Operator-Valued Function Space Integrals

$$\left(I_\lambda(F)\psi\right)(\xi) \equiv \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\,\psi(\lambda^{-1/2}x(b) + \xi)\,dx$$

$\lambda > 0$

F functional on $C_0[a,b]$

ψ function on $(-\infty, \infty)$

ξ real

Assume λ, F, ψ are such that integral exists, then

$I_\lambda(F)\psi$ is a function sending

ξ into $\left(I_\lambda(F)\psi\right)(\xi)$ for $\xi \in S$.

Suppose for some choice of λ, F then

for $\psi \in \mathcal{D}$, $I_\lambda(F)\psi \in \mathcal{E}$

$$I_\lambda(F) : \mathcal{D} \longrightarrow \mathcal{E}$$

$$I_\lambda(F) : L_2(-\infty,\infty) \longrightarrow L_2(-\infty,\infty)$$

e.g.

If $F(x) = \exp\left\{ \int_a^t \theta(s, x(s))\, ds \right\}$

and smoothness & order of growth conditions on θ and ψ, then

$(I_\lambda(F)\psi)(\xi)$ gives a solution to the generalized heat flow equation

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By analytic continuation

$$I_\lambda^{an}(F)$$

for $\operatorname{Re} \lambda > 0$

$$I_\lambda^{seq}(F)$$

e.g. 1

Let $F(y) \equiv 1$, $\psi \in L_2$, then

$$(I_\lambda(F)\psi)(\xi) = \int_{C_0[a,b]} \psi(\lambda^{-1/2} x(b) + \xi)\, dx$$

$$= [2\pi(b-a)]^{-1/2} \int_{-\infty}^{\infty} \psi(\lambda^{-1/2} u + \xi) \exp\left\{-\frac{u^2}{2(b-a)}\right\} du$$

$$= \lambda^{+1/2} [2\pi(b-a)]^{-1/2} \int_{-\infty}^{\infty} \psi(v) \exp\left\{\frac{-\lambda(v-\xi)^2}{2(b-a)}\right\} dv$$

So we have

$(I_\lambda(F)\psi)(\xi) \in L_2(-\infty,\infty)$ if $F \equiv 1$

and

$I_\lambda(F)\psi$ is analytic in λ for $\mathrm{Re}\,\lambda > 0$.

e.g. 2

If $F(y) = \exp\left\{ i\int_a^b y(s)\,ds \right\}$

then $\left(I_\lambda(F)\psi\right)(\xi)$ can be evaluated and gives a function in L_2 whenever $\psi \in L_2$.

also for $\mathrm{Re}\,\lambda > 0$,

$I_\lambda\left[\exp\left\{ i\int_a^b y(s)\,ds \right]\psi\right.$

is analytic in λ

(as a Hilbert space valued function of λ).

Initially, look at

$$F(x) = \exp\left\{\int_a^b \Theta(s, x(s))\,ds\right\}$$

where $\Theta(s, u)$ is almost everywhere continuous on every compact subset of $R = [a,b] \times (-\infty, \infty)$, and $\mathrm{Re}\,\Theta$ is bounded above on R.

Later the condition on continuity is replaced by the much weaker condition of Lebesgue measurability and the condition of boundedness on compact subsets of R is replaced by the condition that the L_2 norm $\|\Theta(s, \cdot)\|$ exist and be bounded for $a \le s \le b$.

The condition $\mathrm{Re}\,\Theta$ be bounded on R remains the same.

The changes in the arguments
are to modify the theory, but
most of the results hold as in
the original or slightly
modified form.

Now take $\varphi \in L_1(-\infty,\infty)$

and study functionals F

that make $I_\lambda(F)$ a bounded

operator from L_1 to L_∞

again define $I_\lambda^{an}(F)$ by

analytic continuation to $\operatorname{Re}\lambda > 0$

and study

$$F(x) = \exp\left\{\int_a^b \theta(s, x(s))\,ds\right\}$$

for θ msble on R

$\theta(t,\cdot) \in L_1$ and $\|\theta(t,\cdot)\|_1 < B$

By a limiting procedure

extend $I_\lambda^{an}(F)$ to the imaginary

axis to obtain

$J_q(F)$ + Integral Eq.

