Ray helicity: a geometric invariant for multi-dimensional resonant wave conversion

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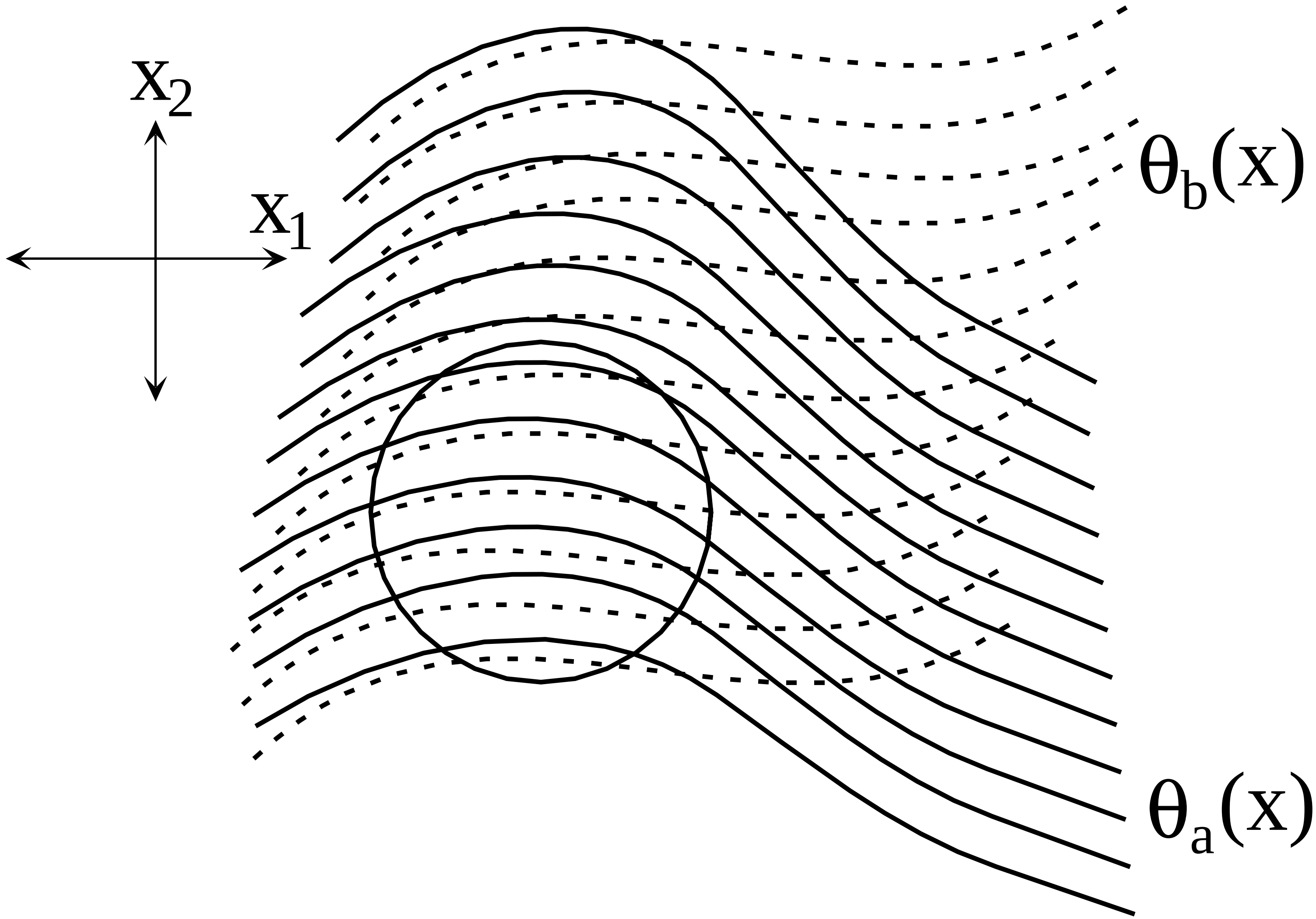
Outline:

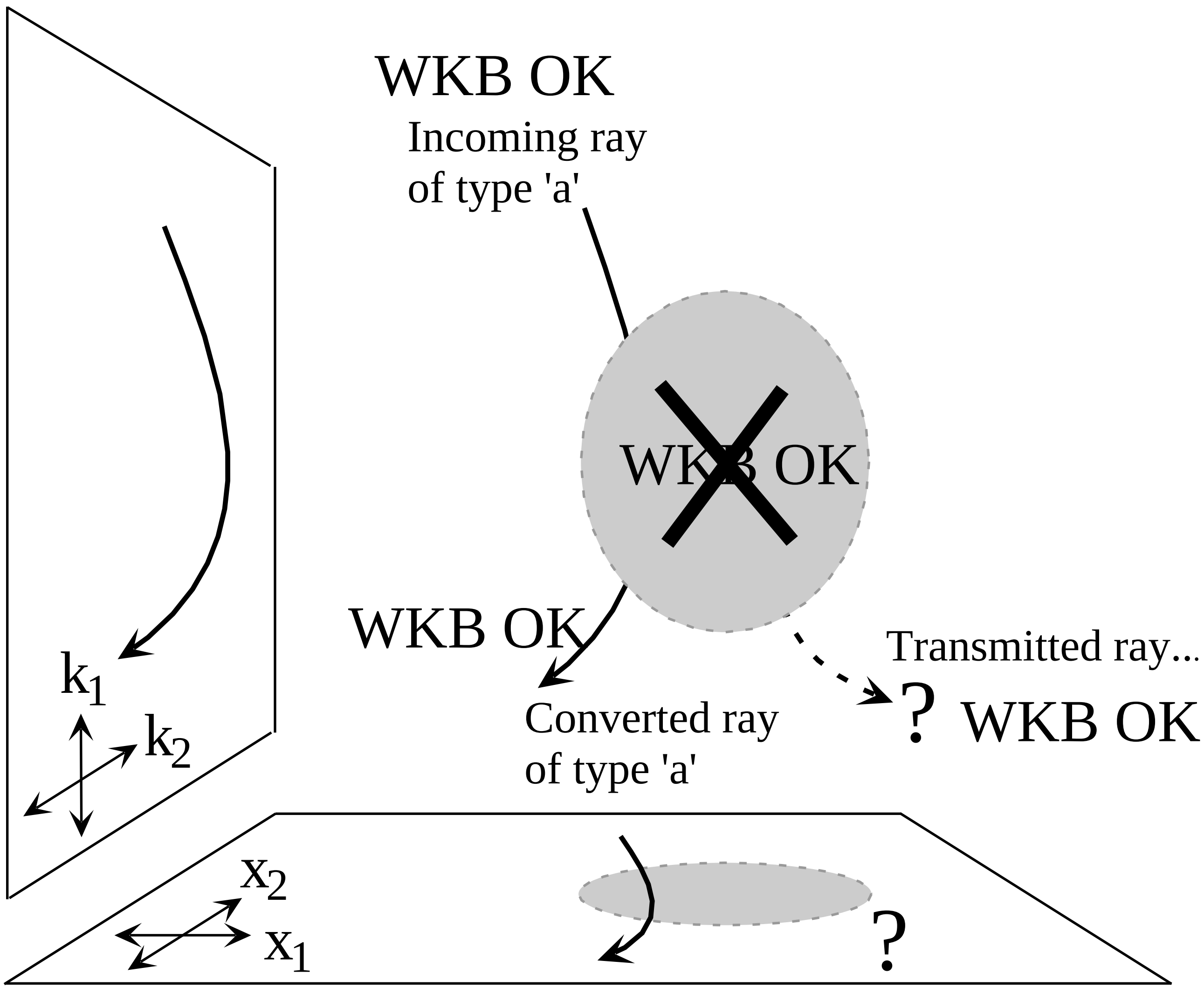
- Introduction and physical motivation: the importance of invariants
- Linear wave conversion:
  1. The 2X2 local wave equation.
  2. The ray equations in non-canonical form.
  3. Geometric invariants in multi-dimensions.
  4. Normal form of the 2X2 wave equation in multi-dimensions.
- Summary and conclusions

