Maslov indices

and

Singularities of Integrable Systems

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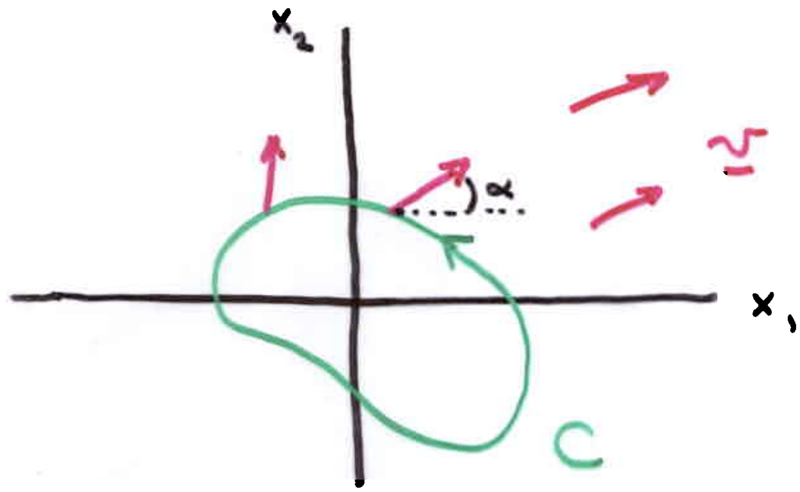
Poincaré Index

$\dot{\underline{x}} = \underline{v}(\underline{x})$, planar system



If $\underline{v} \neq 0$ on C,

$$\boxed{\text{ind } C = \frac{\Delta\alpha}{2\pi}}$$

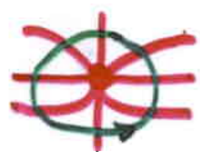
If $\underline{v} \neq 0$ in C,

$$\text{ind } C = 0.$$

Nondegenerate fixed points

$$\underline{v}(\underline{x}_j) = 0, \quad \left|\frac{\partial \underline{v}}{\partial \underline{x}}\right|_j \neq 0$$

$$\left|\frac{\partial \underline{v}}{\partial \underline{x}}\right|_j > 0$$



node

focus

centre

$\operatorname{ind} C = \sigma(C)$, orientation

$$\left|\frac{\partial \underline{v}}{\partial \underline{x}}\right|_j < 0$$

$$\operatorname{ind} C = -\sigma(C)$$

If only fixed points in C are nondegenerate,

$$\boxed{\operatorname{ind} C = \sum_j \sigma(C)\, \operatorname{sgn}\left|\frac{\partial \underline{v}}{\partial \underline{x}}\right|_j}$$

Consequences

1. If C is a periodic orbit,

$\text{ind } C = \sigma(C)$

If only fixed points in C are nondegenerate, their indices sum to +1

2. Index unchanged by local bifurcations

eg, saddle-node

ind C = 0 ind C = 0

For 1-freedom Hamiltonian system,

$$\text{ind } C = \sigma(C) \cdot [\text{\# of elliptic fps in } C - \text{\# of hyperbolic fps in } C]$$

