Path integrals for the Dirac equation on $\mathbb{R}$

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1. **Evolution Processes**

- S: semigroup of continuous linear operators acting on a Banach space X. $S(t+s) = S(t)S(s)$, $s,t \geq 0$, $S(0) = I_X$.
- Q: spectral measure acting on X. $(\Sigma, \mathcal{E})$ measurable space. $Q : \mathcal{E} \to \mathcal{L}(X)$, $Q(A \cap B) = Q(A)Q(B)$, for all $A, B \in \mathcal{E}$, $Q(\Sigma) = I_X$.
- Ω: $\Sigma^{[0,\infty)}$, $\mathcal{S}_t$: algebra generated by all sets

$$E = \{\omega \in \Omega : \omega(t_1) \in B_1, \ldots, \omega(t_n) \in B_n\}$$

for all $0 \leq t_1 < \cdots < t_n \leq t$, all $B_1, \ldots, B_n \in \mathcal{E}$, and all $n = 1, 2, \ldots$

Set $M_t(E) \in \mathcal{L}(X)$ equal to

$$S(t-t_n)Q(B_n)S(t_n-t_{n-1}) \cdots S(t_2-t_1)Q(B_1)S(t_1).$$

This defines an additive set function

$$M_t : \mathcal{S}_t \to \mathcal{L}(X)$$

called the (S, Q, t)-*set function* on $\mathcal{S}_t$ [Kluvanek, C. '78].

2. **Dirac equation in 1 space dimension**

Example (1-D Dirac). Let $X = L^2(\mathbb{R}, \mathbb{C}^2)$

$$
\begin{aligned}
H_0 &= \alpha \frac{1}{i}\frac{\partial}{\partial x} + m\beta, \qquad m \in \mathbb{R} \\
\alpha &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\alpha^2 &= \beta^2 = Id, \qquad \alpha\beta + \beta\alpha = 0.
\end{aligned}
$$

- $S(s) = e^{-iH_0 s}$,
- Q: multiplication by ch'istic functions.

Then the (S, Q, t)-set function M_t is bounded on $\mathcal{S}_t$ for each $t \geq 0$ and defines a σ-additive operator valued measure acting on X.

The space $\Omega \subset \mathbb{R}^{[0,\infty)}$ can be taken to be all functions $\omega : [0, \infty) \to \mathbb{R}$ with speed ± 1 and only finitely many changes in direction in any bounded interval.

The matrix valued measures

$$\nu_{t,x}(E)e_j = M_t(E)(\delta_x e_j), \qquad E \in \mathcal{S}_t,\ j = 1, 2$$

were constructed by T. Ichinose ['82].

The *Feynman-Kac formula*

$$e^{-it(H_0+V)} = \int_\Omega e^{-i\int_0^t V(\omega(s))ds} dM_t(\omega)$$

works for *locally integrable* V.

H_0+V is a first order matrix-valued differential operator which is essentially selfadjoint.

The 1-D Coulomb potential

$$V(x) = \frac{\gamma}{|x|}, \quad x \in \mathbb{R},\ x \neq 0,$$

is not integrable about 0 and the symmetric differential operator H_0+V has a 4-parameter family of selfadjoint extensions.

The Feynman-Kac formula becomes

$$e^{-it(H_0+_\Gamma V)} = \int_\Omega e^{-i\langle\int_0^t V(\omega(s))ds\rangle_\Gamma} dM_t(\omega).$$

Here Γ is a unitary matrix prescribing boundary conditions at 0 and $H_0+_\Gamma V$ is the associated selfadjoint extension.

3. Multiplicative functionals

The integral $\langle \int_0^t V(\omega(s))ds\rangle_\Gamma$ is "renormalised" so that

$$F_t(\omega) = e^{-i\langle \int_0^t V(\omega(s))ds\rangle_\Gamma}, \quad \omega \in \Omega,\ t \geq 0,$$

is a *multiplicative functional*:

$$(F_t \circ \theta_s).F_s = F_{s+t}$$

where $\theta_s\omega : r \mapsto \omega(r+s)$ defines a shift map $\theta_s : \Omega \to \Omega$.

For an integrable multiplicative function $F_t : \Omega \to \mathbb{C}$, $t \geq 0$, the operators

$$R(t) = \int_\Omega F_t(\omega)\, dM_t(\omega), \quad t \geq 0,$$

form a *semigroup*:

$$\begin{aligned} R(s+t) &= \int_\Omega F_{s+t}(\omega)\, dM_{s+t}(\omega) \\ &= \int_\Omega F_t(\theta_s\omega).F_s(\omega)\, dM_{s+t}(\omega) \\ &= \left(\int_\Omega F_t(\omega)\, dM_t(\omega)\right) \int_\Omega F_s(\omega)\, dM_s(\omega) \\ &= R(t)R(s). \end{aligned}$$

The generator of the semigroup R is a "perturbation" of the generator of the original unitary group $S(t) = e^{-itH_0}$, $t \in \mathbb{R}$.

