Feynman (oscillatory) integrals on Hilbert spaces and Schrödinger equation with magnetic field

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Part I. ~~Ex~~ Feynman path integrals of Albeverio - Hoegh Krohn via Elworthy - Truman

Def. 1 If $h \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\operatorname{Im} h \leq 0$, $f : \mathbb{R}^N \to \mathbb{C}$ Borel measurable,

$$\widetilde{\int_{\mathbb{R}^N}} e^{\frac{i}{2h}|x|^2} f(x)\,dx = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} e^{\frac{i}{2h}|x|^2} f(x)\psi(\varepsilon x)\,dx$$

$$\left(\times \left(2\pi i h\right)^{-N/2}\right)$$

provided the limit exists and independent of $\psi \in S(\mathbb{R}^N) : \psi(0) = 1$.

Th. 2. If $L : \mathbb{R}^N \to$ [redacted] $\mathbb{R}^N$ linear

$g = \hat{\mu} \in \mathcal{F}(\mathbb{R}^N)$, μ - complex measure on $\mathbb{R}^N$

and $I + L$ is an isomorphism,

$$\widetilde{\int_{\mathbb{R}^N}} e^{\frac{i}{2h}|x|^2} \underbrace{e^{+\frac{i}{2h}|\text{[redacted]}^2 \langle Lx, x\rangle} g(x)}_{f(x)} dx =$$

$$= \det(1+L)^{-1/2} \int_{\mathbb{R}^N} e^{-\frac{i}{2h}\langle (1+L)^{-1}x, x\rangle} d\mu(x)$$

$$S(\gamma) = \frac{1}{2}\int_0^T |\dot{\gamma}(s)|^2 ds + \int_0^T V_1(\gamma(s))ds$$

$$= \frac{1}{2}|\gamma|^2 + \frac{1}{2}\langle L\gamma, \gamma\rangle + \underbrace{V(\gamma)}_{\int_0^T V_0(\gamma(s))ds}$$

Then

$$\tilde{\int}_{H_T + x} e^{\frac{i}{h}S(\gamma)} g(\gamma) d\gamma$$

$$= \tilde{\int}_{H_T + x} e^{\frac{i}{2h}|\gamma|^2} e^{\frac{i}{2h}\langle L\gamma,\gamma\rangle} f(\gamma) d\gamma$$

with $f(\gamma) = e^{\frac{i}{h}V(\gamma)} g(\gamma)$

provided the oscillatory integral on RHS exists.

Remark. The above is over an affine Hilbert space $H_T + x$, $x \in \mathbb{R}^d$ fixed. Not real difference to the case of H_T.

Def. 3 If H is a separable Hilbert space, $f : H \to \mathbb{C}$ Borel measurable

$$\widetilde{\int_H} e^{\frac{i}{2h}|x|^2} f(x)\,dx := \lim_{P \to I} \widetilde{\int_{P(H)}} e^{\frac{i}{2h}|x|^2} f(x)\,dx$$

where the limit is taken with respect to the net of finite-dimensional orthogonal projections in H.

(Elworthy - Truman)

Th. 4 If $g = \hat{\mu} \in \mathcal{F}(H)$, $L : H \to H$ self-adjoint of trace class, then

$$\widetilde{\int_H} e^{\frac{i}{2h}|x|^2} \underbrace{e^{+\frac{i}{2h}\langle Lx, x\rangle} g(x)\,dx}_{f(x)} = \det(1+L)^{-\frac{1}{2}} \int_H e^{-\frac{i}{2h}\langle (1+L)^{-1}x, x\rangle} d\mu(x)$$

provided $1+L : H \to H$ is an isomorphism.

Appl. 5 Let $H = H_0^{1,2}(0,T;\mathbb{R}^d) = H_T$

$$:= \Big\{ \gamma : (0,T) \to \mathbb{R}^d \text{ abs. cont.}, \underbrace{\int_0^T |\dot\gamma(t)|^2 dt}_{=:|\gamma|^2} < \infty, \gamma(T) = 0 \Big\}$$

$\Omega^2 : \mathbb{R}^d \to \mathbb{R}^d$ self-adjoint

Define

$$L = L_T : H \to H, \text{ s.a.}$$

by
$$\langle L\gamma, \gamma \rangle = -\int_0^T \langle \Omega^2 \gamma(s), \gamma(s) \rangle ds$$

Then L is trace class because $u = L\gamma$ iff

$$\begin{cases} \ddot u(s) = \Omega^2 \gamma(s), & s \in (0,T), \\ \dot u(0) = u(T) = 0. \end{cases}$$

$$\det(I + L) = \det \cos(T\Omega)$$

Let also $V_0 \in \mathcal{F}(\mathbb{R}^d) \ni \psi_0$

$$V(x) = \frac{1}{2}\langle \Omega^2 x, x \rangle + V_0(x)$$

$$g(\gamma) := \psi_0(\gamma(0)), \quad \gamma \in H_T$$

Th. 6 (Alberverio-Högh Krohn, Elworthy Truman)