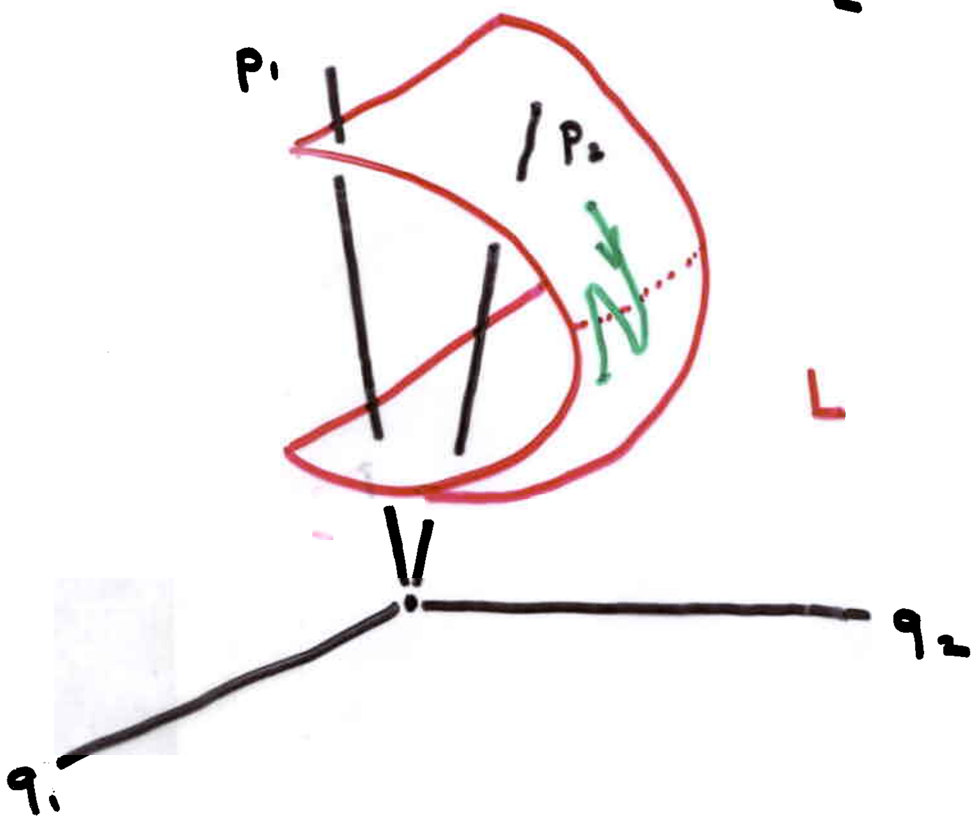
This generalises to formula for Maslov indices of integrable systems in higher dimensions ...

Maslov index

$P = \mathbb{R}^{2n}$, cartesian phase space

$\underline{z} = (\underline{q}, \underline{p})$

An n-dimensional submanifold L is Lagrangian if $\omega|_L = 0$



Caustics: where projection from L to $\underline{q}$-plane is singular

Away from caustics,

$$p_j = p_j(q_1, \ldots, q_n) = \partial S_j(q_1 \ldots q_n)$$

Near caustics,

$$p_{j \neq 1} = p_{j \neq 1}(p_1, q_2, \ldots, q_n)$$
$$q_1 = q_1(p_1, q_2, \ldots, q_n)$$

Given oriented curve γ on L,

$$\mu(\gamma) = \sum_{\substack{\text{caustics} \\ \text{on } \gamma}} \operatorname{sgn}\left(\frac{\partial q_1}{\partial p_1}\right)_+$$

Maslov index as winding number

Arnold (1968)

$\Lambda(n)$, space of Lagrangian planes

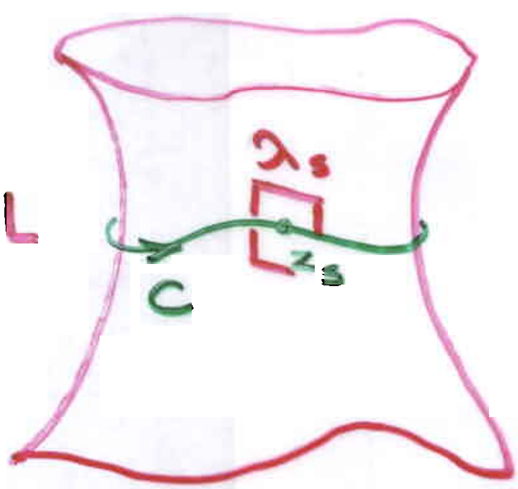
An n-dim plane λ in $\mathbb{R}^{2n}$ is Lagrangian if $v \cdot \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \cdot w = 0$, $v, w \in \lambda$

Tangent planes of Lagrangian submanifold are Lagrangian planes

$$\Lambda(n) = U(n)/O(n)$$

$$\pi_1(\Lambda(n)) = \mathbb{Z}$$

A closed curve λ_s in $\Lambda(n)$ has a winding number, $wn(\lambda_s)$ ...



