The 2-d Yang-Mills functional integral: Physics applied to Geometry

Feynman expressed quantum evolution as a functional integral involving the classical dynamics of a system:

$$\Psi_t = e^{-\frac{i}{\hbar} tH} \psi_0$$

$$K_t(x,y) \sim \int_{\{x \overset{\gamma}{\leadsto} y \text{ paths}\}} e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma$$

propagator/kernel of the quantum evolution operator $e^{-itH/\hbar}$

paths in the classical phase space

Kac transformed this into mathematics,

expressiong the solution of a pde

such as $$\frac{\partial \Psi(t,x)}{\partial t} = \left(\tfrac{1}{2}\Delta + V\right)\Psi(t,x)$$

as an integral over a space of paths

$$\Psi_t(x) \sim \int_{0 \to \gamma x} e^{-\int_0^t V(\gamma_s)\,ds}\, d\mu(\gamma)$$

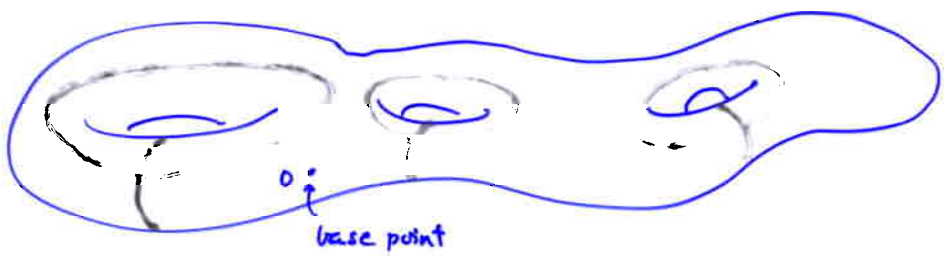
Wiener measure

[Of course, this stands on the shoulders of many great probabilistic predecessors]

The rigorous Feynman-Kac formula stands on its own as a mathematical theorem independent of its quantum theoretic inspiration.

Now let us turn to gauge theory over surfaces.

Consider a surface Σ



oriented genus g
[$g = 3$ in pic]

The fundamental group $\pi_1(\Sigma, o)$ is generated by $2g$ loops $A_1, B_1, A_2, B_2, \ldots, A_g, B_g$

satisfying the relation

$$\bar{B}_g \bar{A}_g B_g A_g \cdots \bar{B}_1 \bar{A}_1 B_1 A_1 = \text{identity in } \pi_1(\Sigma, o).$$

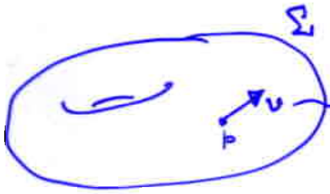
Now consider

G : a compact, connected (say simply-connected) Lie group

LG = the Lie algebra of G

$\langle \cdot, \cdot \rangle$: an Ad-invariant inner-product on LG.

A connection ω over Σ is a smooth LG-valued 1-form on Σ



$\omega(v) \in LG$

$$\boxed{\mathcal{A} = \text{the space of all connections over } \Sigma}$$

This is an ∞-dim affine space (vector space in this set-up)

The group of gauge transformations is

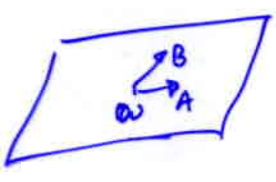
$$\boxed{\mathcal{G} = \{\text{all smooth } \phi : \Sigma \to G\}}$$

This acts on $\mathcal{A}$

$\mathcal{G} \times \mathcal{A} \to \mathcal{A} : (\phi, \omega) \mapsto Ad(\phi^{-1})\omega + \phi^{-1} d\phi$

On $\mathcal{A}$ there is a symplectic structure Ω:

A, B are LG-valued 1-forms on Σ



$$\boxed{\Omega(A,B) = -\frac{k}{2\pi} \int_{\Sigma} \langle A \wedge B \rangle}$$

k constant

oriented 2-d manifold

the 2-form on Σ given by

$\langle A \wedge B \rangle (X,Y)$
$= \langle A(X), B(Y) \rangle_{LG} - \langle A(Y), B(X) \rangle_{LG}$
$\in LG \quad \in LG$

(up to constants) this is the Atiyah–Bott symplectic form on $\mathcal{A}$.

