State Model Representations for the Wilson Loop Observables in Chern-Simons Theory

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December 7, 2002

The heuristic Path Space Measure of a pure Chern-Simons Model

Fix

- M: oriented differentiable 3-manifold.
- G: compact connected Lie subgroup of $U(N)$, $N \geq 2$ (Lie algebra will be denoted by $\mathfrak{g}$)
- $k \in \mathbb{R}\backslash\{0\}$ ("charge"; $\lambda := \frac{1}{k}$ is called "coupling constant")

"Path Space" $\mathcal{A}$: Space of $\mathfrak{g}$-valued 1-forms on M with compact support.

"Action function"

$$S_{CS} : \mathcal{A} \ni A \mapsto \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge dA + \tfrac{2}{3}(A \wedge A) \wedge A) \in \mathbb{C}$$

"Path space measure"

$$\mu_{CS}(dA) := \tfrac{1}{Z} \exp(iS_{CS}(A))DA,$$

where DA is the "Lebesgue measure" on $\mathcal{A}$ and $Z :=$ "$\int \exp(iS_{CS}(A))DA$"

Wilson Loop Observables and Knot Theory

Let $L := (l_1, \ldots, l_n)$, $n \in \mathbb{N}$, be a sufficiently regular link in M.

"Wilson Loop Observable (WLO) associated to L"

$$\mathrm{WLO}(L) := \int \mathrm{WLF}(L)\, d\mu_{CS}, \quad \text{where}$$

$$\mathrm{WLF}(L)(A) := \prod_i \mathrm{Tr}\Big(\mathcal{P}\exp\Big(\int_{l_i} A\Big)\Big)$$

Here $\mathcal{P}\exp(\int_{l_i} A)$ denotes the holonomy of A around l_i, $i \leq n$.

Literature

- Witten '89: Wilson loop observables for $M = S^3$ and $G = SU(N)$ are related to the Homfly polynomial
- Guadagnini/Marellini/Mintchev '90: Computation of the WLOs for $M \in \{S^3, \mathbb{R}^3\}$ using Lorentz Gauge
- Fröhlich/King '89: Computation of the WLOs for $M = \mathbb{R}^3$ and $G = SU(N)$ using Light-cone Gauge

Open Questions

Q1 Can one also work with other gauge like, e.g., axial gauge? Problem with axial Gauge: very singular expressions arise on a formal level

Q2 Can one define the WLOs rigorously in any of the gauges mentioned above?

Partial Answers

1. If $M = \mathbb{R}^3$ it is indeed possible to work with axial gauge using the approach of Albeverio/Sengupta '97. In particular, the WLOs can be defined rigorously in their setting
2. The values of the WLOs can be expressed explicitly; state models arise in a natural way.

1. Step: Axial Gauge Fixing

Set $\mathcal{A}^{ax} := \{A \in \mathcal{A} \mid A_2 = 0\} \qquad (A = \sum_{i=0}^{2} A_i dx_i)$

$$A \in \mathcal{A}^{ax} \Rightarrow (A \wedge A) \wedge A = 0 \Rightarrow$$

$$S_{CS}(A) = \tfrac{k}{4\pi} \int \mathrm{Tr}(dA \wedge A)$$

$$= \tfrac{k}{2\pi} < A_0, \partial_2 A_1 >_{L^2_{\mathfrak{g}}(\mathbb{R}^3)}$$

$$= \tfrac{1}{2\pi\lambda} < (A_0, A_1), S \cdot (A_0, A_1) >_{L^2_{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{R}^3)}$$

with $S := \frac{1}{2}\begin{pmatrix} 0 & \partial_2 \\ -\partial_2 & 0 \end{pmatrix}$

(scalar product $\mathfrak{g} \times \mathfrak{g} \ni (A, B) \mapsto -\mathrm{Tr}(AB) \in \mathbb{R}$)

On the other hand

$$\mathrm{WLO}(L) = \int \mathrm{WLF}(L) d\mu_{CS} = \int \mathrm{WLF}(L) d\mu_{CS}^{ax}$$

with $\mu_{CS}^{ax} := \frac{1}{Z^{ax}} \exp(iS_{CS}(A)) DA_{|\mathcal{A}^{ax}}$,

where "$DA_{|\mathcal{A}^{ax}}$" is the heuristic "Lebesgue measure" on $\mathcal{A}^{ax}$ and "Z^{ax}" a normalization constant.