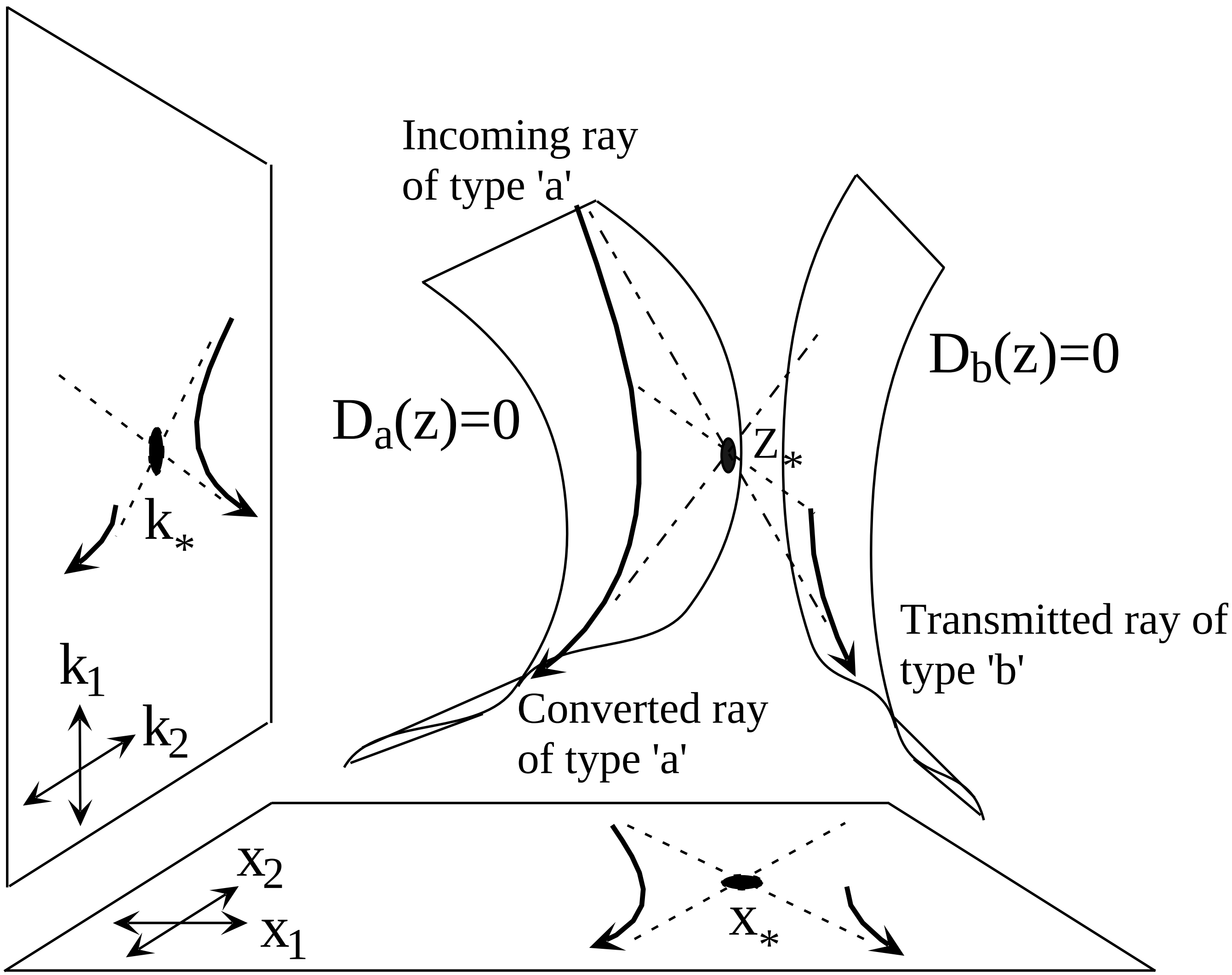
What is linear wave conversion?

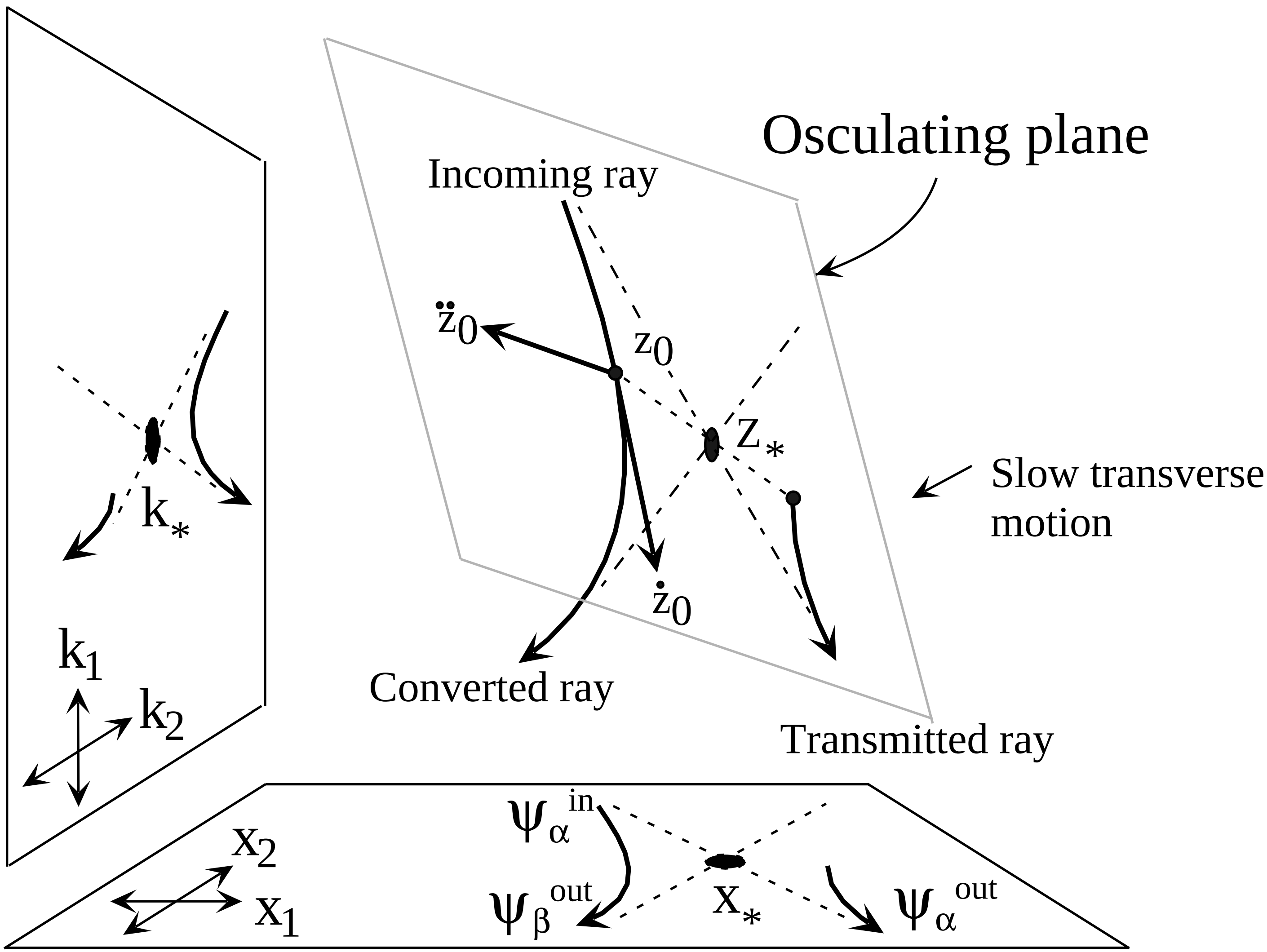
Plasmas, fluids and other media can support a wide variety of *linear* wave types, with different dispersion characteristics and polarizations.