Σ

G : compact simply-connected Lie group

LG : Lie algebra

$\mathcal{A}$ = space of connections over Σ
= { LG-valued C^∞ 1-forms on Σ }

$\mathcal{G}$ = grp of gauge transformations
= { all C^∞ $\phi : \Sigma \to G$ }.

$\mathcal{G} \times \mathcal{A} \to \mathcal{A} : (\phi, \omega) \mapsto \mathrm{Ad}\,\phi^{-1}\,\omega + \phi^{-1} d\phi$

The action of $\mathcal{G}$ on $\mathcal{A}$ is Hamiltonian and has a moment map $J : \mathcal{A} \to (L\mathcal{G})^* \approx L\mathcal{G}$

one can check that (Atiyah-Bott) $\cancel{J : L\mathcal{G}}$:

$$\boxed{J(\omega) = \underbrace{\Omega^\omega}_{\text{the curvature of } \omega} = d\omega + \tfrac{1}{2}[\omega \wedge \omega]}$$

Symplectic reduction (Marsden - Weinstein) produces a symplectic structure $\overline{\Omega}$ on $J^{-1}(0)/\mathcal{G}$

Now $J^{-1}(0)$ = { connections ω with curvature $\Omega^\omega = 0$ }
= { all flat connections }

So $\boxed{J^{-1}(0)/\mathcal{G} = \text{the moduli space of flat } G\text{-connections over } \Sigma}$ $\leftarrow$ denote this by $\mathcal{M}^0$

and $\overline{\Omega}$ is a symplectic structure on $\mathcal{M}^0$

Note : $\mathcal{M}^0$ has singularities (it's not a manifold) and $\overline{\Omega}$ has to be interpreted carefully

There is a concrete description of the moduli space $\mathcal{M}^0$ of flat connections.

Recall that $\pi_1(\Sigma, o)$ is generated by loops $A_1, B_1, \dots, A_g, B_g$ satisfying

$$\bar{B}_g \bar{A}_g B_g A_g \cdots \bar{B}_1 \bar{A}_1 B_1 A_1 = \text{identity in } \pi_1(\Sigma, o)$$

g = genus of Σ

If ω is any connection we have the holonomies

$$\big(\underbrace{h(A_1;\omega)}_{a_1}, \underbrace{h(B_1;\omega)}_{b_1}, \dots, \underbrace{h(A_g;\omega)}_{a_g}, \underbrace{h(B_g;\omega)}_{b_g}\big)$$

If ω is flat then

$$b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1 = e, \text{ the identity in } G.$$

Moreover, a flat ω is determined uniquely up to gauge transformations by $(a_1, b_1, \dots, a_g, b_g) \in G^{2g}$ up to conjugation.

Thus

$$\boxed{\mathcal{M}^0 \approx K_g^{-1}(e)/G}$$

$K_g : G^{2g} \to G : (a_1, b_1, \dots, a_g, b_g) \mapsto b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1$

Note that $K_g^{-1}(e) \subset G^{2g}$.

Thus $\mathcal{M}^0$ is <u>finite-dimensional</u>.

$\mathcal{M}^0 \downarrow$ moduli space of flat connections.

Thus we have

- the moduli space of flat connections $\mathcal{M}^0$
- a symplectic structure $\bar{\Omega}$ on (the generic part of) $\mathcal{M}^0$

A natural question then is:

what is the symplectic volume $\mathrm{vol}_{\bar{\Omega}}(\mathcal{M}^0)$?

Here is the answer (due to Witten)

$$\boxed{\mathrm{vol}_{\bar{\Omega}}(\mathcal{M}^0) = |Z(G)| \cdot \mathrm{vol}(G)^{2g-2} \sum_{\alpha} \frac{1}{(\dim \alpha)^{2g-2}}}$$

genus of Σ ↓ closed, oriented 2-d manifold

technically: this is the set of "irreducible"/generic flat connections

# of elements in the Center $Z(G)$

volume of G w.r.t. the metric $\langle\cdot,\cdot\rangle$ on LG

runs over all irreducible reps of G.