C, closed curve on L

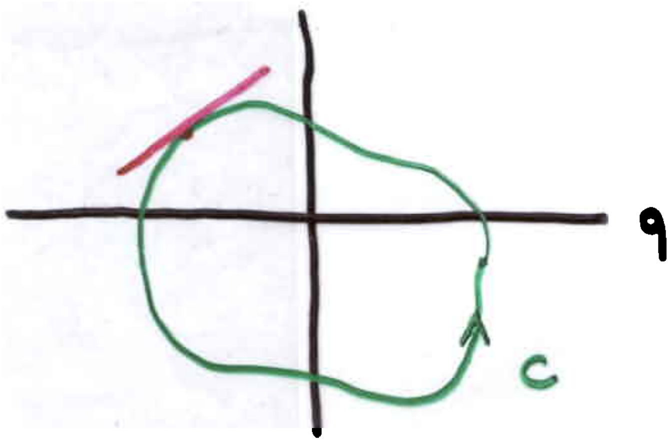
λ_s, tangent plane at z_s

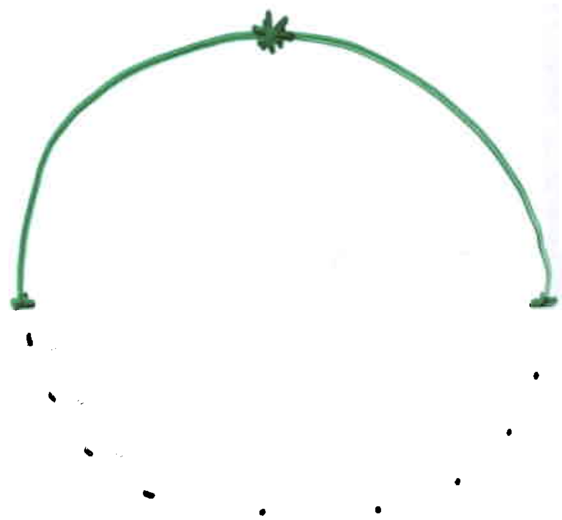
$$\boxed{\mu(C) = wn(\lambda_s)}$$

$\lambda_s \in \Lambda(n)$

Illustration

$n = 1$



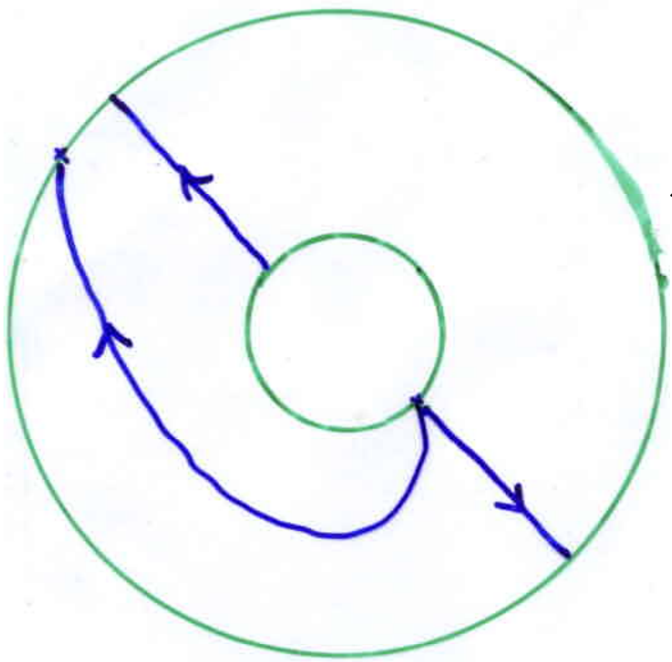


$\Lambda(1)$

(For $n = 1$, $\mu(C) = 2 \operatorname{ind} C$)

For $n=2$, the space of Lagrangian planes is a spherical shell with antipodal points on the inner and outer surfaces identified.