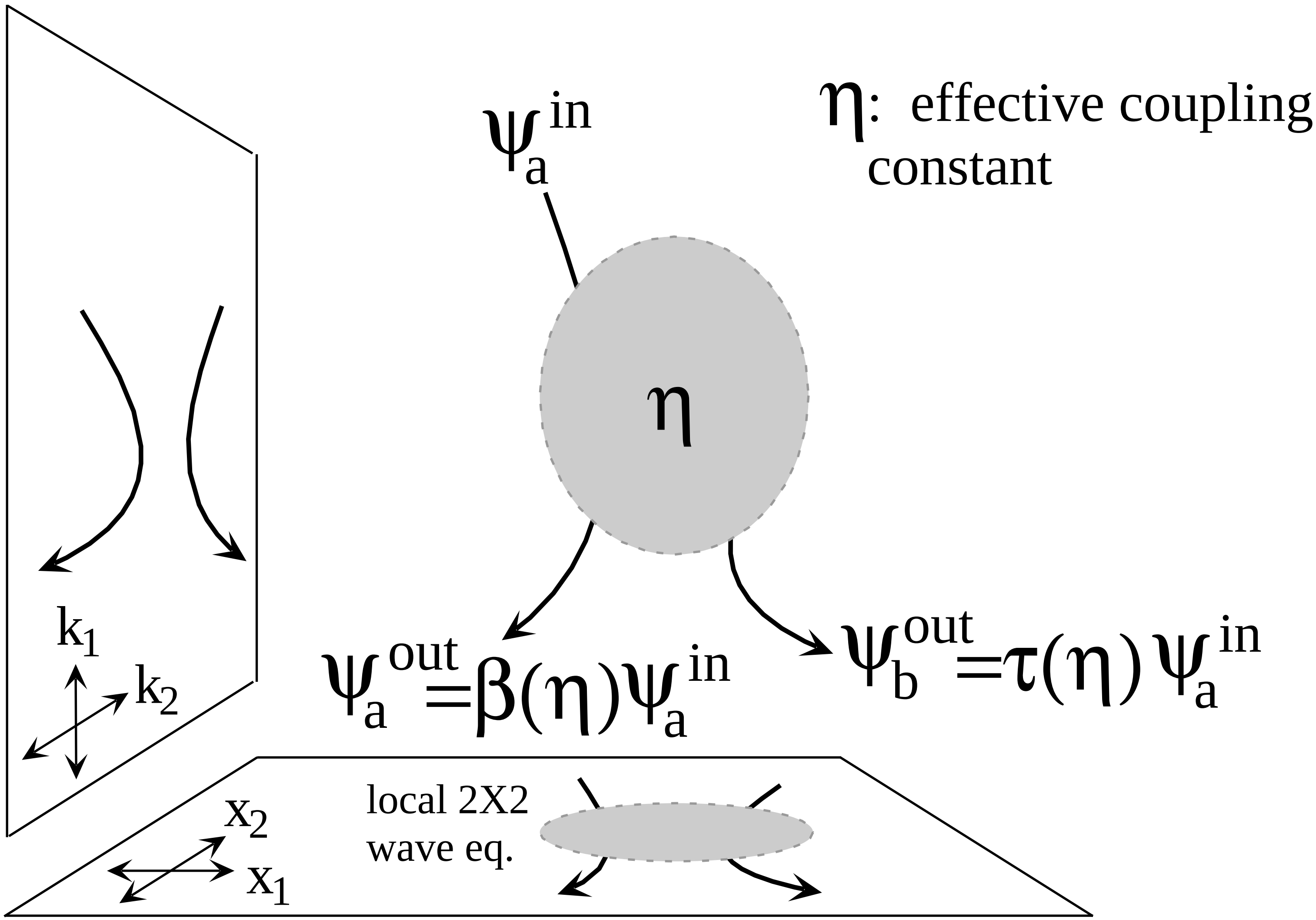
- In a weakly non-uniform medium, the dispersion characteristics and polarizations are *local* objects.
- For fixed frequency, ω, near a point $\mathbf{x}_*$, wave types '*a*' and '*b*' (with different group velocities and polarizations) can have *nearly equal* wavenumbers.
- Linear wave conversion is due to a *local phase resonance*.

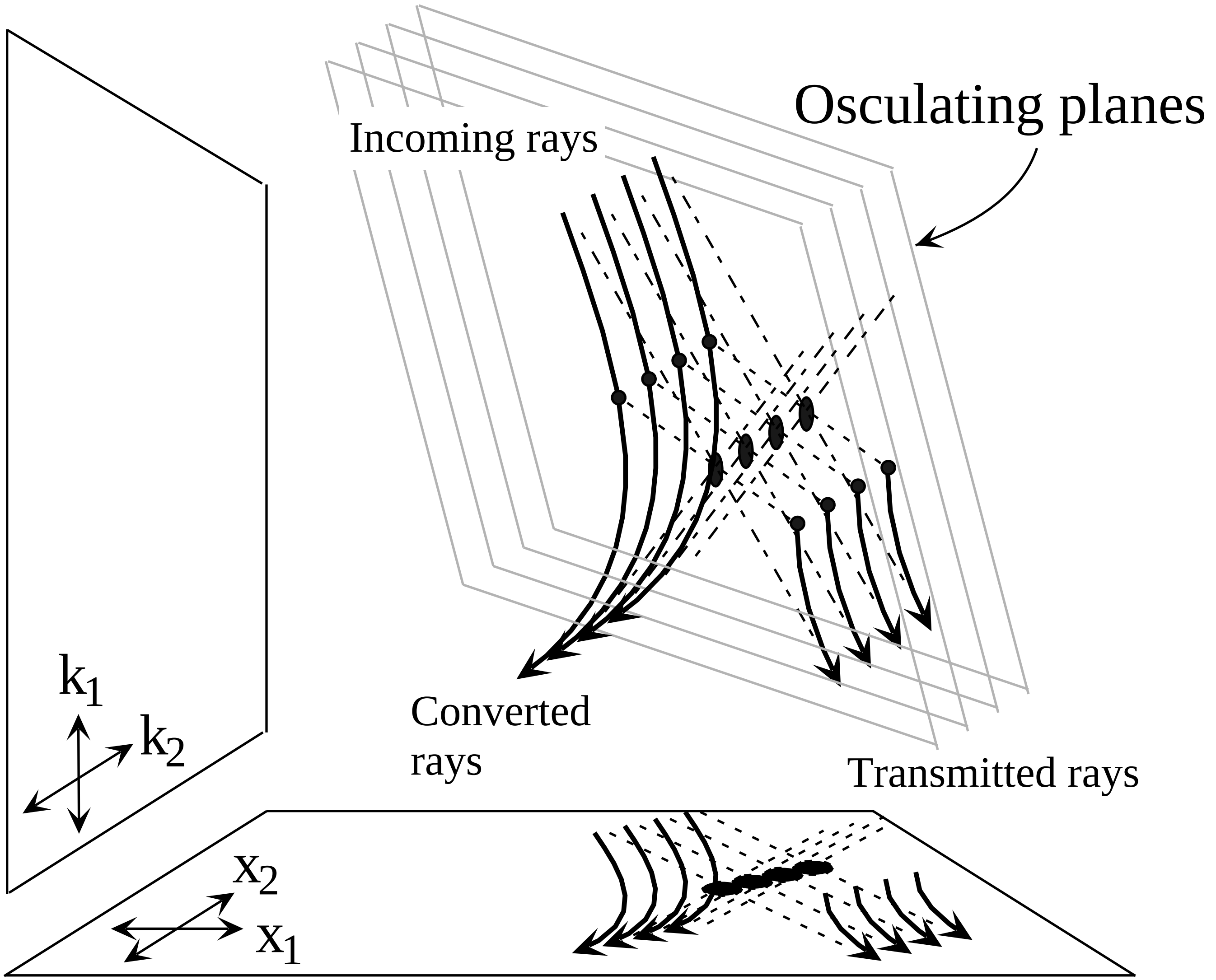

x_2
x_1
$\theta_b(x)$
$\theta_a(x)$


WKB OK
Incoming ray
of type 'a'
WKB OK
WKB OK
Converted ray
of type 'a'
Transmitted ray...
? WKB OK
k1
k2
x2
x1
?


Incoming ray
of type 'a'
$D_a(z)=0$
$D_b(z)=0$
z_*
k_*
k_1
k_2
Transmitted ray of
type 'b'
Converted ray
of type 'a'
x_2
x_1
x_*


Osculating plane
Incoming ray
$\ddot{z}_0$
z_0
Z_*
$\dot{z}_0$
Slow transverse motion
Converted ray
Transmitted ray
k_*
k_1
k_2
x_2
x_1
ψ_α^{in}
ψ_β^{out}
x_*
ψ_α^{out}


ψ_a^{in}
η: effective coupling constant
η
k_1
k_2
$\psi_a^{out} = \beta(\eta)\psi_a^{in}$
$\psi_b^{out} = \tau(\eta)\,\psi_a^{in}$
x_2
x_1
local 2X2
wave eq.


Osculating planes
Incoming rays
Converted rays
Transmitted rays
k_1
k_2
x_2
x_1

Sample application:

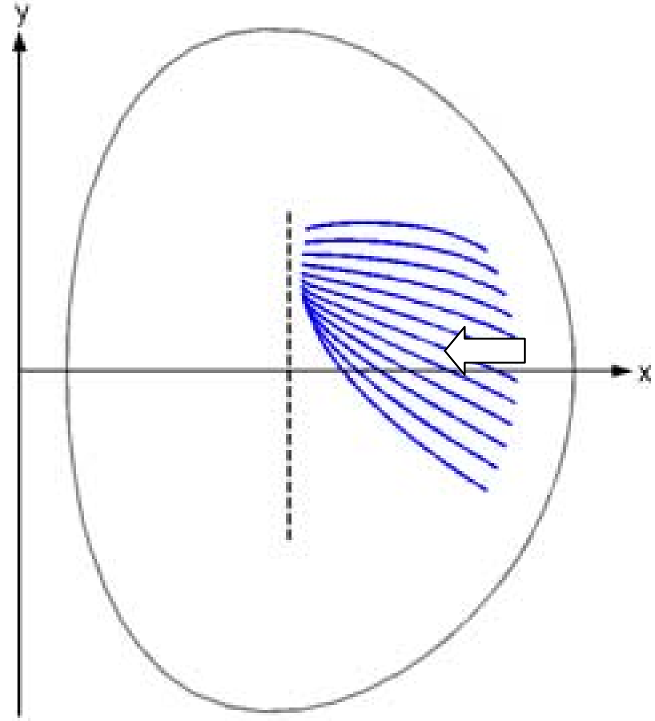
RF heating in fusion devices (*e.g.* tokamaks) (with Andre Jaun, Phys. Lett. A (2001)).

