Semi-classical Methods in Superconductivity.

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(after Helffer-Morame....)

MSRI. April 2003

The question in Superconductivity

Analyze the minima or local minima of the Ginzburg–Landau functional.

$$\mathcal{G}(\psi, A) = \int_\Omega \{ |(\nabla - i\kappa A)\psi|^2 \, dx$$

$$+ \int_\Omega \frac{\kappa^2}{2}(|\psi|^2 - 1)^2) \, dx$$

$$+ \kappa^2 \int_\Omega |\operatorname{curl} A - \sigma \mathcal{H}|^2 \, dx$$

"Parameters"

Ω Bnded open set in $\mathbb{R}^n$ $(n = 2, 3)$

κ Ginzburg–Landau parameter

$\mathcal{H}$, σ External magnetic field (permits to vary the intensity)

Normal state:

$$\psi = 0 \quad ; \quad A = \sigma A_e$$

$$\operatorname{curl} A_e = \mathcal{H}$$

Fact:

For σ large, the normal state is a global minimizer

(Gi-Ph)

<u>Goal</u> When decreasing σ, find the critical value σ_c for which the normal solution is no more a global (local) minimizer.
Analyze the behavior of the minimizer, for σ near σ_c

Asymptotic regime : κ <u>large</u>

Fact: $\sigma_c \underset{\sim}{\approx} \kappa$

Natural step.

Analyze the Hessian of the G-L functional at $(0, \sigma A_e)$

For σ large, the Hessian is definite positive (modulo a gauge degeneracy)

$$\mu^{(1)}(\kappa \sigma_c A_e) \approx \kappa^2$$

where $\mu^{(1)}(A)$ is the lowest eigenvalue of the Neumann realization of

$$(\nabla - iA)^2 \quad \text{in } \Omega$$

Semi-classical parameter:

$$h = 1/\kappa\sigma$$

Semi-classical question

Analyze as $h \to 0$,
the ground-state energy of

$$\sum_{j=1}^{n} (h D_{x_j} - A_j)^2$$

(Neumann realization in Ω)

and the localization of
the corresponding
ground state.

This is at least a first step
for understanding where the
minimizer is no more vanishing,
when bifurcating of the normal
solution (for decreasing σ) ($\sigma \lesssim \sigma_c$)

References (in Mathematics)

$(hD - A)^2 + V$

Helffer – Sjöstrand 1984 – 87

B. Simon in connexion with Solid State Physics

Magnetic bottles 1988 – 89

Montgomery, 1995

Helffer – Mohamed 1996

Can we hear

the zero locus of a magnetic

field

Superconductivity $\boxed{n \geq 2}$ Baumann – Phillips – Tang 1998

Bernoff – Sternberg 1998

Giorgi – Phillips 1999

Delpino – Fellmer – Sternberg 2000

Lu – Pan 1999 – 2001

Helffer – Morame 2001, 2002

Helffer – Pan 2002

Bonnaillie 2002 Pan 2001

Main Results $n=2$

Variable magnetic fields: $B(x) = \operatorname{curl} A$

$$b = \inf_{x \in \Omega} |B(x)|$$

$$b' = \inf_{x \in \partial\Omega} |B(x)|$$

Th 1 (Lu-Pan) $n=2$

$$\lim_{h \to 0} \frac{\lambda^1(h)}{h} = b \quad \text{(Dirichlet)}$$

$$\lim_{h \to 0} \frac{\mu^1(h)}{h} = \min[b, \Theta_0 b'] \quad \text{(Neumann)}$$

$0 < \Theta_0 < 1$

For Neumann:

- Localization at the points where the infimum is attained
- $B(x) =$ cste $\hookrightarrow$ localization at the boundary.

When
κ is large,

Superconductivity appears
first at the
boundary.

The decay is like

$$\exp -\alpha \frac{d[x, \partial\Omega]}{\sqrt{h}}$$

$\alpha > 0$

Exponential decay: proof uses
Agmon estimates, like in He-Sj.

Conjecture

$\alpha \to \theta(\alpha)$ is monotonic

$$\Downarrow$$

The ground state is localized near the corners of minimal angle.

Only proved by Bonnaillie (2003) in the form

The ground state is localized near the corners for which $\theta(\alpha)$ is minimal.

Th2 ([DFS] + [BS] + [LP] + [HeMo])

Ass. Constant magnetic field. $b \neq 0$

$$\mu^1(h) = b\,\Theta_0\, h - C_1(b) \max_{x \in \partial\Omega} K(x)\; h^{3/2} + O(h^{3/2+\eta})$$

NEUMANN !

$\eta > 0$

Any associated ground state is localized at the points of maximal curvature.