Thm. Let $\theta(t,u)$ be measurable on R-strip

Let $\theta(t,\cdot) \in L_1(-\infty,\infty)$

Let $\|\theta(t,\cdot)\|_1 < B$ for a.e. $t \in [0,t_0]$

Let $\psi \in L_1$ then for $\lambda>0$, ξ real $t \in [0,t_0]$,

$$G(t,\xi,\lambda) = \int_{C_0[0,t]} \exp\left\{\int_0^t \theta(t-s,\lambda^{-1/2}x(s)+\xi)\right\} \psi(\lambda^{-1/2}x(t)+\xi)\,dx$$

exists $G(t,\cdot,\lambda) \in L_1(-\infty,\infty)$ and

$G(t,\xi,\lambda)$ satisfies

$$G(t,\xi,\lambda) = \left(\frac{\lambda}{2\pi t}\right)^{1/2}\int_{-\infty}^{\infty} \psi(u)\exp\left\{\frac{-\lambda(\xi-u)^2}{2t}\right\}du$$

$$+\left(\frac{\lambda}{2\pi}\right)^{1/2}\int_0^t (t-s)^{-1/2}\int_{-\infty}^{\infty}\theta(s,u)\,G(s,u,\lambda)$$

$$\exp\left\{\frac{-\lambda(\xi-u)^2}{2(t-s)}\right\}du\,ds$$

Th$^{\underline{m}}$ Under the hyp. of the previous Th$^{\underline{m}}$

for $F_t(x) \equiv \exp\left\{ \int_0^t \theta(t-s, x(s))\, ds \right\}$,

$I_\lambda^{an}(F_t)$ exists for $\mathrm{Re}\,\lambda > 0$ and

for every $\psi \in L_1$, $I_\lambda^{an}(F_t)\psi \in L_\infty$

Moreover for $\mathrm{Re}\,\lambda > 0$

$G(t, \xi, \lambda) \equiv \left(I_\lambda^{an}(F_t)\psi\right)(\xi)$

satisfies the integral equation

for $t \in (0, t_0]$ and almost all ξ.

By a limiting procedure.

$I_\lambda^{an}(F)$ can be extended to the imaginary axis $\mathrm{Re}\,\lambda = 0$. and the resulting operator is called $J_q^{an}(F)$ where q corresponds to $-i\lambda$. and

$$\| J_q(F)\psi \|_\infty \le B^*(q) \|\psi\|_1$$

for each $\psi \in L_1$

Finally we prove that $J_q(F)$ gives the solution to an integral equation formally equivalent to the Schroedinger Equation.

The limiting procedure uses weak analyticity. – i.e.

$$\lim_{\substack{\lambda \to -iq \\ \mathrm{Re}\,\lambda > 0}} \int_{-\infty}^{\infty} \left[(I_\lambda^{an}(F)\psi)(\xi) - (J_q(F)\psi)(\xi) \right] \cdot \varphi(\xi)\, d\xi = 0$$

for every $\varphi \in L_1(-\infty, \infty)$.

Th$^{\underline{m}}$ Under the hyp of the previous Th^{m}s

If $$F_t(x) = \exp\left\{\int_0^t \theta(t-s, x(s))\,ds\right\},$$

for each $t \in (0, t_0]$ and all real q

$J_q(F_t)$ exists and maps L_1 into L_∞

If we set

$$\Gamma(t, \xi, q) = [J_q(F_t)\psi](\xi)$$

then for each $(t, \xi) \in R$ and each real q

$\Gamma(t, \xi, q)$ satisfies

$$\Gamma(t, \xi, q) = \left(\frac{q}{2\pi i t}\right)^{1/2} \int_{-\infty}^{\infty} \psi(u) \exp\left\{\frac{iq(\xi-u)^2}{2t}\right\} du$$

$$+ \left(\frac{q}{2\pi i}\right)^{1/2} \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u)\,\Gamma(s, u, q)$$

$$\exp\left\{\frac{iq(\xi-u)^2}{2(t-s)}\right\} du\,ds.$$

In a major paper in the Journal of Math. Anal + Appl. on the L_1 Theory, Johnson & Skoug gave existence theorems which considerably extended and strengthened these results + Examples

and

In a Memoir of the AMS, Johnson & Lapidus extended the results even further using integrations with respect to Borel measures -