Questions:

1] Where is the energy eventually deposited?

2] What is the global cavity response of the plasma?

Goal: develop practical ray tracing algorithms which include conversion. (Should run much faster than full-wave codes.)



Scenario: a family of rays are launched by an antenna.

Linear wave conversion occurs in many areas of physics

- RF heating in plasmas
- Ionospheric physics
- Atomic, molecular and nuclear physics
(Landau-Zener crossings, spin-orbit resonance)
- Geophysics *(e.g.* equatorial waves*)*
- Neutrino physics ('MSW effect')
- Black hole theory
- Solid mechanics

Distinguish two cases:

CASE I) The two waves undergoing conversion have *different* polarizations. This can be reduced locally to a 2-component *vector wave* problem.

CASE II) The two waves undergoing conversion have *the same* polarization. This can be reduced locally to a *scalar wave* problem. ("*Landau-Zener*", "*avoided crossings*".)

Our work focuses on CASE I.

Multi-dimensional conversion has new physics:

- The resonance condition involves the *phase velocities*, *not* the *group velocities*. Multi-dimensional conversion, in general, *cannot* be reduced to the one-dimensional case (even locally). See, *e.g.* Tracy, Kaufman, Brizard, Phys. Plasmas, Feb.`03.
- ‘Generic’ multi-dimensional conversion will be a hybrid of ‘hyperbolic’ (*i.e.* avoided crossing) and ‘elliptic’ (oscillatory) behaviors. This combination of effects is impossible in one spatial dimension. Preprint: arXiv.org/physics/0303086

$$\int d\mathbf{x}' dt' \, \mathbf{D}(\mathbf{x}, \mathbf{x}', t - t') \cdot \Psi(\mathbf{x}', t') = 0.$$

$$\mathbf{x} = (x_1, x_2), \quad \mathbf{k} = (k_1, k_2).$$

Restrict here to two spatial dimensions for simplicity (four-dim. phase space)

$$\mathbf{D}(\mathbf{x}, \mathbf{x}', t - t') \equiv \begin{pmatrix} D_{11} & \cdots & D_{1N} \\ \vdots & \ddots & \vdots \\ D_{N1} & \cdots & D_{NN} \end{pmatrix} \qquad \Psi(\mathbf{x}, t) = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_N \end{pmatrix}.$$

To simplify the problem, two types of transformations are used:

- Congruence transformations, acting on the vector components of the wave equation, Ψ.
- Canonical transformations, acting on the ray phase space $\mathbf{z=(x,k)}$.

Quantities that are invariant under both sets of transformations have fundamental physical significance.

If the system is *conservative*

$$D_{jk}(\mathbf{x},\mathbf{x}',t-t') = D^{*}{}_{kj}(\mathbf{x}',\mathbf{x},t'-t)$$

Action principle:

$$A \equiv \int dt d\mathbf{x} d\mathbf{x}' dt' \, \Psi^{*t}(\mathbf{x},t) \cdot \mathbf{D}(\mathbf{x},\mathbf{x}',t-t') \cdot \Psi(\mathbf{x}',t')$$

Non-uniform (time-stationary)

$$\int d^2x' dt' \, \mathbf{D}(\mathbf{x}, \mathbf{x}', t - t') \cdot \Psi(\mathbf{x}', t') = 0.$$

$$\mathbf{D}(\mathbf{x}, \mathbf{k}, \omega)$$

Dispersion tensor (Weyl symbol)

$$\mathbf{D}(\mathbf{x}, -i\nabla, i\partial_t) \cdot \Psi(\mathbf{x}, t) = 0.$$

First, try WKB: insert the ansatz:

$$\Psi(\mathbf{x},t) = e^{-i\omega t} e^{i\theta_a(\mathbf{x})} \psi_a(\mathbf{x}) \hat{\mathbf{e}}_a(\mathbf{x}).$$

rapidly varying

slowly varying

$$\mathbf{k}(\mathbf{x}) \equiv \nabla\theta(\mathbf{x})$$ local wavevector

Find, to leading order:

$$\mathbf{D}(\mathbf{x},\nabla\theta,\omega)\cdot\hat{\mathbf{e}}_a(\mathbf{x}) = 0.$$

Non-trivial solutions exist only if

$$D(\mathbf{x},\nabla\theta,\omega) \equiv det(\mathbf{D}(\mathbf{x},\nabla\theta,\omega)) = 0.$$

$\theta(\mathbf{x})$ is *unknown* at this point.

- This is a PDE that $\theta(\mathbf{x})$ must satisfy (*the eikonal equation*).
- In practice, solutions are found by *ray tracing* with a *family* of rays.

Trouble in Paradise:

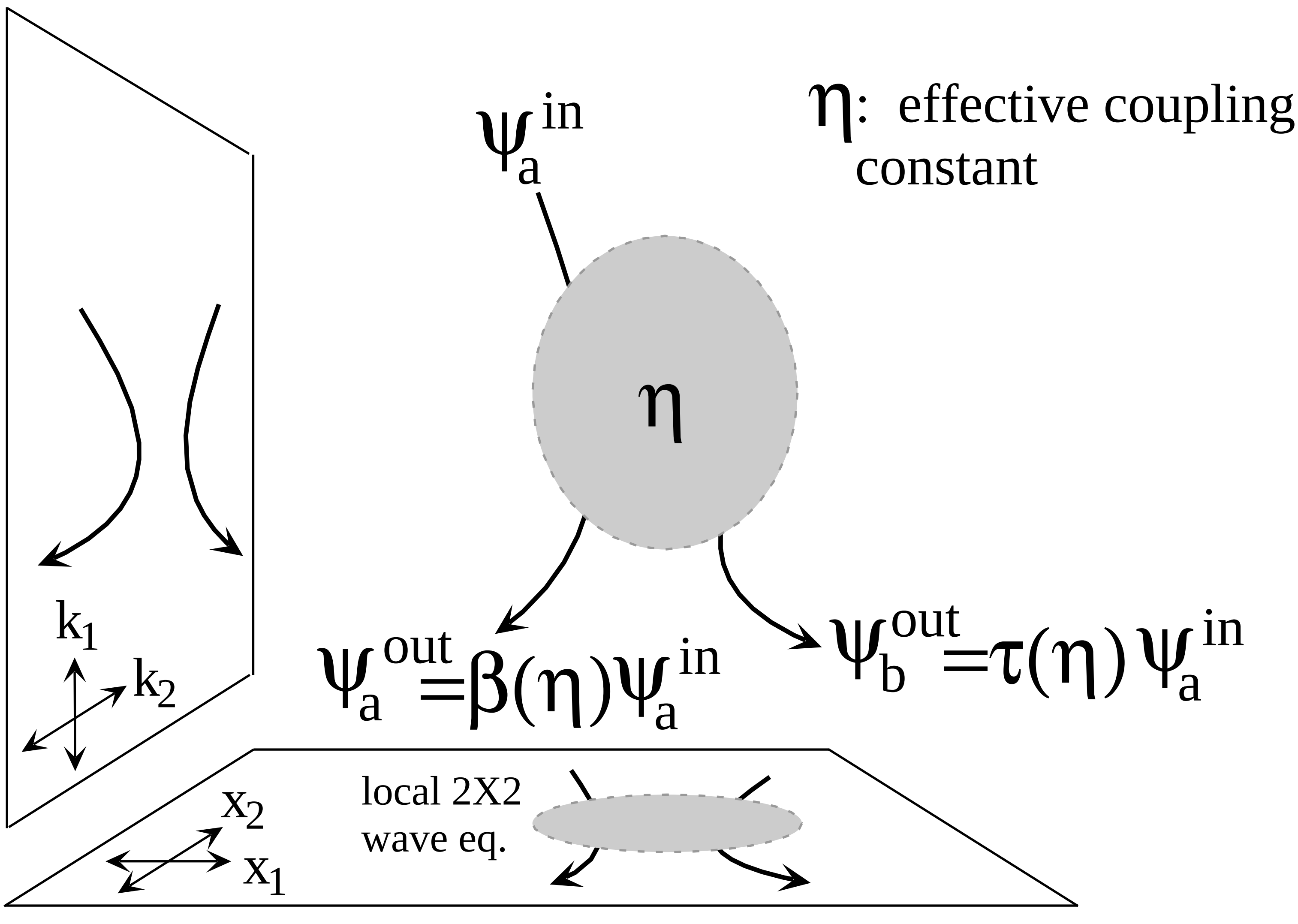
- In *phase space* **z**=(**x**,**k**) the rays don't cross, but in **x**-space they can. If this occurs, we get two phases for each **x**! This is a *caustic*, and is dealt with using *Maslov* methods (Littlejohn, Delos...).

- If the rays encounter a linear conversion region, then the polarization and amplitude vary rapidly: the WKB ansatz is *not* valid there.

