Fourier-Feynman Transforms, Convolution Products and First Variations in Function Space

by

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1. Fourier-Wiener Transforms (FWT's).

Let

(1.1) $C_0 \equiv C_0[0,T] = \{x : x$ is a continuous $\mathbb{R}$-valued function on $[0,T]$ with $x(0) = 0\}$,

and let

(1.2) $K_0 \equiv K_0[0,T] = \{z : z$ is a continuous $\mathbb{C}$-valued function on $[0,T]$ with $z(0) = 0\}$.

Note. $z \in K_0 \Leftrightarrow \exists\ a, b \in C_0$ such that $z(t) = a(t) + ib(t)$ on $[0,T]$.

<u>Definition</u>(Cameron-1945). Let $F : K_0 \to \mathbb{C}$ be such that $F(x+iy)$ is Wiener integrable in x over $C_0[0,T]$ for each fixed $y \in K_0[0,T]$.

Then

$$(1.3)\ \mathcal{F}(F)(y) \equiv \int_{C_0[0,T]} F(x+iy)m(dx), y \in K_0$$

and

$$(1.4)\ \mathcal{F}^{-1}(F)(y) \equiv \int_{C_0[0,T]} F(x-iy)m(dx), y \in K_0$$

are called the FWT and the inverse FWT of F, respectively.

$$\mathcal{F}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(x)\, dx$$

$$\mathcal{F}^{-1}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(x)\, dx$$

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Definition. Let E_m be the class of functionals F defined on K_0 which are

(a) mean continuous, i.e., $\|z_n - z\|_2 \to 0 \Rightarrow F(z_n) \to F(z)$,

(b) entire, i.e., $F(z + \lambda y)$ is an entire function of λ for all $z, y \in K_0$,

and
(c) of mean exponential type , i.e., $| F(z) | \leq A\exp\{B\|z\|_2\}$ for all $z \in K_0$.

Note. E_m is dense in $L_2(C_0[0,T])$.

Theorem (Cameron and Martin - 1945).
$\mathcal{F} : E_m \rightarrow E_m$ is onto.

Theorem (Cameron and Martin - 1946).
$\mathcal{F} : L_2(C_0[0,T]) \rightarrow L_2(C_0[0,T])$ is onto. Furthermore,

$$\int_{C_0} \mid F(x) \mid^2 m(dx) = \int_{C_0} \mid \mathcal{F}(F)(y) \mid^2 m(dy).$$

They also point out that $F \in L_2(C_0[0,T])$ only needs to be defined a.e. on $C_0[0,T]$ and not on $K_0[0,T]$.

In 1965, James Yeh (Convolution in FWT - Pacific J. of Math.) gave the following definition of a convolution product:

$$(F_1 * F_2)(y) \equiv \int_{C_0} F_1\left(\frac{x+y}{\sqrt{2}}\right) F_2\left(\frac{x-y}{\sqrt{2}}\right) m(dx)$$

for all $y \in K_0$.

Note that $(F_1 * F_2)(y) = (F_2 * F_1)(-y)$.

Main Result. For $F_1, F_2 \in E_m$, $(F_1 * F_2)(y)$ exists for every $y \in K_0$. Furthermore,

$$\mathcal{F}((F_1 * F_2))(y) = \mathcal{F}(F_1)\left(\frac{y}{\sqrt{2}}\right) \mathcal{F}(F_2)\left(\frac{-y}{\sqrt{2}}\right)$$

for all $y \in K_0$.

3. Fourier-Feynman Transforms (FFT's).

The basic idea is to use the "i" in the analytic Feynman integral to define an integral transform on $C_0[0,T]$.

- 1972 - Brue : an L_1-FFT
- 1976 - Cameron and Storvick : an L_2-FFT
- 1979 - Johnson and Skoug : an L_p-FFT
- 1979 - Johnson and Skoug : scale-invariant measurability in Weiner space.

Assume that

$F : C_0[0,T] \to \mathbb{C}$ is defined s-a.e. and is s.i.m.,

$$\int_{C_0[0,T]} |F(\rho x)| m(dx) < \infty \text{ for each } \rho > 0,$$

and that for each $\lambda \in \mathbb{C}_+$,

$$T_\lambda(F)(y) = \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x+y) m(dx)$$

exists for s-a.e. $y \in C_0[0,T]$.