2. Step: Making sense of $\int \cdots d\mu_{CS}^{ax}$

μ_{CS}^{ax} is formally Gaussian with covariance operator

$$C := 2\pi i\lambda \cdot \begin{pmatrix} 0 & \partial_2 \\ -\partial_2 & 0 \end{pmatrix}^{-1}$$

If one can make sense of C as a continuous operator $\mathcal{S}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{R}^3) \to \mathcal{S}'_{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{R}^3)$ then

$$< \cdot >_{CS}^{ax} := \int \cdots d\mu_{CS}^{ax}$$

can be defined rigorously as a generalized distribution on

$$\overline{\mathcal{A}^{ax}} := \mathcal{S}'_{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{R}^3) \quad \text{instead of } \mathcal{A}^{ax} \cong C_c^\infty(\mathbb{R}^3, \mathfrak{g}\oplus\mathfrak{g})$$

Ansatz: $\begin{pmatrix} 0 & \partial_2 \\ -\partial_2 & 0 \end{pmatrix}^{-1} := \begin{pmatrix} 0 & \partial_2^{-1} \\ -\partial_2^{-1} & 0 \end{pmatrix}$ where

$\partial_2^{-1} = r \cdot \hat{\partial}_2^{-1} + (1-r) \cdot \tilde{\partial}_2^{-1}, \quad (r \in \mathbb{R}$ fixed but arbitrary)

and where $\hat{\partial}_2^{-1}, \tilde{\partial}_2^{-1} : \mathcal{S}_{\mathfrak{g}}(\mathbb{R}^3) \to C_b^\infty(\mathbb{R}^3, \mathfrak{g})$ are given by

$$(\hat{\partial}_2^{-1} f)(x) = \int_{-\infty}^{x_2} f(x_0, x_1, s) ds$$

$$(\tilde{\partial}_2^{-1} f)(x) = -\int_{x_2}^{\infty} f(x_0, x_1, s) ds$$

Summary of Steps 1 + 2

$$
\begin{aligned}
&\text{“WLO}(L)\text{''} \longleftrightarrow \text{Knot polynomials} \\
&= \text{“}\int \text{WLF}(L) d\mu_{CS}\text{''} \\
&\overset{1.Step}{=} \text{“}\int \text{WLF}(L) d\mu_{CS}^{ax}\text{''} \\
&\overset{2.Step}{=} \text{“} < \text{WLF}(L) >_{CS}^{ax} \text{''} \\
&\overset{3.Step}{=} \lim_{\epsilon \to 0} < \text{WLF}(L, \epsilon) >_{CS}^{ax}
\end{aligned}
$$

3. Step: Regularize WLF(L) by using “smeared loops”

4. Step: Introduce “deformations” $< \cdot >_{\phi_s}^{ax}$ of $< \cdot >_{CS}^{ax}$ w.r.t. a family $(\phi_s)_{s>0}$ of diffeomorphisms of $\mathbb{R}^3$ such that $\phi_s \to \mathrm{id}_{\mathbb{R}^3}$ as $s \to 0$ in a certain sense (“Framing”)

5. Step + 6. Step: Existence proof and computation of $\lim_{s \to 0} \lim_{\epsilon \to 0} < \text{WLF}(L, \epsilon) >_{\phi_s}^{ax}$

3. Step: Loop Smearing

Recall $\mathrm{WLF}(L)(A) = \prod_i \mathrm{Tr}(\mathcal{P}\exp(\int_{l_i} A))$

with $\mathcal{P}\exp(\int_l A) = P_1^l(A)$ where

$\frac{d}{dt}P_t^l(A) + A_{l(t)}(l'(t))) \cdot P_t^l(A) = 0, \;\; P_0^l(A) = 1$

Problem: The expression $A_{l(t)}(l'(t))$ does not make sense for a general element A of $\overline{\mathcal{A}^{ax}} = \mathcal{S}'_{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{R}^3)$.