A ray-based approach to multi-dimensional linear conversion (with A. Jaun)

Although WKB is invalid within the conversion region, the ray geometry leading into - and out of - the conversion region can be used to

- Detect the presence of conversion,
- Find the outgoing transmitted and converted rays,
- Extract the local polarizations for reduction to 2X2 form,
- Guide the asymptotic matching to incoming and outgoing WKB waves.


ψ_a^{in}
η: effective coupling constant
η
k_1
k_2
$\psi_a^{out} = \beta(\eta)\psi_a^{in}$
$\psi_b^{out} = \tau(\eta)\,\psi_a^{in}$
x_2
x_1
local 2X2
wave eq.

Outside the conversion region ('exterior' WKB solutions) (Note: there are 'incoming' and 'outgoing' versions with 'connection' coefficients relating them.)

$$\boxed{\Psi(\mathbf{x},t) = e^{-i\omega t}\left[e^{i\theta_a(\mathbf{x})}\psi_a(\mathbf{x})\hat{\mathbf{e}}_a(\mathbf{x}) + e^{i\theta_b(\mathbf{x})}\psi_b(\mathbf{x})\hat{\mathbf{e}}_b(\mathbf{x})\right]}$$

Inside the conversion region (interior solution)

Local field amplitudes

$$\Psi(\mathbf{x},t) = e^{-i\omega t}\left[\psi_\alpha(\mathbf{x})\hat{\mathbf{e}}_\alpha + \psi_\beta(\mathbf{x})\hat{\mathbf{e}}_\beta\right]$$

'uncoupled' polarization basis

Reduction to local 2X2 form

$$A \equiv \int dt d\mathbf{x} \Psi^{*t}(\mathbf{x},t) \cdot \mathbf{D}(\mathbf{x},-i\nabla,i\partial_t) \cdot \Psi(\mathbf{x},t) =$$

$$\int dt d\mathbf{x} \left[\psi_\alpha^* \hat{D}_{\alpha\alpha} \psi_\alpha + \psi_\beta^* \hat{D}_{\beta\beta} \psi_\beta + \psi_\alpha^* \hat{D}_{\alpha\beta} \psi_\beta + \psi_\beta^* \hat{D}_{\beta\alpha} \psi_\alpha \right].$$

$$\hat{D}_{\alpha\alpha} \equiv \hat{\mathbf{e}}_\alpha^{*t} \cdot \mathbf{D}(\mathbf{x},-i\nabla,i\partial_t) \cdot \hat{\mathbf{e}}_\alpha .$$

$$\hat{D}_{\alpha\beta} \equiv \hat{\mathbf{e}}_\alpha^{*t} \cdot \mathbf{D}(\mathbf{x},-i\nabla,i\partial_t) \cdot \hat{\mathbf{e}}_\beta = \hat{D}_{\alpha\beta}^{*t} .$$

$$\begin{pmatrix} \hat{D}_{\alpha\alpha} & \hat{D}_{\alpha\beta} \\ \hat{D}_{\alpha\beta}^{*t} & \hat{D}_{\beta\beta} \end{pmatrix} \begin{pmatrix} \psi_{\alpha} \\ \psi_{\beta} \end{pmatrix} = 0.$$

The 2X2 wave operator has the related disperson matrix:

$$\begin{pmatrix} D_{\alpha\alpha}(\mathbf{z}) & D_{\alpha\beta}(\mathbf{z}) \\ D_{\alpha\beta}^{*}(\mathbf{z}) & D_{\beta\beta}(\mathbf{z}) \end{pmatrix} \qquad \mathbf{z}=(\mathbf{x},\mathbf{k})$$

The 2X2 wave equation is simplified *via* congruence transformations: Ψ'=QΨ

$$\begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \end{pmatrix} = \begin{pmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} \\ Q_{\beta\alpha} & Q_{\beta\beta} \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix}, \quad det(\mathbf{Q}) \neq 0.$$

Q_{jk} are complex *constants*

$$\mathbf{D}'(\mathbf{z}) = \mathbf{Q}^{*t}\,\mathbf{D}(\mathbf{z})\mathbf{Q}, \quad det(\mathbf{D}') = |\det \mathbf{Q}|^2\, det(\mathbf{D})$$

Following Littlejohn and Flynn, write:

$$\mathbf{D} = \begin{pmatrix} D_{\alpha\alpha} & D_{\alpha\beta} \\ D_{\alpha\beta}{}^{*} & D_{\beta\beta} \end{pmatrix} = \begin{pmatrix} B_0 + B_3 & B_1 + iB_2 \\ B_1 - iB_2 & B_0 - B_3 \end{pmatrix} = B_\mu \boldsymbol{\sigma}^\mu$$

Where σ^μ $\mu=0,1,2,3$ are the Pauli matrices, and the 'four-vector' $B(\mathbf{z})$ is:

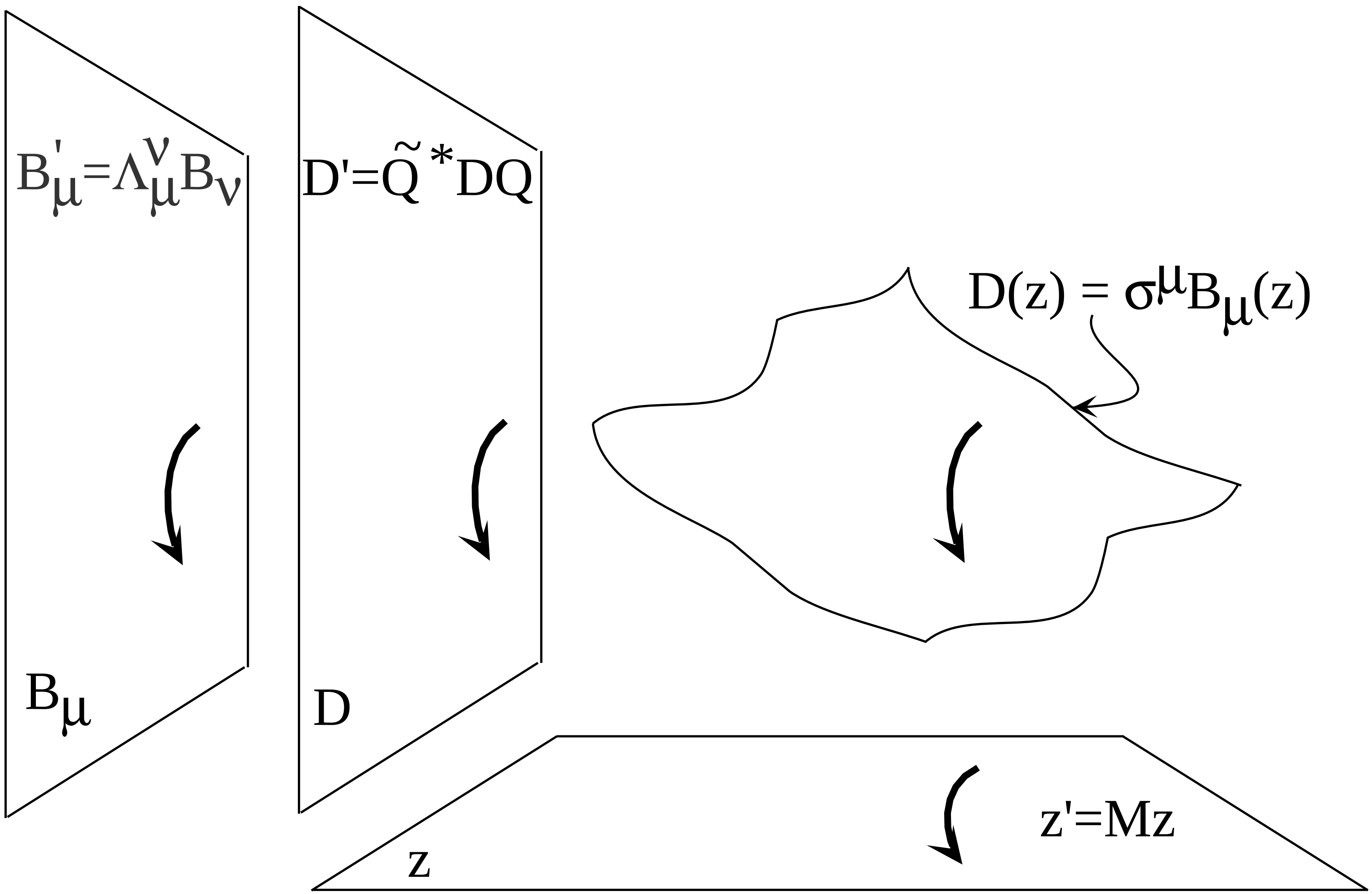
$$B(\mathbf{z}) = (B_0(\mathbf{z}), B_1(\mathbf{z}), B_2(\mathbf{z}), B_3(\mathbf{z})).$$

Under a congruence transformation:

$$\mathbf{D}' = \mathbf{Q}^{*t}\,\mathbf{D}\mathbf{Q} = \begin{pmatrix} B'_0 + B'_3 & B'_1 + iB'_2 \\ B'_1 - iB'_2 & B'_0 - B'_3 \end{pmatrix} = B'_\mu\,\sigma^\mu$$

$$B'_\mu = \Lambda^\nu_\mu B_\nu\,, \quad \left(\Lambda^{-1}\right)^\nu_\mu = \tfrac{1}{2}\,tr\left(\sigma_\nu \mathbf{Q}^{*t}\,\sigma_\mu \mathbf{Q}\right)$$

If |det(Q)|=1, Λ is a Lorentz transformation, otherwise it is *conformal*. Both preserve the 'light cone' in B-space.