Recently Bong Jin Kim and Kun Sik Ryu have published results about the existence of the operator valued function space integral for an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ functionals involving double integrals with respect to Borel measures –

I should also mention the work of J. S. Chang & G W Johnson and the work of K. S. Chang and K. S. Ryu.

$\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$

and

$\mathcal{L}(L_p, L_{p'})$.

An Operator Valued Function Space Integral applied to Multiple Integrals of functions of class L_1

I $\quad F(x) = f\left\{\int_a^b\int_a^b \theta(s, t, x(s), x(t))\, ds\, dt\right\}$

f an entire function

II $\quad F(x) = f\left\{\int_a^b \theta(s, x(s))\, ds,\ \int_a^b \varphi(t, x(t))\, dt\right\}$

f is an entire function of <u>two</u> complex variables.

Need a Growth condition on f.

A pair of simultaneous integral equations -

Consider a function space of functions of two variable

Let $R = R_a \equiv \{(s,t) / a \le s \le b,\ \alpha \le t \le \beta\}$

$$C_2^*[R] = \{x(s,t) / \text{cont. on } R,\ x(\cdot,\alpha) = 0\}$$

$$C_2[R] = \{x(s,t) / \text{cont on } R \quad x(a,\cdot) = x(\cdot,\alpha) = 0\}$$

Let F be a functional on $C_2^*[R]$

ψ be functional on $C_1[\alpha,\beta]$

$\lambda > 0$ and define for $\eta \in C_1[\alpha,\beta]$

$I_\lambda(F) \equiv I_{\lambda,R}(F)$ by

$$\big(I_\lambda(F)\psi\big)(\eta(\cdot)) =$$

$$\int_{C_2[R]} F(\lambda^{-1/2} x(\cdot,\cdot) + \eta(\cdot))\ \psi(\lambda^{-\frac{1}{2}} x(b,\cdot) + \eta(\cdot))\ dx$$

Yeh - Wiener integral on $C_2[R]$

Let the functional F be given by

$$F(x) \equiv F_a(x) \equiv \exp\left\{\int_a^b \Theta(\sigma, x(\sigma,\cdot))\, d\sigma\right\}$$

where Θ is bdd + continuous on $[a,b] \times C_1[\alpha,\beta]$ Then if

$$G(s,\eta(\cdot)) \equiv \left(I_{\lambda, R_s}(F_s)\varphi\right)(\eta(\cdot))$$

for $\eta \in C_1[\alpha,\beta]$ G satisfies the Wiener Integral Equation

$$G(s,\eta(\cdot)) = \int_{C_1[\alpha,\beta]} \varphi\left(\lambda^{-1/2}\left(\tfrac{b-s}{2}\right)^{1/2} w + \eta(\cdot)\right) dw$$

$$+ \int_a^b \int_{C_1[\alpha,\beta]} \Theta\left(\sigma, \lambda^{-1/2}\left(\tfrac{\sigma-s}{2}\right)^{1/2} w + \eta(\cdot)\right) \cdot$$

$$G\left(\sigma, \lambda^{-1/2}\left(\tfrac{\sigma-s}{2}\right)^{1/2} w + \eta(\cdot)\right) dw\, d\sigma.$$

Furthermore the solution is unique and can be expressed as a sum of a Neumann Series whose terms are multiple Wiener integrals

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Thus this Yeh-Wiener integral can be written as a sum of multiple (ordinary) Wiener Integrals.

An operator-valued Yeh-Feynman Integral and a Feynman Integral Equation