Solution: We "smear" the loops:

1. We replace $A_{l(t)}(l'(t))$ by

$$A_{l^\epsilon(t)}(l'(t)) := \sum_{i=0}^{1} A(l^\epsilon(t))_i \cdot l'_i(t)$$

where $l^\epsilon(t) = \psi^\epsilon(\cdot - l(t))$, $(\psi^\epsilon)_{\epsilon>0}$ being a suitable "Dirac-family" (see below)

2. Set $\mathrm{WLF}(L,\epsilon)(A) := \prod_i \mathrm{Tr}(\mathcal{P}\exp(\int_{l_i^\epsilon} A))$

where $\mathcal{P}\exp(\int_{l^\epsilon} A) := P_1^{l^\epsilon}(A)$,

$\frac{d}{dt}P_t^{l^\epsilon}(A) + A_{l^\epsilon(t)}(l'(t)) \cdot P_t^{l^\epsilon}(A) = 0, \;\; P_0^{l^\epsilon}(A) = 1$

3. Replace $< \mathrm{WLF}(L) >_{CS}^{ax}$ by $\lim_{\epsilon\to 0} < \mathrm{WLF}(L,\epsilon) >_{CS}^{ax}$

Remark: Naive approach $\psi^\epsilon(x) := \frac{1}{\epsilon^3}\psi(\frac{x}{\epsilon})$ with fixed $\psi \in C^\infty(\mathbb{R}^3, \mathbb{R}_+)$ fulfilling $\int \psi(x)dx = 1$, $\text{supp}(\psi) \subset B_1(0)$ does not work. Instead take

$$\psi^\epsilon(x) := \frac{1}{\epsilon^{3+\vartheta}}\psi(\frac{x\cdot\hat{a}}{\epsilon}\cdot\hat{a} + \frac{x\cdot\hat{b}}{\epsilon^{1+\vartheta}}\cdot\hat{b} + \frac{x\cdot e_2}{\epsilon}\cdot e_2)$$

where $(\hat{a}, \hat{b})$ is a fixed oriented ONB of $\mathbb{R}^2$ ($\hat{a}$: "loop-smearing axis") and $0 < \vartheta < \frac{1}{2}$ ("smearing exponent")

4. Step: Framing

Already in the simple case $G = U(1)$ "Framing" is necessary to avoid the so-called "Self-Linking Problem":

Let $G = U(1)$. It can be shown that

$$\begin{aligned}\mathrm{WLO}(L,\epsilon) &:= < \mathrm{WLF}(L,\epsilon) >_{CS}^{ax} \\ &= \prod_{i,j} \exp(\pi i \lambda Q^{ax}_{l_i,l_j}(\epsilon))\end{aligned}$$

for suitable expressions $Q^{ax}_{l_i,l_j}(\epsilon)$ with the property

$$\lim_{\epsilon \to 0} Q^{ax}_{l_i,l_j}(\epsilon) = \mathrm{LK}(l_i, l_j) \quad \text{if } i \neq j$$

However, $\lim_{\epsilon \to 0} Q^{ax}_{l_i,l_i}(\epsilon)$ is not related to a link invariant.

Remedy: Additional Regularization: "Framing":

1. Choose a family $(\phi_s)_{s>0}$ of diffeomorphisms of $\mathbb{R}^3$ with $\phi_s \to \mathrm{id}_{\mathbb{R}^3}$ as $s \to 0$ in a certain (very weak) sense and certain additional natural properties
2. Deform $< \cdot >_{CS}^{ax}$ by ϕ_s in a natural way and denote the deformation by $< \cdot >_{\phi_s}^{ax}$
3. Define regularized WLOs by

$$\mathrm{WLO}(L; (\phi_s)_s) := \lim_{s \to 0} \lim_{\epsilon \to 0} < \mathrm{WLF}(L, \epsilon) >_{\phi_s}^{ax}$$

5. Step: Existence of $\lim_{s\to 0}\lim_{\epsilon\to 0} < \mathrm{WLF}(L,\epsilon) >^{ax}_{\phi_s}$

Theorem 1 *For every admissible link* L *in* $\mathbb{R}^3$ *and every admissible framing* $\phi := (\phi_s)_{s>0}$ *of* L

$$\mathrm{WLO}(L;\phi) := \lim_{s\to 0}\lim_{\epsilon\to 0} < \mathrm{WLF}(L,\epsilon) >^{ax}_{\phi_s}$$

exists

6. Step: Computation of $\lim_{s\to 0}\lim_{\epsilon\to 0} < \mathrm{WLF}(L,\epsilon) >^{ax}_{\phi_s}$

Before we state the main result (Theorem 2 below) let us first look at two examples:

1. Example Let $G = U(1)$. Then for every admissible link L and every admissible framing $\phi := (\phi_s)_{s>0}$ we have

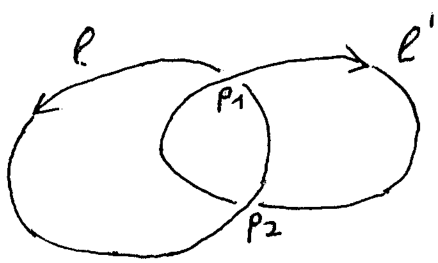
$$WLO(L;\phi) = \exp(\lambda\pi i \sum_j lk_j) \exp(\lambda\pi i \sum_{j\neq k} LK(l_j,l_k))$$

with $lk_j = \lim_{s\to 0} LK(l_j, \phi_s \circ l_j)$.

