A Flexible Tool *

C/Δ-M Scheme

$$\text{Gaussian} \swarrow \qquad \searrow \text{Gaussian} + ?$$

Parabolic — Elliptic, Parabolic, Hyperbolic

(2nd order linear)

$? \leadsto \tau : [t_a, t_b] \subseteq \mathbb{R} \to [0, \tau_b] \subseteq \mathbb{R}_+$

$\Pi \equiv$ space of pointed paths with $\tau \in T$

$\mathcal{D}\gamma_{\omega,\sigma}(\tau) \equiv$ "gamma" integrator

Motives:

- Physics
- Variational Principle
- Stochastic

Applications:

- $B^n \subset \mathbb{R}^n$
- ΨDOs

C/D-M Scheme

Domain: $z \in \mathbb{Z}_0$, $z' \in \mathbb{Z}_0'$; $\langle z', z \rangle$

$\mathbb{Z}_0$ a Banach space

$\mathbb{Z}_0'$ separable Banach space with defined norm which admits complex Borel measures μ.

(usually a result of parametrization

$$dx(t) - Y(x(t))\,dt = X(x(t))\,dz \;)$$

Integrator: $\Theta : \mathbb{Z}_0 \times \mathbb{Z}_0' \to \mathbb{C}$, $Z : \mathbb{Z}_0' \to \mathbb{C}$

$$\int_{\mathbb{Z}_0} \Theta(z,z')\, \mathcal{D}_{\Theta,Z}\, z := Z(z')$$

Exs. 1) $\Theta(z,z') = e^{-\pi/s\, Q(z) - 2\pi i \langle z', z \rangle}$ $\quad s \in \mathbb{C}_+$

$Z(z') = e^{-\pi s W(z')}$ (Gaussian integrator)

2) $\lim\limits_{s \to 0}$ of 1) (Delta integrator)

3) $\Theta(z,z') = e^{L_\omega(\tau) - \langle \tau', \tau \rangle}$

$Z(z') = \dfrac{1}{(\underset{\sim}{\tau}' - \omega)^{\nu}}$

$\langle \tau', \tau \rangle = \underset{\sim}{\tau}' \int_{t_a}^{t_b} \dot{\tau}\, dt$, $\underset{\sim}{\tau}', \omega \in \mathbb{C}$ (Gamma integrator)

Integrand: $F_\mu(z) = \int_{z'} \Theta(z,z')\, d\mu(z')$

$$\boxed{\int_{\mathcal{P}_a N} F(x)\, \mathcal{D}x := \int_{Z_0} F_\mu(x(z))\, \mathcal{D}_{0,z} z = \int_{Z'_0} Z(z')\, d\mu(z')}$$

Important Properties of path integrals

Change of variable:

Gaussian

$$\int_{Z_0} F(z(t))\, \mathcal{D}\omega(z)$$

$$\Big\downarrow \; z \longmapsto z(t) \in \mathbb{R}^n$$

$$C \int_{\mathbb{R}^n} F(u)\, e^{-\frac{\pi}{s}\, u^2/t}\, d^n u \qquad C = (st)^{-n/2}$$

Delta

$$\int_{Z_0} F(z(t))\, \mathcal{D}\delta(z)$$

$$\Big\downarrow$$

$$\int_{\mathbb{R}^n} \delta^n(u)\, F(u)\, d^n u$$

Gamma

$$\int_{\mathbb{T}} F(\tau(t))\, \mathcal{D}\gamma_{\omega,\nu}(\tau)$$

$$\Big\downarrow \; \tau \longmapsto \tau(t)$$

$$C \int_{\mathbb{R}_+} F(\rho)\, \rho^{\nu-1} e^{\omega\rho}\, d\rho \qquad C^{-1} = \Gamma(\nu)$$

$$\int_{\Pi} \mathcal{D}\tau := \lim_{\substack{\nu\to 0^+ \\ \omega\to\infty}} \int_{\Pi} \mathcal{D}\gamma_{\omega,\nu}(\tau) \qquad (\mathrm{Im}\,\omega = 0)$$

$\mathcal{D}\tau$ is the integrator associated with τ that is useful for solving PDEs with B.C.s .

$$\begin{array}{c} \displaystyle\int_{\Pi} F(\tau(t))\,\mathcal{D}\tau \\ \Big\downarrow \; \tau \longmapsto \tau(t) \\ \displaystyle N \int_{\mathbb{R}_+} F(\rho)\,\frac{d\rho}{\rho} \end{array} \qquad \left(N \sim \frac{e^{\infty}}{\Gamma(0^+)} \right)$$

Integration by parts

$$0 = \int \frac{\delta F(z)}{\delta z(t)} \mathcal{D}z = -\int F(z) \frac{\delta}{\delta z(t)} \mathcal{D}z$$

(requires $\int \frac{\delta}{\delta z(t)} [F \mathcal{D}z] = 0$)

Gassian

$$\int_{Z_0} F(z+z^0) \mathcal{D}z = \int_{Z_0} F(z) \mathcal{D}z$$

(i.e. $\mathcal{D}(z-z^0) = \mathcal{D}z$)

Gamma

$$\int_{T} F(\epsilon\tau) \mathcal{D}\tau = \int_{T} F(\tau) \mathcal{D}\tau \qquad \epsilon \in R_+$$

(i.e. $\mathcal{D}(\epsilon\tau) = \mathcal{D}\tau$)