$$wn = \#(\text{inner} \to \text{outer}) - \#(\text{outer} \to \text{inner})$$



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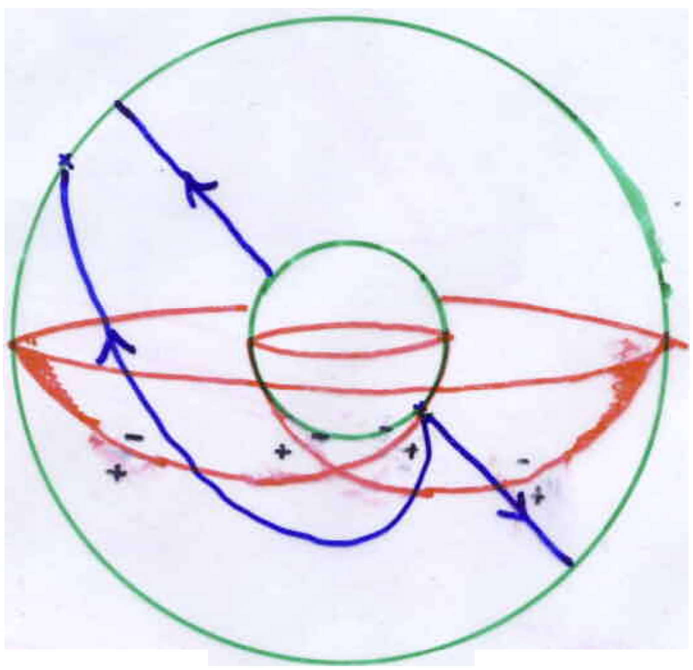
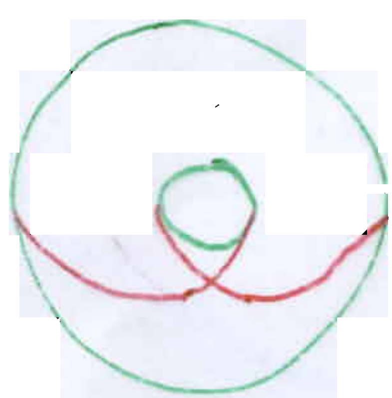




caustic

Integrable systems

$$H = H(F_1, \dots, F_n)$$

F_j's are functionally independent and

$$\{F_j, F_k\} = 0$$

Level sets $\underline{F} = \text{const}$ are (typically) Lagrangian submanifolds.

$$\lambda(\underline{z}) = \text{span}\{J\,dF_1(\underline{z}), \dots, J\,dF_n(\underline{z})\},$$

$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ tangent planes

$$\boxed{\mu(c) = \text{wn}\,\lambda(z_0)}$$

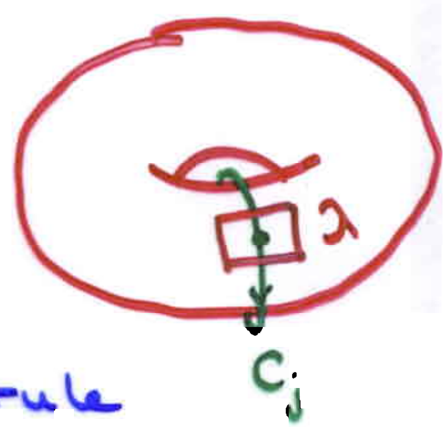
for closed curves c on level set

If level sets are compact, they are n-tori

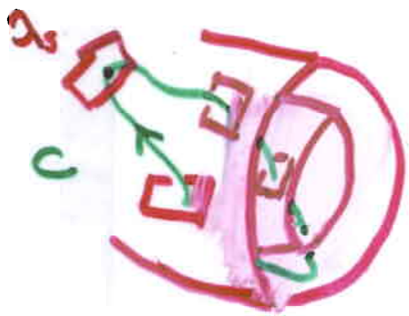
$$H = H(\underline{I}),$$

$$I_j = \left(n_j + \frac{\mu(c_j)}{4}\right)\hbar,$$

EBK quantisation rule



General closed curve



C, closed curve (not necessarily on $\underline{F}$ = const)

If $\lambda(\underline{z})$ is defined along C, let

$$\mu(C) = wn(\lambda_s)$$

If C is contractible, and $\lambda(\underline{z})$ defined throughout, then

$$\mu(C) = 0$$

$\mu(C) \neq 0$ implies that C encloses singularities of $\lambda(\underline{z})$, where $J\,dF_1(\underline{z}), \ldots, J\,dF_n(\underline{z})$ are linearly ~~in~~ dependent

$$\boxed{\text{Sources of the Maslov index are critical points of } d\underline{F}}$$

Nondegenerate singularities

Σ, critical points of $\underline{F}$

$$\Sigma_1 = \{ y \mid \underset{\substack{\| \\ \dim \lambda(y)}}{\dim \operatorname{span} dF_j(y)} = n-1 \}$$