Define $\psi(t,x) = \widetilde{\int}_{H_t + x} e^{\frac{i}{\hbar} S(\gamma)} g(\gamma)\, d\gamma$

Then ψ is the unique solution to Schrödinger equation

$$\begin{cases} i\hbar \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2} \Delta \psi(t,x) + V_1(x)\, \psi(t,x) \\ \psi(0,x) = \psi_0 \end{cases}$$

Applications: Method of stationary phase (Alberverio, Högh Krohn, Brzezniak - Alberverio)

Example 7. If $C^* = -C$ in $\mathbb{R}^d$, consider

$$\begin{cases} i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2}(-i\hbar\nabla + Cx)^2\psi + V_0\psi \\ \psi(0,x) = \psi_0(x) \end{cases}$$

One would expect

$$\psi(t,x) = \int_{H_t + x} e^{\frac{i}{\hbar}S(\gamma)} g(\gamma)\, d\gamma$$

with

$$S(\gamma) = \frac{1}{2}\int_0^t |\dot\gamma(s)|^2 - \int_0^t \langle C\gamma(s), \dot\gamma(s)\rangle ds - \int_0^t V_0(\gamma(s))ds$$

Difficulty:

Operator $K = K_t : H_t \longrightarrow H_t$ defined by

$$\langle K\gamma, \gamma\rangle = 2\int_0^t \langle C\gamma(s), \dot\gamma(s)\rangle ds$$

is not trace class.

but is Hilbert-Schmidt!

Define

$$\widetilde{\int_H} e^{\frac{i}{2h}|x|^2} e^{\frac{i}{2h}\langle Lx, x\rangle} g(x)\,dx =$$

$$\lim_{P \to I} \underbrace{e^{-\frac{i}{2}\mathrm{tr}(I-PLP^*)}}_{} \widetilde{\int_{P(H)}} e^{\frac{i}{2h}|x|^2} e^{\frac{i}{2h}\langle Lx, x\rangle} g(x)\,dx$$

$$e^{-\frac{i}{2}\mathrm{tr}(I_{P(H)} - PLP^*)}$$

Note: If $L \notin$ trace class, then

$$\lim_{P \to I} \mathrm{tr}\left(I_{P(H)} - PLP^*\right)$$

doesn't exist.

Nevertheless

Theorem (Albeverio - B)

$$\widetilde{\int_H} e^{\frac{i}{2h}\left(|x|^2 + \langle Lx, x\rangle\right)} g(x)\,dx =$$

$$\det\nolimits_{(2)}(I+L)^{-1/2} \int_H e^{-\frac{ih}{2}\langle (I+L)^{-1}x, x\rangle} d\mu(x)$$

$$\det_{(2)}(I+L) = \det\left[(I+L)e^{I-L}\right]$$

Carleman-Fredholm determinant

By similar limiting procedure one can define

$$\widetilde{\int_H} e^{-\frac{\lambda}{2h}|x|^2} e^{-\frac{\lambda}{2h}\left[\frac{1}{\lambda^2}\langle Lx,x\rangle + \frac{i}{\lambda}\langle Kx,x\rangle\right]} g(x)\,dx$$

where $h>0$, $\operatorname{Re}\lambda \geq 0$.

$L = L^* \geq 0$, $K = K^*$ – both HS

Then (Albeverio - B)

$$\ldots = \det_{(2)}\left(I + \frac{1}{\lambda^2}L + \frac{i}{\lambda}K\right)^{-\frac{1}{2}} \int_H e^{-\frac{h}{2\lambda}\left\langle\left(I + \frac{L}{\lambda^2} + \frac{iK}{\lambda}\right)^{-1}x,x\right\rangle} d\mu(x)$$

Remark: One assumes that $\det_{(2)}(\dots) \neq 0$.

Analytic-continuation

(generalization of Kallianpur et al.)

If also $\lambda > 0$ then

$$\dots = \int_W e^{-\frac{1}{2\lambda^2} U(L)(x)} e^{-\frac{i}{2\lambda} U(K)(x)} \tilde{g}\left(\sqrt{\frac{\hbar}{\lambda}}\, x\right) \, dP(x)$$

where (H, W, P) is the corresponding AWS

and $U(L) : W \to \mathbb{R}$

is the 'Ramer' functional associated with L.

Ex. If $H = H_t$, $(K\gamma, \gamma) = 2\int_0^t \langle C\gamma, \dot{\gamma}\rangle ds$

then

$$U(K) = 2\int_0^t \langle Cw(s), dw(s) \rangle$$

"Lèvy-area"