[~~both~~ $\frac{k}{4\pi} \to 1$ in def of Ω]

A rigorous proof of this formula can be given without using any "physics", but using a method provided by physics. [AS, to appear in JGP]

Here I would like to discuss some of the 'physics' connected with the above formula.

Recall that $\mathcal{M}^0 = J^{-1}(0)/\mathcal{G}$

where J is the moment map (curvature function) of the action of $\mathcal{G}$ on $\mathcal{A}$.

Heuristics show that

$$\mathrm{vol}(\mathcal{M}^0) \sim \lim_{t \to 0} \int_{\mathcal{A}} e^{-|J(\omega)|^2/t} \, \mathcal{D}\omega$$

[localizes on $J^{-1}(0)$ then quotient by $\mathcal{G}$]

but this is an ∞-dim functional integral which is the "partition function" of 2d Yang-Mills.

Both a heuristic/'physical' analysis (Polyakov, Migdal, Witten) and rigorous math suggests that

$$\int_{\mathcal{A}} e^{-|J(\omega)|^2/t} \, \mathcal{D}\omega \sim c \cdot \int_{G^{2g}} Q_t\left(b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1\right) da_1 \cdots db_g$$

$$= \int_{G^{2g}} Q_t(k_g(x)) \, dx$$

where Q_t is the heat kernel on G (the main pt is $Q_0 = \delta_e$)

This leads to the conjecture that

$$\mathrm{vol}\,(\mathcal{M}^0) \sim \lim_{t\to 0} \int_{G^{2g}} \Theta_t\big(\underset{\substack{\uparrow \\ \text{product commutator}}}{K_g(x)}\big)\,dx$$

this is a fin-dim integral.

This indeed turns out to be true. (to appear in JGP ~~[illegible]~~ summing over all topological bundle types & assuming $g \geq 2$)

Expanding Θ_t in characters $\sum_\alpha e^{-tc_\alpha}\chi_\alpha$

leads then to Witten's formula:

$$\mathrm{vol}_{\tilde\Omega}\,(\mathcal{M}^0) = |Z(G)|\,\mathrm{vol}(G)^{2g-2} \sum_\alpha \frac{1}{(\dim \alpha)^{2g-2}}$$

$\mathcal{M}^0$ is a very interesting space:

- moduli space of flat connections
- $\mathrm{Hom}(\pi_1(\Sigma), G)/G$: "moduli space" of representations of $\pi_1(\Sigma)$ in G
- moduli space of flat (vector) bundles (algebraic geometry)
- phase space of Chern-Simons theory
- phase space of 3d gravity (according to Ashtekar)

Lets take a look at the relationship with Chern-Simons theory [Witten's "Jones polynomial" paper CMP 1989]

(The CS functional integral will be explained by Atle Hahn)

CS theory is concerned with connections over a 3-manifold M.

We take

$$M = \underbrace{\Sigma}_{\text{2-d surface}} \times \underbrace{\mathbb{R}^1}_{\text{"time"}}$$

If $\mathcal{A}_M$ = space of connections over M.

For $A_M \in \mathcal{A}_M$:

$$\boxed{S_{cs}(A_M) = \frac{k}{4\pi} \int_M T_{cs}(A_M)} \leftarrow \text{Chern-Simons action}$$

$$T_{cs}(\omega) = (\omega \wedge d\omega) + \underset{\uparrow \; \frac{2}{3} \text{ if matrix notation is used}}{\frac{1}{3}} (\omega \wedge [\omega \wedge \omega])$$

The game is to "quantize" the classical system given by the action S_{cs}.

This involves

- the Feynman integrals $\int_{\mathcal{A}_M} e^{iS_{cs}(A_M)} \mathcal{D}A_M$
- finding a Hilbert space for the quantum theory.

It is convenient (& okay) to consider only connections A on $\Sigma \times \underset{\substack{\uparrow \\ \text{time}}}{\mathbb{R}^1}$ whose time-component is 0 (gauge fixing).