"Corners"

(Bonnaillie [2002], Jadallah [2001], Pan [2001])

$$\lim_{h \to 0} \frac{\mu^{(1)}(h)}{h} = \min_{\alpha_j} b\, \Theta(\alpha_j)$$

$$0 < \Theta(\alpha_j) \leq \Theta_0$$

angle of the corner

conjectured to be $< \Theta_0$

$$\lim_{\alpha \to 0} \Theta(\alpha) = 0$$

Main Results $n=3$ (HeMo, Pan?)

Theorem 3 [Pan]

If the magnetic field is constant, then

i) $$\lim_{h\to 0} \frac{\mu^{(1)}(h)}{h} = \Theta_0\, b$$

ii) Any ground state is localized at the boundary [Pan]

iii) Any ground state is localized at the points of $\partial\Omega$ (denoted Γ_B) where $B = \mathrm{curl}\, A(x)$ is tangent to $\partial\Omega$.

Ex:

$$\alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 \leq 1$$

$$\alpha x_1 B_1 + \beta x_2 B_2 + \gamma x_3 B_3 = 0$$

Curvature effects in dimension 3. He-Mo Pan?

Invariants.

$$\Gamma_B = \{ x \in \partial\Omega \ ; \ \vec{B}_0 \cdot n_x = 0 \}$$

normal at x

Assume Γ_B is regular, and $\vec{B}$ not tangent to Γ_B

$$K_{n,B}(x) = K_x \left(T(x) \wedge n_x \, , \, \frac{\vec{B}}{|B|} \right)$$

second fundamental form

Effective potential on Γ_B:

$$\tilde{\gamma}_0(x) = \left(\frac{1}{2}\right)^{2/3} \hat{\nu}_0 \ \delta_0^{1/3} \Big(|K_{n,B}(x)|^{2/3} \big(\delta_0 + (1-\delta_0) \, | \langle T(x), \frac{B}{|B|} \rangle |^2 \big)^{1/3} \Big)$$

$$\hat{\gamma}_0 = \inf_{x \in \Gamma_B} \tilde{\gamma}_0(x) \ , \quad \delta_0 \in \,]0,1[$$

Theorem 4

$$\mu^1(h) = b\,\Theta_0\, h + \hat{\gamma}_0\, b^{2/3} h^{4/3} + O(h^{\eta + 4/3})$$

$\eta > 0$

Expected :
Localization near the points of Γ_B
where $\hat{\gamma}_0 = \hat{\gamma}_0(x)$

Unexplained constants
Θ_0, $\Theta(\alpha)$, $\hat{\gamma}_0$, δ_0
Sketch of the proof through analysis of models.

Analysis of the models

$n=2$

Exercise 1

$D_t^2 + t^2$ harmonic oscillator

Application

$D_{x_1}^2 + (D_{x_2} - x_1)^2$ bottom 1

Exercise 2

$D_t^2 + (t-\xi)^2$ harmonic oscillator on $\mathbb{R}^+$, Neumann

$\xi \to \mu(\xi)$

unique

$\inf_\xi \mu(\xi) = \Theta_0 = \mu(\xi_0)$

$0 < \delta_0 = \frac{1}{2}\mu''(\xi_0) < 1$

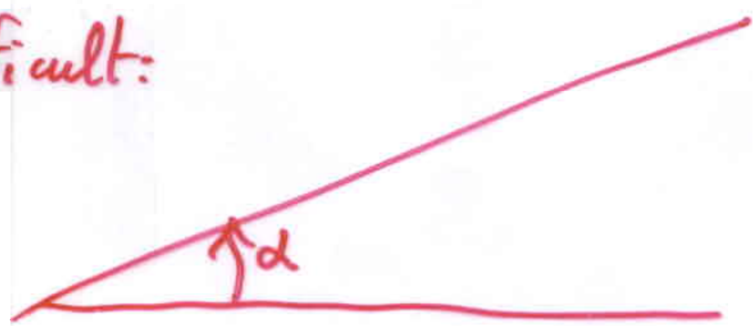
Application

$D_{x_1}^2 + (D_{x_2} - x_1)^2$ on $\mathbb{R}^+ \times \mathbb{R}$, Neumann

bottom Θ_0

More difficult:

Exercise 3



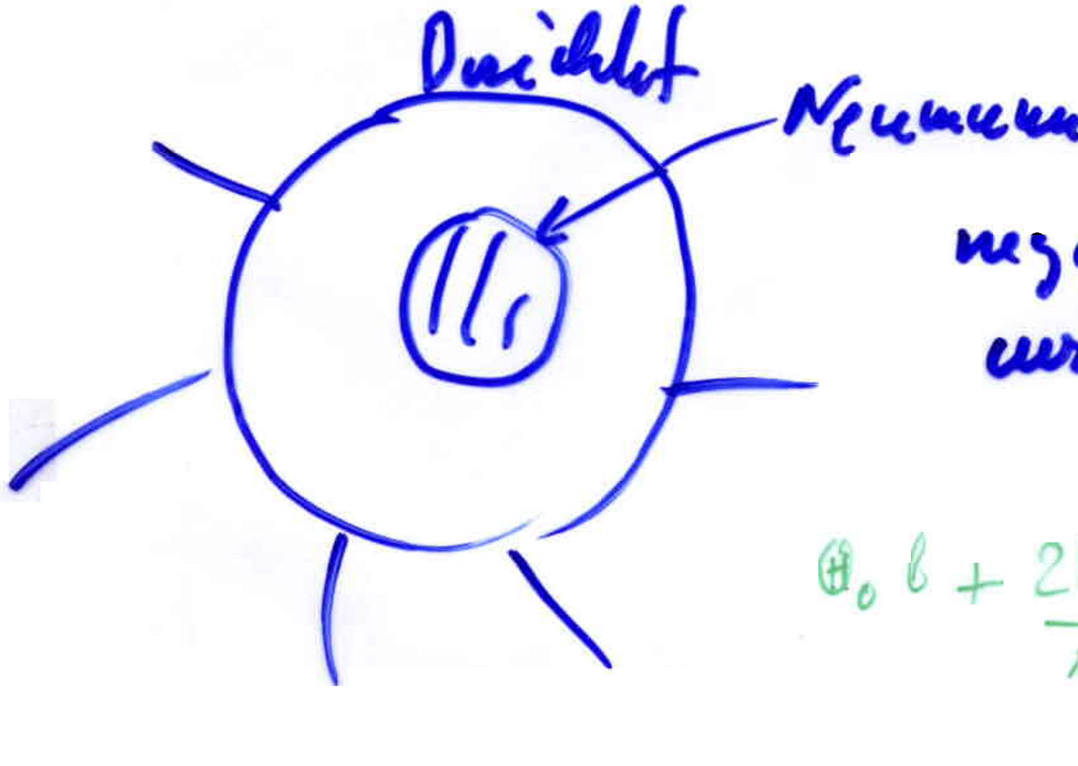
Bottom of essential spectrum is Θ_0

$\Theta(\alpha)$ bottom of the spectrum

positive curvature

$$\Theta_0 b - \frac{2M_3}{R} b^{1/2} + O(1)$$

b large



negative curvature

$$\Theta_0 b + \frac{2M_3}{R} b^{1/2} + O(1)$$

b large

Exercise 4

$$D_{x_1}^2 + D_{x_2}^2 + (\tau - \cos\vartheta\, x_1 - \sin\vartheta\, x_2)^2 ; \quad x_1 > 0$$

+ Neumann at $x_1 = 0$

$$\sigma(\vartheta) = \inf_{\tau} \sigma(\vartheta, \tau)$$

$$\sigma(0) = \Theta_0 \; ; \quad \sigma\left(\frac{\pi}{2}\right) = 1$$

$$\sigma \nearrow \quad \text{on } \left[0, \frac{\pi}{2}\right]$$

Application

$$D_{x_1}^2 + D_{x_2}^2 + (D_{x_3} - \cos\vartheta\, x_1 - \sin\vartheta\, x_2)^3$$

The bottom of the spectrum is minimal for $\vartheta = 0$

Curvature model.

$$(hD_r - \sin\theta\, t)^2$$

angle between T and $\frac{\vec{B}}{|B|}$

$$+ \left(hD_s + \cos\theta\, t + \kappa \frac{r^2}{2}\right)^2$$

parametrization of Γ_B

curvature $\kappa_{m,B}(x)$

$$+ h^2 D_t^2$$

— normal variable to $\partial\Omega$

$$\mu\left(h^{1/6}\sin\theta\, D_{\tilde{r}} + \cos\theta\left(\sigma + \frac{\kappa}{2}\tilde{r}^2\right)\right)$$

$$+ \left(h^{1/6}\cos\theta\, D_{\tilde{r}} - \sin\theta\left(\sigma + \frac{\kappa}{2}\tilde{r}^2\right)\right)$$

$$\Theta_0 + h^{1/3}\Big[\delta_0\left[\sin\theta\, D_r + \cos\theta\left(\sigma + \frac{\kappa}{2}r^2\right)\right]^2$$

$$+ \left(\cos\theta\, D_r - \sin\theta\left(\sigma + \frac{\kappa}{2}r^2\right)\right)^2\Big]$$