Here $LK(l,l')$ is the “linking number” of l and l'

- $\epsilon(p) \in \{-1, 1\}$ for $p \in V(L)$ where $V(L)$ is the set of all crossings of L:

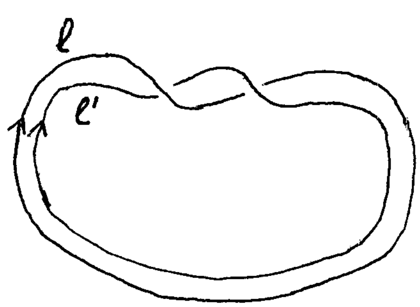
- $LK(l, l') := \frac{1}{2} \sum_{p \in cr(l,l')} \epsilon(p)$ where $cr(l, l')$ is the set of all crossings of l and l'.



$$LK(l, l') = \frac{1}{2}\left[\varepsilon(p_1) + \varepsilon(p_2)\right]$$

$$\frac{1}{2}[1 + 1] = 1$$

- $w(L) := \sum_{p \in V(L)} \epsilon(p)$ ("wraith" of L)
- $twist(l, l')$: Number of "twists" of the ribbon obtained by interpolating $l(t)$ and $l'(t)$ for every $t \in [0, 1]$ (defined if l is sufficiently close to l')



$$twist(l, l') = 1$$

2. Example Let $G = SU(N)$. Then $\mathrm{WLO}(L;\phi)$ is independent of $(\hat{a},\hat{b})$ for every admissible link L and every admissible framing $\phi = (\phi_s)_{s>0}$ if and only if $\lambda \in 2\mathbb{Z}$. In this case

$$\mathrm{WLO}(L;\phi) = N^{\#L} \prod_{j \leq n} \exp(-\tfrac{\lambda \pi i}{N}\, t_j)\ \exp(-\tfrac{\lambda \pi i}{N}\ \mathrm{w}(L))$$

for all L and $\phi := (\phi_s)_{s>0}$ as above where $\#L$ is the number of components of L and $t_j := \lim_{s\to 0} \mathrm{twist}(l_j, \phi_s \circ l_j)$.

Comparison of Example 2 with the standard physics literature:

Standard literature expects (†) for $\lambda \in \Lambda := \{\pm\frac{1}{N+1}, \pm\frac{1}{N+2}, \ldots\}$

$$\begin{aligned}\mathrm{WLO}(L;\phi) &\overset{(\dagger)}{=} \mathrm{Homfly}_L(\exp(\lambda \pi i N), 2i\sin(\lambda\pi)) \\ &\qquad \prod_{j\leq n} \exp(\lambda \pi i\, \tfrac{N^2-1}{N} t_j)\exp(\lambda\pi i\, \tfrac{N^2-1}{N}\,\mathrm{w}(L)) \\ &\overset{(*)}{=} N^{\#L} \prod_{j \leq n} \exp(-\tfrac{\lambda \pi i}{N}\, t_j)\ \exp(-\tfrac{\lambda \pi i}{N}\ \mathrm{w}(L))\end{aligned}$$

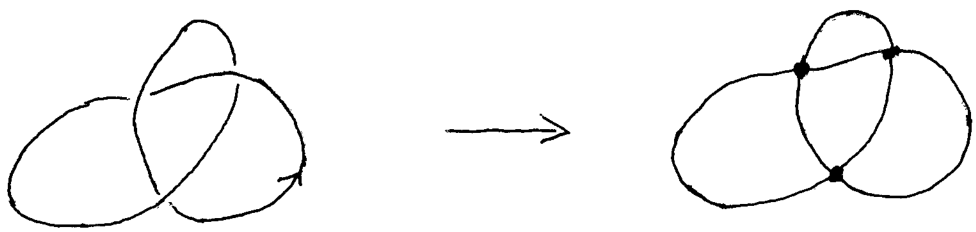
$(*)$ holds if $\lambda \in 2\mathbb{Z}$ $\quad\Rightarrow$

Example 2 suggests that Λ should be replaced by $2\mathbb{Z}$.