The Coulomb potential $\frac{\gamma}{|x|}$, $x \neq 0$, is a singular perturbation of H_0.

3. Zero mass: $m = 0$

$$H_0 = \alpha \frac{1}{i}\frac{\partial}{\partial x}, \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\Gamma = \eta \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$ is a unitary matrix, with $\alpha, \beta, \eta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$ and $|\eta| = 1$,

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \left(e^{-it(H_0 + \Gamma V)} \phi \right)(x), \quad x \in \mathbb{R},$$

satisfies the boundary conditions

$$\begin{pmatrix} \lim_{x\to 0+} |x|^{i\gamma} u(x) \\ \lim_{x\to 0-} |x|^{i\gamma} v(x) \end{pmatrix} = \Gamma \begin{pmatrix} \lim_{x\to 0-} |x|^{-i\gamma} u(x) \\ \lim_{x\to 0+} |x|^{-i\gamma} v(x) \end{pmatrix}$$

and

$$u(x,t) = \begin{cases} e^{-i\int_{x-t}^{x} V(s)\,ds} \phi_1(x-t), \\ \qquad \text{for all } x > t,\ x < 0, \\ \eta e^{-i\gamma(\ln|x| + \ln|x-t|)} [\alpha \phi_1(x-t) - \overline{\beta} \phi_2(-(x-t))], \\ \qquad \text{for all } 0 < x < t. \end{cases}$$

$$v(x,t) = \begin{cases} e^{-i\int_{x}^{x+t} V(s)\,ds} \phi_2(x+t), \\ \qquad \text{for all } x < -t,\ x > 0, \\ \eta e^{-i\gamma(\ln|x| + \ln(x+t))} [\beta \phi_1(-(x+t)) + \overline{\alpha} \phi_2(x+t)] \\ \qquad \text{for all } -t < x < 0. \end{cases}$$

Suppose that $\beta = 0$ and $\eta\alpha = e^{-i\kappa_1}$, $\eta\overline{\alpha} = e^{-i\kappa_2}$ for numbers $0 \leq \kappa_j < 2\pi$.

- Ω: all ω for which $\exists x \in \mathbb{R}$ s.t. $\omega(s) = x + s$ or $\omega(s) = x - s$, $s \geq 0$. $X_s(\omega) = \omega(s)$, $\omega \in \Omega$.

If $X_0(\omega)X_t(\omega) < 0$, set

$$\left\langle \int_0^t V \circ X_s(\omega)\, ds \right\rangle_\Gamma =$$
$$\begin{cases} \gamma(\ln|\omega(0)| + \ln|\omega(t)|) + \kappa_1 & \text{if } \omega'(s) = 1,\ s > 0, \\ \gamma(\ln|\omega(0)| + \ln|\omega(t)|) + \kappa_2 & \text{if } \omega'(s) = -1,\ s > (\end{cases}$$

If $0 < x < t$ and $\omega(s) = x - s$, then

$$\int_{[0,t]\cap\{|\omega(s)|>\epsilon\}} V(\omega(s))\, ds$$
$$= \int_{x+\epsilon}^{t} \frac{\gamma}{|x-s|}\, ds + \int_0^{x-\epsilon} \frac{\gamma}{|x-s|}\, ds$$
$$= \gamma(\ln|\omega(0)| + \ln|\omega(t)|) - 2\gamma \ln \epsilon,$$

as $\epsilon \to 0+$, so we are subtracting a logarithmic divergence.

Then $F_t^\Gamma : \Omega \to \mathbb{C}$ is defined by

$$F_t^\Gamma = \chi_{\{X_0X_t>0\}} . e^{-i\int_0^t V\circ X_s\, ds} + \chi_{\{X_0X_t<0\}} . e^{-i\langle \int_0^t V\circ X_s\, ds\rangle_\Gamma}$$

and $e^{-it(H_0 + \Gamma V)} = \int_\Omega F_t^\Gamma\, dM_t$.

4. Nonzero mass: $m \neq 0$

$$H_0 = \alpha \frac{1}{i}\frac{\partial}{\partial x} + m\beta, \qquad m \in \mathbb{R},\ m \neq 0.$$

- Ω: paths $\omega : [0,\infty) \to \mathbb{R}$ with speed ± 1 and finitely many changes in direction in any bounded interval.