Then $S_{CS}(A) = \int_{\mathbb{R}} \mathcal{L}(A(t), \overset{\dot{A}(t)}{\dot{A}(t)})\, dt$

$$\mathcal{L}(A, \dot{A}) = -\frac{k}{4\pi} \int_{\Sigma} (A \wedge \dot{A})_{LG}$$

The equations of motion turn out to say:

A is a flat connection on Σ

$\dot{A} = 0$

Thus the phase space of Chern-Simons theory is the moduli space of flat connections

[space of solutions of the Euler-Lagrange eqns]

(we reduce by the action of $\mathcal{G}$)

The Hilbert space of the quantum theory is then the space of holomorphic sections of a certain Hermitian line bundle over $\mathcal{M}^0$. ~~whose~~ This Hermitian line bundle has a connection whose curvature descends to the symplectic structure $\bar{\Omega}$ on $\mathcal{M}^0$.

"geometric quantization" of $\mathcal{M}^0$.

The Hilbert space is finite-dimensional.

~~$\mathcal{E}$ "If vol (phase~~

Recall the "physical" principle that the number of quantum states "is" the number of Planck cells in phase space. So if vol(phase space) $< \infty$ then the quantum Hilbert space is also finite dimensional.

~~To summarize:~~

~~the moduli~~

To summarize:

Σ

- associated to Σ and G is the important space $\mathcal{M}^0$
  the moduli space of flat G-connections over Σ

- $\mathcal{M}^0$ has a natural symplectic structure (on strata)

- $\mathrm{vol}(\mathcal{M}^0)$ can be computed explicitly as $\lim_{t \to 0}$ (∞-dim functional integral for 2-d Yang-Mills theory)

- $\mathcal{M}^0$ is the phase space of Chern-Simons theory

Appendix: Technical comments on the proof of the volume formula.

Recall our strategy

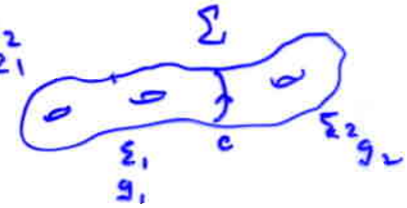
$$\mathrm{vol}(\mathcal{M}^0) = c \cdot \lim_{t \to 0} \int_{G^{2g}} Q_t(K_g(x))\, dx$$

$$K_g(a_1, b_1, \ldots, a_g, b_g) = b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1$$

The map $K_g : G^{2g} \to G$ has critical points on $K_g^{-1}(e)$ and this complicates evaluation of the limit. The way around is:

$$\int Q_t(K_g(x))\, dx = \int_G \Big[\int_{G^{2g_1}} Q_{t/2}(K_{g_1}(x) c)\, dx \Big] \Big[\int_{G^{2g_2}} Q_{t/2}(c^{-1} K_{g_2}(y))\, dy \Big]\, dc$$

$g = g_1 + g_2$, $g_1 \geq 1$, $g_2 \geq 1$



these limits are easier to evaluate by taking c non-critical.

Necessary: prove that certain sets have measure 0

- $\{$critical pts of $K_g\}$
- $\{$pts of non-minimal isotropy for the conjugation action of G on $G^{2g}\}$

A second ingredient is needed <u>after</u> proving

$$\lim_{t\to 0} \int_{G^{2g}} \Theta_t\left(\kappa_g(x)\right)dx \quad \sim \quad \int_{\kappa_g^{-1}(e)} \frac{1}{|\det d\kappa_g(y)^*|} \, \mathrm{vol}_{\kappa_g^{-1}(e)}(y)$$

We have to pass to the quotient

$$\mathcal{M}^0 \cong \kappa_g^{-1}(e)/G$$

This gives the integral

$$(*) \qquad \int_{\kappa_g^{-1}(e)/G} \frac{|\det \gamma_y'|}{|\det d\kappa_g(y)^*|} \, \mathrm{vol}_{\kappa_g^{-1}(e)/G}(y)$$

$$\gamma_y : G \to \kappa_g^{-1}(e)$$
$$x \mapsto x y x^{-1}$$
$$(y \in \kappa_g^{-1}(e))$$

In work with C. King [King & S. JMP '94]

we showed that

$$\frac{|\det \gamma_y'|}{|\det d\kappa_g(y)^*|} = \mathrm{Pfaffian}(\bar{\Omega})$$

This makes $(*) = \mathrm{vol}_{\bar{\Omega}}(\mathcal{M}^0)$.

Thank you !