mathcal{C}^{t_1+t_2}$ )



$x_1 \in \mathcal{C}^{t_1}$ ,  $x_1(s) = x(s)$ ,  $s \in [0, t_1]$ .

$x_2 \in \mathcal{C}^{t_2}$ ,  $x_2(s) = x(t_1+s)$ ,  $s \in [0, t_2]$ .

$F: \mathcal{C}^{t_1} \rightarrow \mathbb{C}$ ,  $G: \mathcal{C}^{t_2} \rightarrow \mathbb{C}$ .

Pointwise product

$(F * G)(x) = F(x_1) \cdot G(x_2)$ .

$(F + G)(x) = F(x_1) + G(x_2)$ .

Pointwise addition

Rk.: Actually, due to measure-theoretic technicalities, we work with equivalence classes of Wiener functionals. Of course, the above operations are compatible with this eq. rel.

As before:

$F \sim G$  iff  $\forall \lambda > 0, \exists \delta > 0$ ,  $F(\lambda^{-1/2} x + \xi) = G(\lambda^{-1/2} x + \xi)$   
 for  $(x, \xi) \in \mathcal{C}^t \times \mathbb{R}^N$ .

$([F] * [G]) = [F * G]$   
 $([F] + [G]) = [F + G]$

# The Disentangling Algebras $\{A_t\}_{t>0}$ and the Noncommutative Operations $*$ and $+$

$(A_t, ||| \cdot |||_t)$ , commutative Banach algebras  
These ops. are compatible with the Banach algebra structure of the  $A_t$ 's.

Thm. 1. If  $F \in A_{t_1}$  and  $G \in A_{t_2}$ ,  
then  $F * G$  and  $F + G$  are in  $A_{t_1+t_2}$ ,  
further,

$$||| F * G |||_{t_1+t_2} \leq ||| F |||_{t_1} ||| G |||_{t_2}$$

, norm on  $A_t$

and

$$||| F + G |||_{t_1+t_2} \leq ||| F |||_{t_1} + ||| G |||_{t_2}$$

Ex:

Also: If  $F_j \in A_{t_j}$  ( $j=1, \dots, n$ ), then  
 $F_1 * \dots * F_n$  &  $F_1 + \dots + F_n$  are in  $A_{t_1+\dots+t_n}$   
and the associative (and) holds.

Main Theorem:

Thm. 2 [J12]. If  $F \in \mathcal{A}_{t_1}$  and  $G \in \mathcal{A}_{t_2}$

then for all  $\lambda \in \mathbb{C}_+^{\sim}$ ,  $K_{\lambda}^{t_1}(F)$ ,

$K_{\lambda}^{t_2}(G)$  and  $K_{\lambda}^{t_1+t_2}(F * G)$

exist and

$$K_{\lambda}^{t_1+t_2}(F * G) = K_{\lambda}^{t_1}(F) K_{\lambda}^{t_2}(G).$$

Cor. a. If  $F \in \mathcal{A}_{t_1}$  &  $G \in \mathcal{A}_{t_2}$ , then  
for all  $\lambda \in \mathbb{C}_+^{\sim}$ ,  $K_{\lambda}^{t_1}(F) K_{\lambda}^{t_2}(G)$

can be disentangled via a generalized  
Dyson series (G-DS) for  $K_{\lambda}^{t_1+t_2}(F * G)$ .

Cor. 6. Let  $F \in \mathcal{A}_{t_1}$  &  $G \in \mathcal{A}_{t_2}$ .

Then  $[F, G] = F * G - G * F$  is in  $\mathcal{A}_{t_1+t_2}$  and, for all  $\lambda \in \mathbb{C}_+^2$ ,

$$K_{\lambda}^{t_1+t_2}([F, G]) = [K_{\lambda}^{t_1}(F), K_{\lambda}^{t_2}(G)]$$

*Commutator of F & G*

$$(\equiv K_{\lambda}^{t_1}(F)K_{\lambda}^{t_2}(G) - K_{\lambda}^{t_2}(G)K_{\lambda}^{t_1}(F) \in \mathcal{L}(\mathcal{H}))$$

*Commutator of two operators*

Rules of  $\dagger$  and  $*$  combined:

Thm. 3. Let  $F \in \mathcal{A}_{t_1}$  &  $G \in \mathcal{A}_{t_2}$ . Then the following equality holds in  $\mathcal{A}_{t_1+t_2}$ :

(1)  $\exp(F \dagger G) = \exp(F) * \exp(G)$ ;

further, for all  $\lambda \in \mathbb{C}_+^2$ :

(2)  $K_{\lambda}^{t_1+t_2}(\exp(F \dagger G)) = K_{\lambda}^{t_1}(\exp(F)) K_{\lambda}^{t_2}(\exp(G))$

RK.: Compare with Feynman's 'paradoxical formula'  $e^{A+B} = e^A \cdot e^B$

Here, <sup>in (1),</sup> we work with functionals ("symbols") and not operators.