Suppose ω changes direction at consecutive times $\tau_k(\omega)$, $k = 1, \ldots, K(\omega)$ with $\tau_{K+1}(\omega) = t$. On the set of ω with $K(\omega) = n$, set

$$F_t^\Gamma = \exp\Bigg[-i \sum_{k=0}^{n} \chi_{\{X_{\tau_k} X_{\tau_{k+1}} > 0\}} \left(\int_{\tau_k}^{\tau_{k+1}} V \circ X_s \, ds \right) + \chi_{\{X_{\tau_k} X_{\tau_{k+1}} < 0\}} \left\langle \int_{\tau_k}^{\tau_{k+1}} V \circ X_s \, ds \right\rangle_\Gamma \Bigg].$$

The expression $\left\langle \int_{\tau_k}^{\tau_{k+1}} V \circ X_s \, ds \right\rangle_\Gamma$ is given by

$$\begin{cases} \gamma(\ln|\omega(\tau_k)| + \ln|\omega(\tau_{k+1})|) + \kappa_1, \\ \qquad \text{if } \omega'(s) = 1,\ \tau_k(\omega) < s < \tau_{k+1}(\omega), \\ \\ \gamma(\ln|\omega(\tau_k)| + \ln|\omega(\tau_{k+1})|) + \kappa_2, \\ \qquad \text{if } \omega'(s) = -1,\ \tau_k(\omega) < s < \tau_{k+1}(\omega). \end{cases}$$

and

$$e^{-it(H_0 + \Gamma V)} = \int_\Omega F_t^\Gamma \, dM_t.$$

Sketch of proof.

$$S(t) \;=\; e^{-it(\alpha\frac{1}{i}\frac{\partial}{\partial x}+m\beta)} = e^{-it\alpha\frac{1}{i}\frac{\partial}{\partial x}} + \sum_{n=1}^{\infty}(-im)^n R_n(t)$$

with

$$R_n(t) \;=\; \int_0^t \cdots \int_0^{s_2} e^{-(t-s_n)\alpha\frac{\partial}{\partial x}}\beta e^{-(s_n-s_{n-1})\alpha\frac{\partial}{\partial x}} \cdots \beta e^{-(s_2-s_1)\alpha\frac{\partial}{\partial x}}\beta e^{-s_1\alpha\frac{\partial}{\partial x}}\, ds_1 \ldots ds_n$$

converges absolutely in the operator norm of $\mathcal{L}(L^2(\mathbb{R},\mathbb{C}^2))$. Put $R_0(t) = e^{-t\alpha\frac{\partial}{\partial x}}$ for $t \geq 0$.

For each cylinder set

$$E = \{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\},\ \text{set}$$

$$M_t^{(n)}(E) = \sum_{\substack{n_0+\cdots+n_k=n \\ n_0,\ldots,n_k\geq 0}} R_{n_k}(t-t_k)Q(B_k)R_{n_{k-1}}(t_k-t_{k-1})\cdots \cdots Q(B_2)R_{n_1}(t_2-t_1)Q(B_1)R_{n_0}(t_1).$$

Then $M_t = \sum_{n=0}^{\infty}(-im)^n M_t^{(n)}$ and $\int_\Omega F_t^{\ulcorner}\, dM_t^{(n)} =$

$$\int_0^t \cdots \int_0^{s_2} e^{-i(t-s_n)(A+\ulcorner V)}\beta e^{-i(s_n-s_{n-1})(A+\ulcorner V)} \ldots \beta e^{-i(s_2-s_1)(A+\ulcorner V)}\beta e^{-is_1(A+\ulcorner V)}\, ds_1 \cdots ds_n$$

where $A = \alpha\frac{1}{i}\frac{\partial}{\partial x}$.

5. Suggestions

- The Feynman-Kac functional

$$F_t(\omega) = e^{-i\int_0^t V(\omega(s))\,ds}, \qquad \omega \in \Omega,$$

may not be the appropriate *multiplicative functional* (MF) for singular interactions V: boundary conditions at the singularities may need to be incorporated via 'renormalisation'.

- Construction of MF F_t with $|F_t(\omega)| = 1$ only requires a notion of *measurability* e.g. scale invariant measurability.

- Dynamics given by a 'Feynman integral'

$$e^{-itH} = \int_\Omega F_t\,dM_t, \qquad t > 0.$$

- The Feynman integral $\int_\Omega e^{-i\int_0^t V\circ X_s\,ds}\,dM_t$ need not be a unitary operator [Nelson '64].

- Construction of MF F_t, $t > 0$ representing an interaction in quantum field theory requires 'renormalisation' in phase *everywhere*, but still $|F_t(\phi)| = 1$.