$B'_\mu = \Lambda_\mu^\nu B_\nu$
$D' = \tilde{Q}^* D Q$
$D(z) = \sigma^\mu B_\mu(z)$
B_μ
D
z
z'=Mz

$$\det(\mathbf{D}) \equiv D(\mathbf{z}) = B_0^2 - B_1^2 - B_2^2 - B_3^2 = \eta^{\mu\nu} B_\mu B_\nu .$$

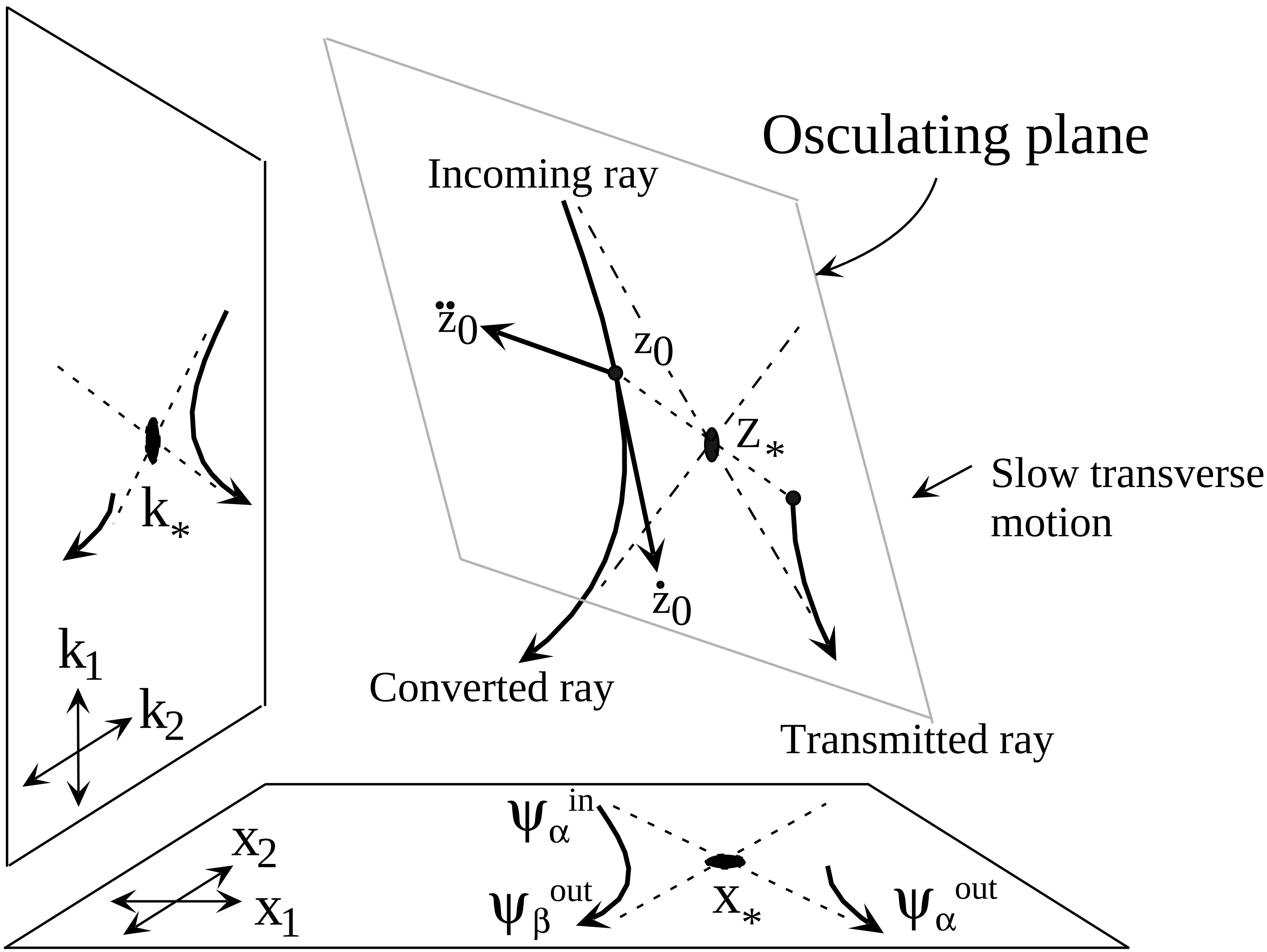
$\eta = diag\,(1,-1,-1,-1)$. (The Minkowski tensor.)

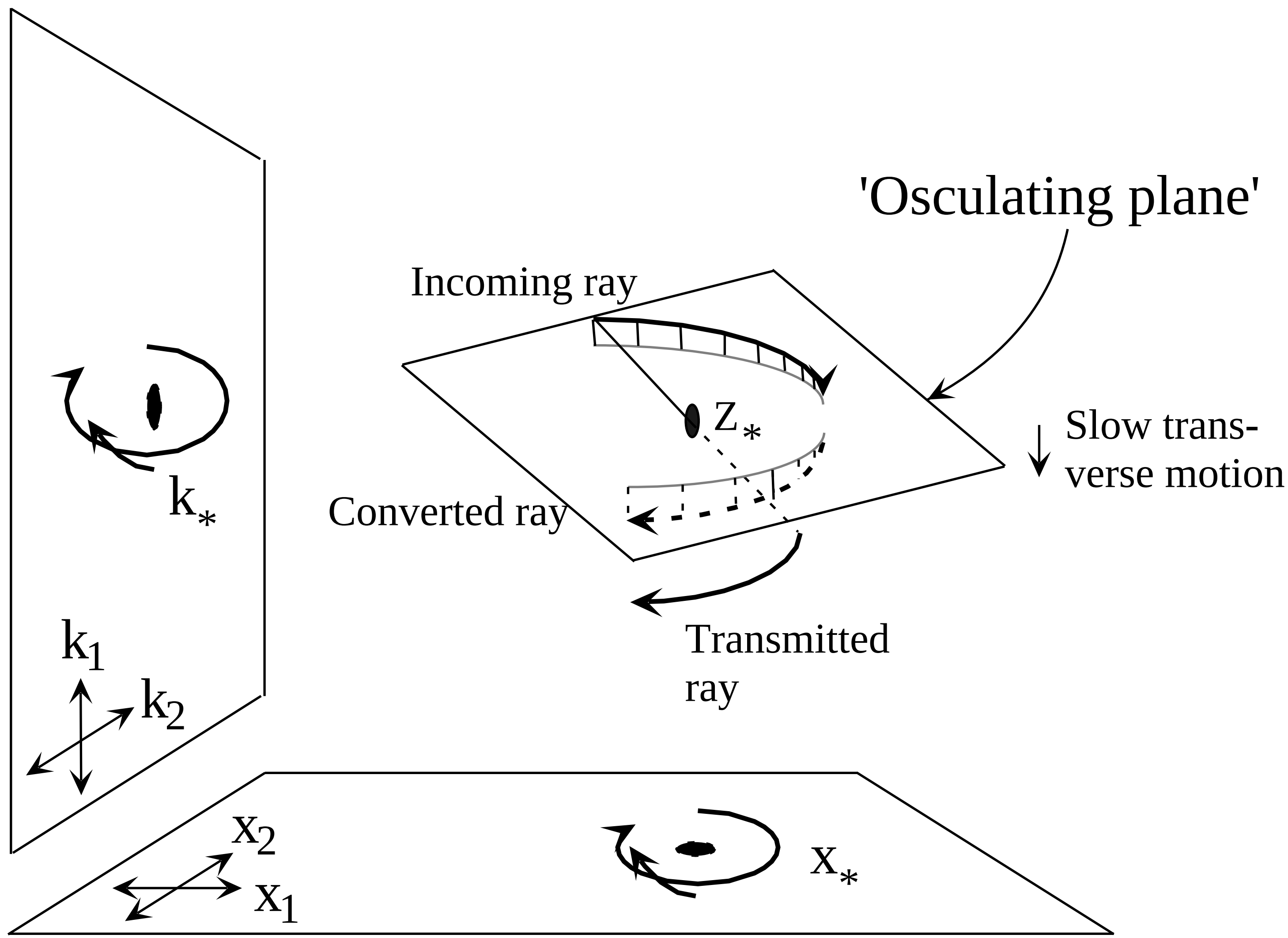
$$\det(\mathbf{D}) = 0 \quad \Rightarrow \quad \boxed{\eta^{\mu\nu} B_\mu B_\nu = 0.}$$

- ‘B’-space: dispersion surface (D(**z**)=0) is the ‘light cone’.
- <u>‘Genericity’ defined</u>: Assume the four $B_\mu(\mathbf{z})$ are independent and can be used as local (non-canonical!) coordinates.

$$rank(\nabla B_0, \nabla B_1, \nabla B_2, \nabla B_3) = ?$$

- rank = 1: no conversion, usual WKB.
- rank = 2: local confinement to 2-dim plane, (very slow transverse motion)
    - "avoided crossing" with constant coupling (Phys. Lett. A, 2002) hyperbolic ray motion (locally)
    - "effective cavity" with elliptic ray motion (locally)
- rank = 3: Braam-Duistermaat-type, variable coupling versions of rank 2 cases, linear 'transverse' motion.
- rank = 4: full 'generic' conversion with combination of hyperbolic and elliptic ray motion.