We then deduce a corresponding formula for operators; see (2).

Note that in (1) we use the

noncommutative addition  $\dagger$  (and multiplication  $\times$ ).

$$e^{F+G} = e^F \times e^G.$$

Corollary. For  $F \in \mathfrak{A}_{t_1}$ , and  $G \in \mathfrak{A}_{t_2}$ ,

$$[\exp(F), \exp(G)] = \exp(F+G) - \exp(G+F)$$

and  $[K_{\lambda}^{t_1}(\exp(F)), K_{\lambda}^{t_2}(\exp(G))]$

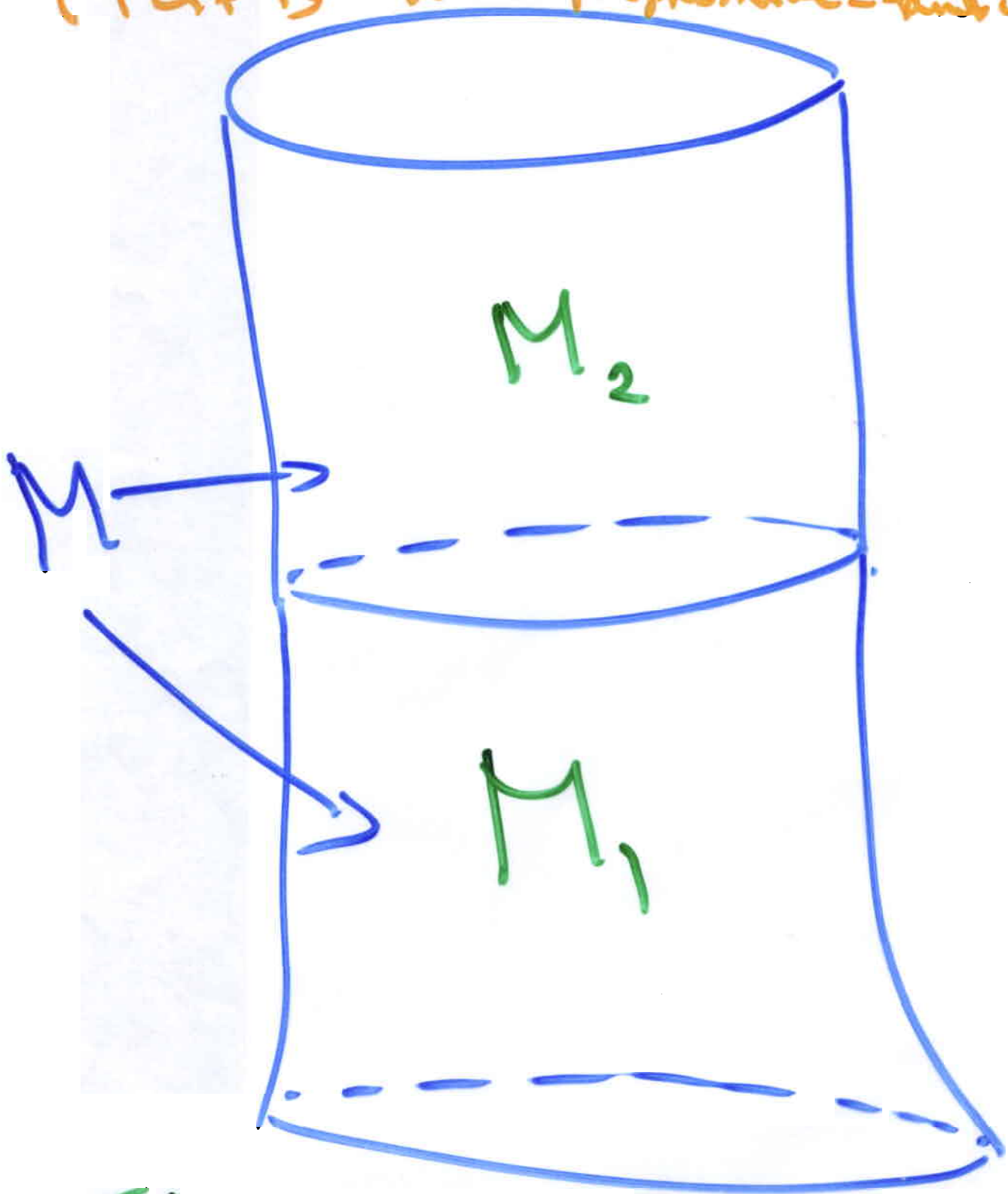
$$= K_{\lambda}^{t_1+t_2}(\exp(F+G)) - K_{\lambda}^{t_1+t_2}(\exp(G+F)).$$

Exponential defined via the functional calculus in the Banach algebra  $\mathfrak{A}_{t_1, t_2}$ .



# Analogy:

(TQFT, Jones polynomial - but invert)



$$K(M) = K(M_1) \cdot K(M_2)$$