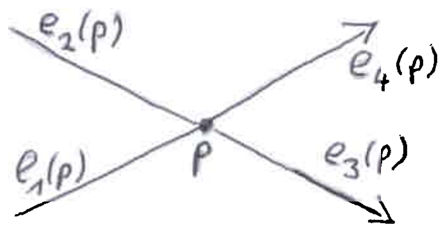
States and State Models

For a sufficiently regular link L in $\mathbb{R}^3$ consider

- $(V(L), E(L))$: Planar graph obtained by projecting L onto the plane $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\}$.



- $St(L)$: Set of all mappings $E(L) \mapsto \{1, 2, \ldots, N\}$
- For $p \in V(L)$ define $e_i(p) \in E(L)$, $1 \leq i \leq 4$, by



- For $R^+, R^- \in \otimes^2 \operatorname{Mat}(N, \mathbb{C})$ we set

  $\sigma_{(R^+,R^-)}(L) := \sum_{f \in St(L)} \prod_{p \in V(L)} (R^{\epsilon(p)})^{f(e_1(p))f(e_2(p))}_{f(e_3(p))f(e_4(p))}$

  where $(R^{ab}_{cd})_{a,b,c,d}$ for $R \in \{R^+, R^-\}$ is given by

$$R = \sum_{a,b,c,d} R^{ab}_{cd} \cdot e_a \otimes e_c \otimes e_b \otimes e_d$$

  (after identification $\otimes^2 \operatorname{Mat}(N, \mathbb{C}) \cong \otimes^4 \mathbb{C}^N$)

Set

$$\mathfrak{E} := \sum_{a,b} E_{aa} \otimes E_{bb} \in \otimes^2 \mathrm{Mat}(N, \mathbb{C})$$
$$\mathfrak{P} := \sum_{a,b} E_{ab} \otimes E_{ab} \in \otimes^2 \mathrm{Mat}(N, \mathbb{C})$$

where $E_{ab} \in \mathrm{Mat}(N, \mathbb{C})$ is given by $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

Theorem (Kauffman): Let $A \in \mathbb{C}\backslash\{0\}$ such that $A^2 + A^{-2} = -N$ and set

$$R^+ := A \cdot \mathfrak{E} + A^{-1} \cdot \mathfrak{P},$$
$$R^- := A^{-1} \cdot \mathfrak{E} + A \cdot \mathfrak{P}$$

Then

$$\sigma_{(R^+,R^-)}(L) = \sum_{f \in St(L)} \prod_{p \in V(L)} (R^{\epsilon(p)})^{f(e_1(p))f(e_2(p))}_{f(e_3(p))f(e_4(p))}$$

$$= Jones_L(A^{-4}) \cdot (-A^3)^{-w(L)}$$

Main Result

Theorem 2 *In the special case where $(\phi_s)_{s>0}$ is "vertical" we have for arbitrary G*

$$WLO(L;\phi) = \sum_{f\in St(L)} \prod_{p\in V(L)} (R^{\epsilon(p)}_{type(p)})^{f(e_1(p))f(e_2(p))}_{f(e_3(p))f(e_4(p))}$$

with

$$R_1^{\pm} := \exp\big(\mp\lambda\pi i(\textstyle\sum_a T_a \otimes T_a)\big) \cdot \mathfrak{T}$$
$$R_2^{\pm} := \Xi(\exp\big(\mp\lambda\pi i \cdot \Xi(\textstyle\sum_a T_a \otimes T_a)\big) \cdot \mathfrak{T}$$

where

- $(T_a)_{a\leq\dim(\mathfrak{g})}$ *is an arbitrary* $(\cdot,\cdot)_{\mathfrak{g}}$*-orthonormal basis of* $\mathfrak{g}$
- Ξ *is the linear automorphism of* $\otimes^2 \mathrm{Mat}(N,\mathbb{C})$ *given by* $\Xi(A\otimes B) = A\otimes B^t$ *for all* $A,B\in \mathrm{Mat}(N,\mathbb{C})$
- $\mathfrak{T} := \sum_{a,b} E_{ab}\otimes E_{ba} \in \otimes^2 \mathrm{Mat}(N,\mathbb{C})$
- $type(p) = 1$ *if the tangent vectors of* $e_1(p)$ *and* $e_2(p)$ *in the point* p *lie both in* H_+ *or both in* H_- *where* $H_\pm := \{x\in\mathbb{R}^2 \mid \pm x\cdot\hat{b} > 0\}$
- $type(p) = 2$ *otherwise*