(a) For $1 < p \leq 2$ and $q \in \mathbb{R} - \{0\}$, the L_p-FFT, $T_q^{(p)}(F)$ of F, is defined by the formula

$$T_q^{(p)}(F)(y) \equiv \underset{\lambda \to -iq}{\text{l.i.m.}}\, T_\lambda(F)(y)$$

for s-a.e. $y \in C_0[0,T]$) whenever this limit exists (scale-invariant limit in the mean of order p').

(b) For $p = 1$,

$$\begin{aligned} T_q^{(1)}(F)(y) &\equiv \lim_{\lambda \to -iq} T_\lambda(F)(y) \\ &= \int_{C_0[0,T]}^{\text{anf}_q} F(x+y)m(dx) \end{aligned}$$

Note. For all $p \in [1,2]$, $T_q^{(p)}(F)$ exists for functionals of the form

$$F(x) = \exp\left\{\int_0^T \theta(s, x(s))ds\right\}$$

for appropriate θ.

4. In a unifying paper, Yuh Jia Lee (J. of Functional Analysis - 1982) defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on abstract Wiener space which included the Gauss transform, the FWT, and the FFT, as special cases.

Recently, Byoung Soo Kim and Skoug obtained a necessary and suficient condition that a functional F in $L_2(C_0[0,T])$ has an integral transform

$$\mathcal{F}_{\alpha,\beta}F(y) = \int_{C_0[0,T]} F(\alpha x + \beta y) m(dx),$$

$y \in C_0[0,T]$, also belonging to $L_2(C_0[0,T])$.

5. Convolution Products (1993).

For $q \in \mathbb{R} - \{0\}$, we define the CP, $(F*G)_q \equiv (F*G)_{-iq}$, of F and G by the formula (if it exists)

$$(F*G)_q(y) \equiv \int_{C_0[0,T]}^{\text{anf}_q} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx).$$

Then $(F*G)_q = (G*F)_q$. Also under reasonable conditions on F and G,

$$T_q^{(p)}((F*G)_q)(y) = T_q^{(p)}(F)\left(\frac{y}{\sqrt{2}}\right) T_q^{(p)}(G)\left(\frac{y}{\sqrt{2}}\right)$$

for s-a.e. $y \in C_0[0,T]$.

A tempting conjecture:

$$(5.1) \qquad T^{(1)}_{q_1}((F * G)_{q_2})(y) =$$

$$T^{(1)}_{\frac{2q_1q_2}{q_1+q_2}}(F)\left(\frac{y}{\sqrt{2}}\right) T^{(1)}_{\frac{2q_1q_2}{q_1+q_2}}(G)\left(\frac{y}{\sqrt{2}}\right).$$

But in general, the equality (5.1) holds if and only if $q_1 = q_2 = q$. However,

$$T^{(1)}_{q_2}(T^{(1)}_{q_1}(F))(y) = T^{(1)}_{\frac{q_1q_2}{q_1+q_2}}(F)(y)$$

$$= T^{(1)}_{q_1}(T^{(1)}_{q_2}(F))(y).$$

6. First Variation.

Let F be a Wiener measurable (s.i.m.) functional on $C_0[0,T]$, and let $w \in C_0[0,T]$. Then

$$\delta F(x|w) = \frac{\partial}{\partial k} F(x+kw)\Big|_{k=0}$$

(if it exists) is called the <u>first variation</u> of $F(x)$.

Usually we will take w to be of the form

$$w(t) = \int_0^t z(s)ds, \qquad 0 \le t \le T$$

for some $z \in L_2[0,T]$. Note that for F of the form

$$F(x) = f(\langle \alpha_1, x\rangle, \langle \alpha_2, x\rangle)$$

where $\{\alpha_1, \alpha_2\}$ are orthonormal functions in $L_2[0,T]$,

$$\delta F(x|w) = \langle \alpha_1, w\rangle f_1(\langle \alpha_1, x\rangle, \langle \alpha_2, x\rangle)$$

$$+ \langle \alpha_2, w\rangle f_2(\langle \alpha_1, x\rangle, \langle \alpha_2, x\rangle)$$

where

$$\langle \alpha, x \rangle = \int_0^T \alpha_1(t) dx(t)$$

and

$$\langle \alpha, w \rangle = \int_0^T \alpha_1(t) dw(t) = \int_0^T \alpha_1(t) z(t) dt.$$

7. The basic formula

$$\int_{C_0[0,T]}^{\text{anf}_q} \delta F(x|w) m(dx) = -iq \int_{C_0[0,T]}^{\text{anf}_q} F(x) \langle z, x \rangle m(dx)$$

has led to many interesting results; for example, the parts formula

$$\int_{C_0[0,T]}^{\text{anf}_q} [F(x)\delta G(x|w) + G(x)\delta F(x|w)] m(dx)$$
$$= -iq \int_{C_0[0,T]}^{\text{anf}_q} F(x) G(x) \langle z, x \rangle m(dx).$$

8. Relationships among the First Variation $\delta F(x|w)$, the Fourier-Feynman Transform $T_q(F)$, and the Convolution Product $(F * G)_q$.