Osculating plane
Incoming ray
$\ddot{z}_0$
z_0
Z_*
Slow transverse motion
k_*
$\dot{z}_0$
Converted ray
Transmitted ray
k_1
k_2
x_2
x_1
$\psi_\alpha{}^{in}$
$\psi_\beta{}^{out}$
x_*
$\psi_\alpha{}^{out}$


'Osculating plane'
Incoming ray
z_*
Slow trans-
verse motion
Converted ray
Transmitted
ray
k_*
k_1
k_2
x_2
x_1
x_*

The ray equations: D($\mathbf{z}$) is the ray Hamiltonian

$$\dot{\mathbf{z}} = \{D, \mathbf{z}\} = -J\nabla D \qquad J^{4\times4} = \begin{pmatrix} 0^{2\times2} & 1^{2\times2} \\ -1^{2\times2} & 0^{2\times2} \end{pmatrix}$$

For any scalar function $f(\mathbf{z})$

$$\dot{f} = \{D, f\}$$

In particular:

$$\dot{B}_\mu = \{D, B_\mu\} = 2\Omega_\mu^\nu B_\nu$$

(Hamilton's eqs. In *non-canonical* coordinates.)

where

$$\Omega_\mu^\nu = \eta^{\nu\rho}\Omega_{\rho\mu} = \eta^{\nu\rho}\{B_\rho, B_\mu\}$$

$$\Omega_{\rho\mu} = \{B_\rho, B_\mu\} \Rightarrow \quad \Omega_{\rho\mu}(\mathbf{z}) = -\Omega_{\mu\rho}(\mathbf{z})$$

- The entries are Poisson brackets, hence the 4X4 matrix Ω is *automatically* invariant under all *canonical* transformations (which act on **z**).
- However, under *congruence* transformations

$$\Omega'_{\rho\mu} = \{B'_\rho, B'_\mu\} \Rightarrow \quad \Omega' = \Lambda\Omega\Lambda^t$$

Restrict to congruence transformations with |det(**Q**)|=1 first. Then Λ is a Lorentz matrix:

$$\Lambda^t \eta \Lambda = \eta$$

$$\eta\Omega' = \eta\Lambda\Omega\Lambda^t = \left(\Lambda^t\right)^{-1}\eta\Omega\Lambda^t$$

Therefore, $\eta\Omega$ transforms via *similarity*.

The characteristic polynomial of ηΩ is invariant:

$$P(\lambda) \equiv det(\eta\Omega' - \lambda) = det\left[\left(\Lambda^t\right)^{-1}(\eta\Omega - \lambda)\Lambda^t\right] = det(\eta\Omega - \lambda)$$

$$P(\lambda) = \lambda^4 - \tfrac{1}{2} tr\left(\left(\eta\Omega\right)^2\right)\lambda^2 + \det\left(\eta\Omega\right).$$

2 congruence invariants

$$tr\left((\eta\Omega)^2\right) = \{B^\mu, B_\nu\}\{B^\nu, B_\mu\}.$$

Previously found by Littlejohn & Flynn, 'Generic mode conversion in one-dimension', PRL & Annals of Phys.

$$det(\eta\Omega) = det(\Omega)$$

This new invariant can only be non-zero when Ω is of full rank (4), which can only occur in multi-dimensional conversion.

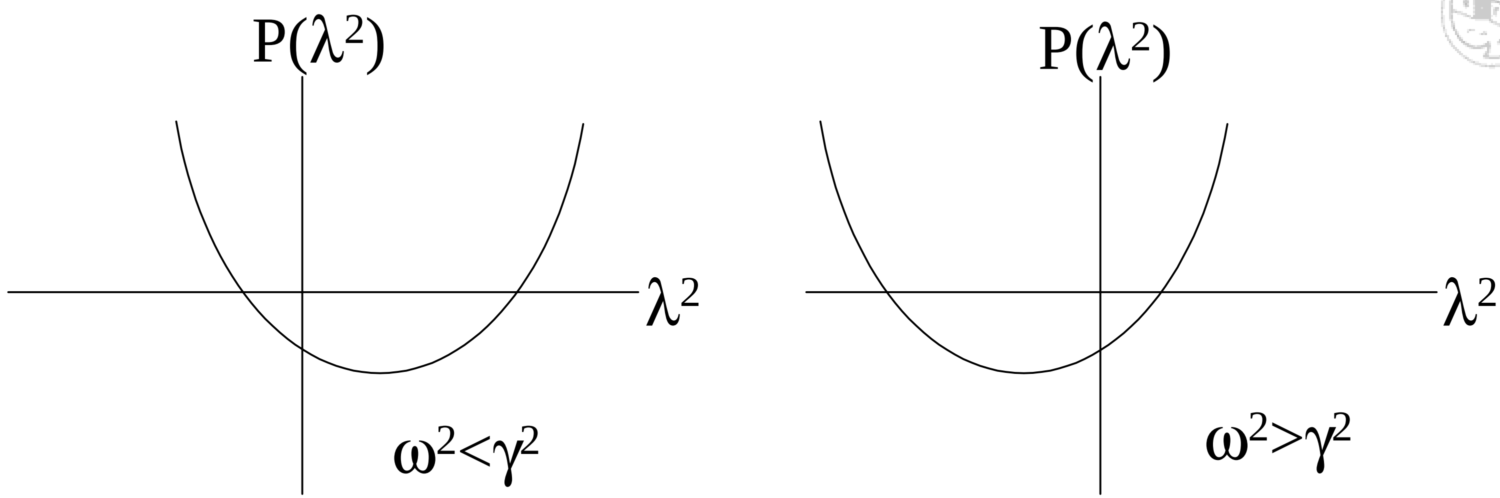
Physical interpretation
From the theory of Lorentz tranformations:

$$\Omega = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_1 & 0 & -\omega_3 & \omega_2 \\ -\gamma_2 & \omega_3 & 0 & -\omega_1 \\ -\gamma_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

tr$((\eta\Omega)^2)$ can be positive or negative

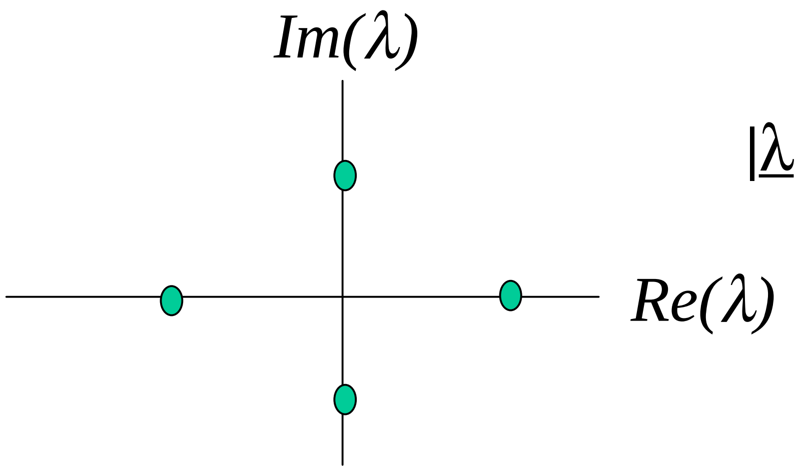
$$P(\lambda) = det(\eta\Omega - \lambda) = \lambda^4 + \left(\omega^2 - \gamma^2\right)\lambda^2 - (\omega \cdot \gamma)^2$$

But, det$(\eta\Omega)$ can *never be positive*



Both cases have a mixture of ‘hyperbolic’ and ‘elliptic’behavior

Roots of $P(\lambda)=0$ on the complex λ-plane



Geometric interpretation: choose a point $\mathbf{z}_0$ on the ray. The matrix $\eta\Omega_0=\eta\Omega(\mathbf{z}_0)$ generates the Lorentz transformation:

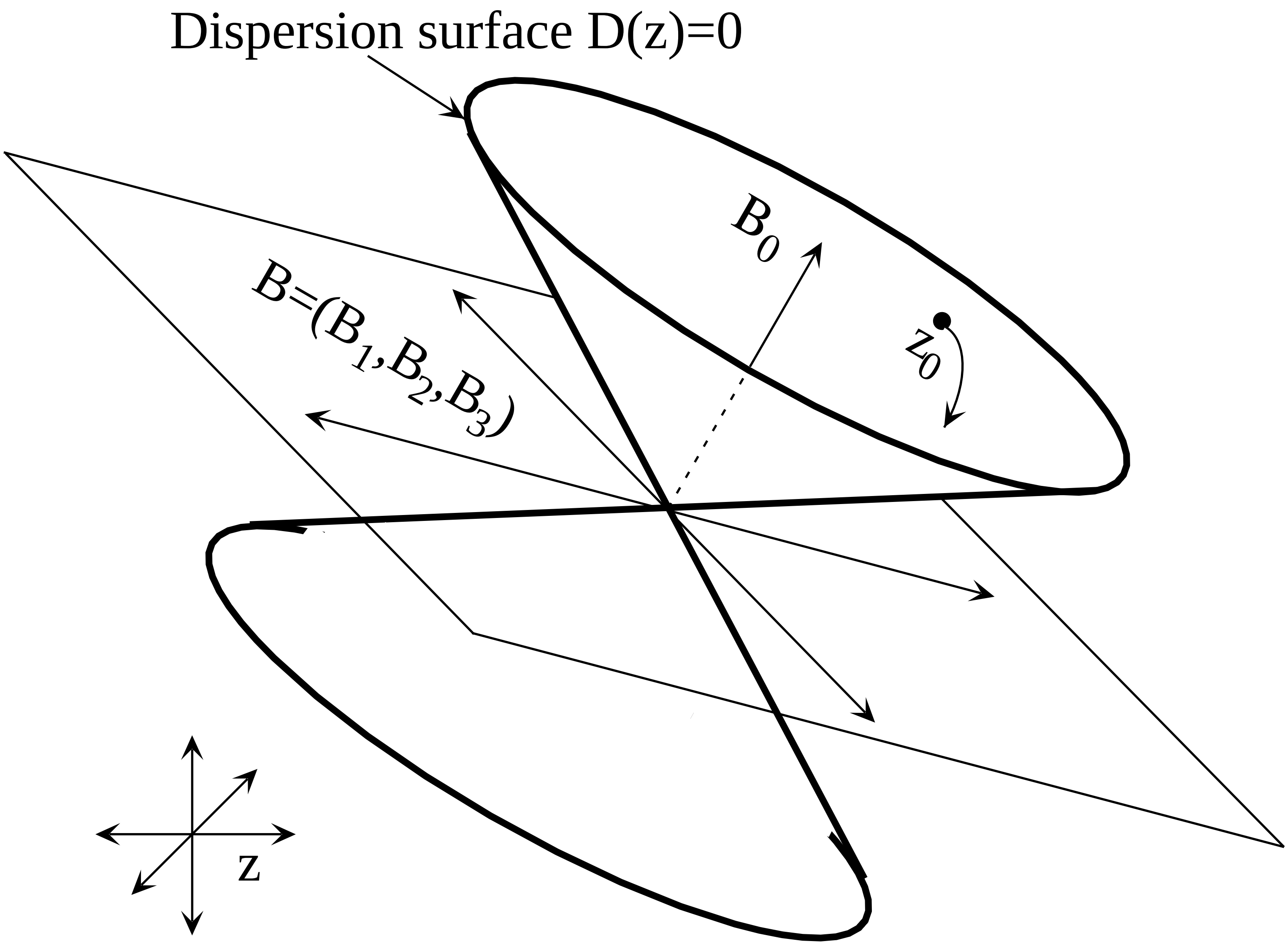
$$\Lambda_0(\sigma) \equiv exp(\sigma\eta\Omega_0)$$

σ is 'ray orbit parameter'

Acting upon $B(\mathbf{z}_0)$:

$$B(\sigma) = \Lambda_0(\sigma)B(\mathbf{z}_0) = exp(\sigma\eta\Omega_0)B(\mathbf{z}_0)$$

This is a local approximation of the ray orbit in 'B-coordinates'.


Dispersion surface D(z)=0
B0
B=(B1,B2,B3)
z0
z

$$\Lambda_0(\sigma) \equiv exp(\sigma\eta\Omega_0)$$

$$\Lambda_0(\sigma) = \left\{ \begin{array}{c} rotation\ about\ \ \omega \\ \& \\ boost\ in\ direction\ \ \gamma \end{array} \right\} in\ B - coordinates$$

Therefore, the ‘generic’ ray motion in multi-dimensional conversion will be a combination of hyperbolic and elliptic motions.

What about congruence transformations with

$$|\,det(\mathbf{Q})\,| = Q \neq 1?$$

$$\Lambda^t \eta \Lambda = Q^2 \eta$$

Λ is *conformal*

$$\eta \Omega' = \eta \Lambda \Omega \Lambda^t = Q^2 \left(\Lambda^t\right)^{-1} \eta \Omega \Lambda^t$$

The characteristic polynomial is no longer invariant, but...

$$P'(\lambda) = det(\eta\Omega' - \lambda) = \lambda^4 + \left(\omega^2 - \gamma^2\right)\lambda^2 Q^4 - (\omega\cdot\gamma)^2 Q^8$$

$$\Rightarrow \boxed{K \equiv \frac{\omega^2 - \gamma^2}{\omega\cdot\gamma} = -\frac{\frac{1}{2}tr((\eta\Omega)^2)}{\sqrt{-det(\eta\Omega)}}}$$

K is invariant under *all* (constant) congruence transformations.

Physical interpretation: using congruence transformations, can find B-coordinates where ω||γ

$$K \equiv \frac{\omega}{\gamma} - \frac{\gamma}{\omega}$$

$$\frac{\omega}{\gamma} \equiv intrinsic\ \ ray\ \ helicity$$

In this coordinate frame:

$$\Omega(\mathbf{z}) = \begin{pmatrix} 0 & \{B_0, B_1\} & \{B_0, B_2\} & \{B_0, B_3\} \\ \{B_1, B_0\} & 0 & \{B_1, B_2\} & \{B_1, B_3\} \\ \{B_2, B_0\} & \{B_2, B_1\} & 0 & \{B_2, B_3\} \\ \{B_3, B_0\} & \{B_3, B_1\} & \{B_3, B_2\} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & -\omega & 0 \\ 0 & \omega & 0 & 0 \\ -\gamma & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{D}(\mathbf{z}) = \begin{pmatrix} B_0 + B_3 & B_1 + iB_2 \\ B_1 - iB_2 & B_0 - B_3 \end{pmatrix}$$

Diagonal elements commute with off-diagonals

If we expand about the 'apex' of the light cone in B-coordinates

$$\mathbf{D}(\mathbf{z}) = \begin{pmatrix} \mathbf{b}_0 \cdot \mathbf{z} + \mathbf{b}_3 \cdot \mathbf{z} & \mathbf{b}_1 \cdot \mathbf{z} + i\mathbf{b}_2 \cdot \mathbf{z} \\ \mathbf{b}_1 \cdot \mathbf{z} - i\mathbf{b}_2 \cdot \mathbf{z} & \mathbf{b}_0 \cdot \mathbf{z} - \mathbf{b}_3 \cdot \mathbf{z} \end{pmatrix} + O(z^2)$$

A linear canonical transformation (**z'=Mz**) gives

$$\mathbf{D}(\mathbf{z}') = \begin{pmatrix} q_1 & q_2 + i\omega p_2 \\ q_2 - i\omega p_2 & \gamma p_1 \end{pmatrix} + O(z'^2)$$

If γ is non-zero, can perform a further combination of congruence and canonical transformations to find the 'normal' form:

$$\mathbf{D}(\mathbf{z}') = \begin{pmatrix} q_1 & q_2 + i\kappa p_2 \\ q_2 - i\kappa p_2 & p_1 \end{pmatrix} + O(z'^2) \qquad \kappa = \frac{\omega}{\gamma}$$

The related 2X2 wave equation

$$\begin{pmatrix} \hat{q}_1 & \hat{q}_2 + i\kappa\,\hat{p}_2 \\ \hat{q}_2 - i\kappa\,\hat{p}_2 & \hat{p}_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

Can be solved by separation of variables and a generalization of the Fourier transformation. (Littlejohn, de Verdiere)

Summary and conclusions:

- Linear wave conversion in multi-dimensions has new physics not present in one-dimensional version, such as ray helicity.
- There is an invariant that characterizes the helicity of rays in conversion regions.
- Methods are constructive and should lead to explicit solutions (work in progress).
- Preprint available at arXiv.org/physics/0303086