Comparison with Kauffman's State Models

For the special case $G = SO(N)$ and $\lambda = n + \frac{1}{2}$, $n \in \mathbb{Z}$, we have

$$R_1^+ = A \cdot \mathfrak{E} + B \cdot \mathfrak{P}, \qquad R_1^- = (R_1^+)^{-1}$$
$$R_2^- = B \cdot \mathfrak{E} + A \cdot \mathfrak{P}, \qquad R_2^+ = (R_2^-)^{-1}$$

with certain $A, B \in \mathbb{C}$. It is easy to see that

$$A \neq 0, B = A^{-1}, A^2 + A^{-2} = -N \iff R_1^\pm = R_2^\pm$$

However, $R_1^\pm \neq R_2^\pm$ $(A = i,\ B = i\frac{(-i)^N - 1}{N}$ if n is even$)$

When does $R_1^{\pm} = R_2^{\pm}$ hold?

- G abelian $\Longrightarrow$ $R_1^{\pm} = R_2^{\pm}$ holds.
- $G = SU(N)$, $N \geq 2$, or $G = SO(N)$, $N \geq 3$,

  $R_1^{\pm} = R_2^{\pm}$ holds $\Longleftrightarrow$ $\lambda \in 2\mathbb{Z}$

In both situations the WLOs can be expressed by linking numbers, $w(L)$, and twist expressions.

Comparison with State Models of Jones/Turaev

State Models of Jones/Turaev can be used to give an explicit representation of $\text{Homfly}_L(\exp(\lambda\pi i N), 2i\sin(\lambda\pi))$ for $\lambda \in \Lambda := \{\pm\frac{1}{N+1}, \pm\frac{1}{N+2}, \pm\frac{1}{N+3}, \dots\}$

State sums are of the form

$$\sum_{f\in St(L)} \prod_{p\in V(L)} (R^{\epsilon(p)})^{f(e_1(p))f(e_2(p))}_{f(e_3(p))f(e_4(p))} \cdot \Psi(f)$$

with

$$\begin{aligned} R^+ &:= +q\sum_a E_{aa}\otimes E_{aa} + \sum_{a\neq b} E_{ab}\otimes E_{ba} \\ &\quad + (q^{-1}-q)\sum_{a<b} E_{aa}\otimes E_{bb} \\ R^- &:= +q^{-1}\sum_a E_{aa}\otimes E_{aa} + \sum_{a\neq b} E_{ab}\otimes E_{ba} \\ &\quad + (q-q^{-1})\sum_{a>b} E_{aa}\otimes E_{bb} \end{aligned}$$

$q := e^{i\pi\lambda}$, and where $\Psi(f)$ depends on the "turning points" of $(V(L), E(L))$ w.r.t. a fixed "time axis".

R^+, R^- are related to quantum group $SU_q(N)$

Note: $\lambda \in 2\mathbb{Z} \;\Leftrightarrow\; q = 1$ (i.e. $SU_q(N) = SU(N)$)

Hypothesis: It is not original Chern-Simons theory on $\mathbb{R}^3$ with structure group $G = SU(N)$ that is related to non-trivial HOMFLY polynomial expressions but a suitable deformation of this theory.

What kind of deformation?

- Deformation involving Quantum Groups?
- Quantum Chern-Simons Models?
- Deformation involving Quantum Probability?
- Or do we simply have to "deform" the base manifold, i.e. replace $M = \mathbb{R}^3$ by S^3 or another compact manifold?

Conclusions

Results:

- If $M = \mathbb{R}^3$ it is possible to work with axial gauge using the approach of Albeverio/Sengupta '97. In particular, the WLOs can be defined rigorously.
- The values of the WLOs can be expressed explicitly; state models arise in a natural way.

Issues open for Discussion/Open Questions:

- Is $\lambda \in \{\pm\frac{1}{N+1}, \pm\frac{1}{N+2}, \pm\frac{1}{N+3}, \dots\}$ really the "correct" charge quantization condition if $M = \mathbb{R}^3$ and $G = SU(N)$ or should it be replaced by $\lambda \in 2\mathbb{Z}$?
- Is the Hypothesis on the previous slide true?