(a) There are seven relationships involving exactly two of the three concepts of "transform", "convolution", and "first variation", including:

- $T_q(\delta F(\cdot|w))(y) = \delta T_q(F)(y|w)$
- $T_q(\delta F(y|\cdot))(w) = \delta(F)(y|w)$
- $T_q((F * G)_q)(y) = T_q(F)\left(\frac{y}{\sqrt{2}}\right) T_q(G)\left(\frac{y}{\sqrt{2}}\right)$

(b) There are nine distinct relationships involving all three concepts where each concept is used exactly once. One such example is:

- $T_q(\delta(F * G)_q(\,\cdot \mid w))(y) = \delta T_q((F * G)_q)(y|w)$

$$= T_q(F)\left(\frac{y}{\sqrt{2}}\right) T_q(\delta G(\,\cdot \mid w/\sqrt{2}))\left(\frac{y}{\sqrt{2}}\right)$$

$$+ \quad T_q(\delta F(\,\cdot \mid w/\sqrt{2}))\left(\frac{y}{\sqrt{2}}\right) T_q(G)\left(\frac{y}{\sqrt{2}}\right)$$

9. Other Gaussian Processes.

(a) The Wiener Process

$$W(x,t) = x(t)$$

is free of drift and is stationary in time with mean function zero and covariance function $s \wedge t$.

(b) Let $h \in L_2[0,T]$ with $\|h\|_2 > 0$. Then the Gaussian process

$$Z(x,t) = \int_0^t h(s)dx(s)$$

is free of drift and is nonstationary in time with mean function zero and covariance function

$$\int_0^{s\wedge t} h^2(u)du.$$

(c) Let $a(t)$ be an absolutely continuous function on $[0, T]$ with $a(0) = 0$ and with $a'(t)$ belonging to $L_2[0, T]$. Then the Gaussian process

$$X_a(x, t) = \int_0^t h(s)dx(s) \; + \; a(t)$$

is subject to the drift $a(t)$ and is nonstationary in time with mean function $a(t)$ and covariance function

$$\int_0^{s \wedge t} h^2(u)du.$$

Note. If $h(t) \equiv 1$ and $a(t) \equiv 0$, then

$$X_a(x, t) = Z(x, t) = W(x, t) = x(t).$$

Next defining

$$_aT_\lambda(F)(y) = \int_{C_0[0,T]}^{\text{anw}_\lambda} F(y + X_a(x,\ \cdot\ ))m(dx),$$

etc., we can study the relationships among

$_aT_q(F)(y)$, $_a(F * G)_q(y)$, and $\delta F(y|w)$.

For example,

- $_aT_q(_a(F * G)_q(y) = T_q(F)\left(\frac{y+2a}{\sqrt{2}}\right) T_q(G)\left(\frac{y}{\sqrt{2}}\right)$
- $T_q(_a(F * G)_q(y) = T_q(F)\left(\frac{y+a}{\sqrt{2}}\right) T_q(G)\left(\frac{y-a}{\sqrt{2}}\right)$
- $_aT_q(_a(\delta F(y \mid \cdot\ ) * \delta G(y \mid \cdot\ ))_q)(w)$

$$= \delta(F)\left(y \mid \frac{w+2a}{\sqrt{2}}\right) \delta G\left(y \mid \frac{w}{\sqrt{2}}\right)$$

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with density function

$$
\begin{aligned}
K(\vec{t},\vec{\eta}) &= \Big((2\pi)^n \prod_{j=1}^{n}(b(t_j)-b(t_{j-1}))\Big)^{-1/2} \\
&\quad \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j)-b(t_{j-1})}\right\}
\end{aligned} \tag{2.1}
$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [13, p.18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [13, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable [9] provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.).

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$
L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\} \tag{2.2}
$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let