For $y \in \Sigma_1$, let

$$\sum_j c_j(y)\, dF_j(y) = 0,$$

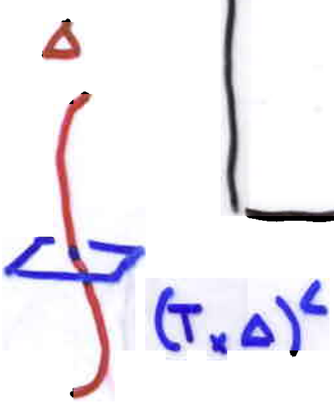
$$K(y) = \sum_j c_j(y)\, J F_j''(y)$$

(determined up to nonzero scalar)

$$\Delta = \{ \underline{x} \in \Sigma_1 \mid \operatorname{Tr} K^2(\underline{x}) \neq 0 \},$$

nondegenerate singularities

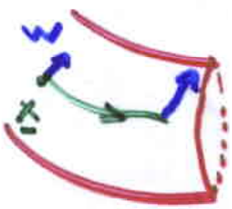
$$\boxed{\begin{array}{l} \text{1. } \Delta \text{ is an invariant codimension-2} \\ \quad \text{symplectic submanifold, and} \\ \qquad T_x\Delta = \ker K^2(\underline{x}), \\ \qquad (T_x\Delta)^{\angle} = \operatorname{Im} K^2(\underline{x}) \end{array}}$$



Transverse stability of nondegenerate singularities

Suppose the orbit of $\underline{x} \in \Delta$ under the flows of the F_j's is compact. Then $\underline{x}$ lies on an invariant $(n-1)$-torus

$\underline{z}^H(\underline{z},t)$, flow $\quad S^H(\underline{z},t)$, linearised flow



Transverse Liapunov exponent:

$$\mathcal{K}^H(\underline{x}) = \sup_{w} \lim_{T\to\infty} \frac{1}{T} \left| S(\underline{x},t)\cdot w \right|$$

$$\boxed{\begin{array}{l} \text{2. If } \operatorname{Tr} K^2(\underline{x}) < 0,\ \mathcal{K}^H(\underline{x}) = 0. \\[1em] \quad \text{If } \operatorname{Tr} K^2(\underline{x}) > 0, \\[1em] \qquad \mathcal{K}^H(\underline{x}) = \displaystyle\sum_j \frac{\partial H}{\partial F_j} \mathcal{K}^j(\underline{x}) \\[1em] \mathcal{K}^j(\underline{x}) \text{ is Liapunov exponent for } F^j. \\[1em] \sum c_j(\underline{x}) \mathcal{K}^j(\underline{x}) = \operatorname{Tr} K^2(\underline{x}) \end{array}}$$

$\operatorname{Tr} K^2(\underline{x}) < 0 \Rightarrow$ stable w.r.t. transverse displacements

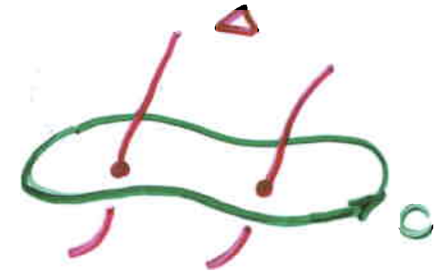
$\operatorname{Tr} K^2(\underline{x}) > 0 \Rightarrow$ unstable "

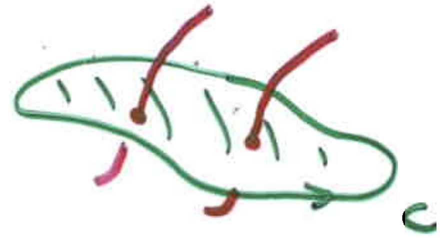
$$\mathcal{K}^{j}(\underline{x}) = \left| \left\langle \mathrm{Tr}\, P J F_j'' \right\rangle_{T(\underline{x})} \right| ,$$

$$P(\underline{x}) = \frac{Q Q^T}{\mathrm{Tr}\, Q Q^T}(\underline{x}) ,$$

$$Q(\underline{x}) = \left(K(\underline{x}) + \left(K^2(\underline{x})\right)^{1/2} \right) K^2(\underline{x})$$

Transversality.



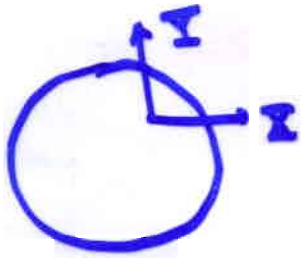


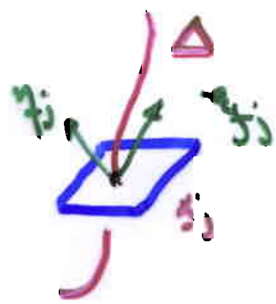
S, smooth map from oriented 2-disk to $\mathbb{R}^n$

C, boundary of image

S is transverse if:

- finite # of singular points $\underline{x}_j$ in image
- $\underline{x}_j \in \Delta$





- image of tangent map is plane transverse to Δ at $\underline{x}_j$

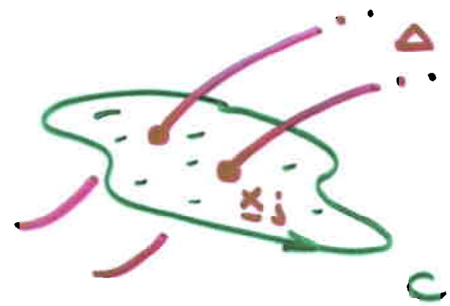
Orientation

$$\boxed{\sigma_j = \operatorname{sgn}\left[\xi_j, \eta_j\right]_{(T_{x_j}\Delta)^\perp}}$$

Maslov index formula

C, oriented closed curve in $\mathbb{R}^{2n}$

Suppose C is the boundary of transverse S.



$$\boxed{3. \quad \mu(C) = \sum_j \sigma_j \quad \begin{cases} +1, & x_j \text{ stable} \\ -1, & x_j \text{ unstable} \end{cases}}$$

$$= \sum_j \operatorname{sgn} \xi_j \cdot JK^2(x_j) \cdot \eta_j$$

- If degenerate singularities $\Sigma - \Delta$ are contained in submanifold of codim 3, then C is boundary of some transverse S
- Number of singularities, and their stabilities + orientations, depend on S

Related work

Global topology of integrable systems...

Monodromy (talks of Child, Faure, Zhelinskii)

Fomenko (Morse theory of integrable systems)

Eliasson, Vey

Tien Zung

F., Bolsinov, Richter

Maslov classes and residue classes

H Suzuki (1994)

Formula

$$\mu(c) = \Delta\left(\arg\det\left(\frac{\partial F_j}{\partial p_k} - i\,\frac{\partial F_j}{\partial q_k}\right)\right)$$

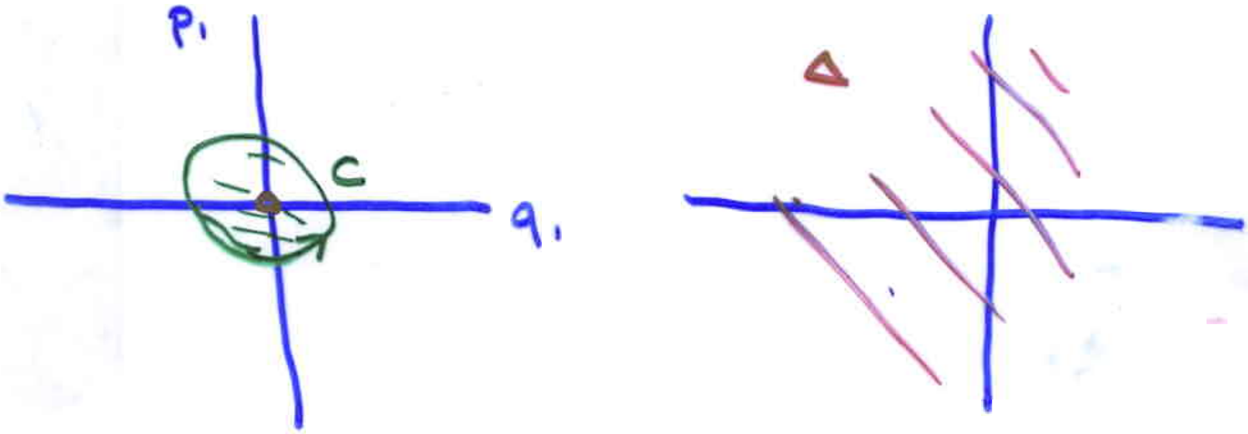
Example - Two harmonic oscillators

$$F_1 = \tfrac{1}{2}(p_1^2 + q_1^2), \quad F_2 = \tfrac{1}{2}(p_2^2 + q_2^2)$$

$$\Delta : \quad (q_1, p_1) = 0 \quad \text{or} \quad (q_2, p_2) = 0$$

e.g.,

$$q_2 = p_2 = 0$$



Clearly, Δ is codimension-2 symplectic.

$$\operatorname{Tr} K^2 = -2, \quad \text{(stable)}$$

$$\sigma = 1$$

$$\mu(C) = 2.$$

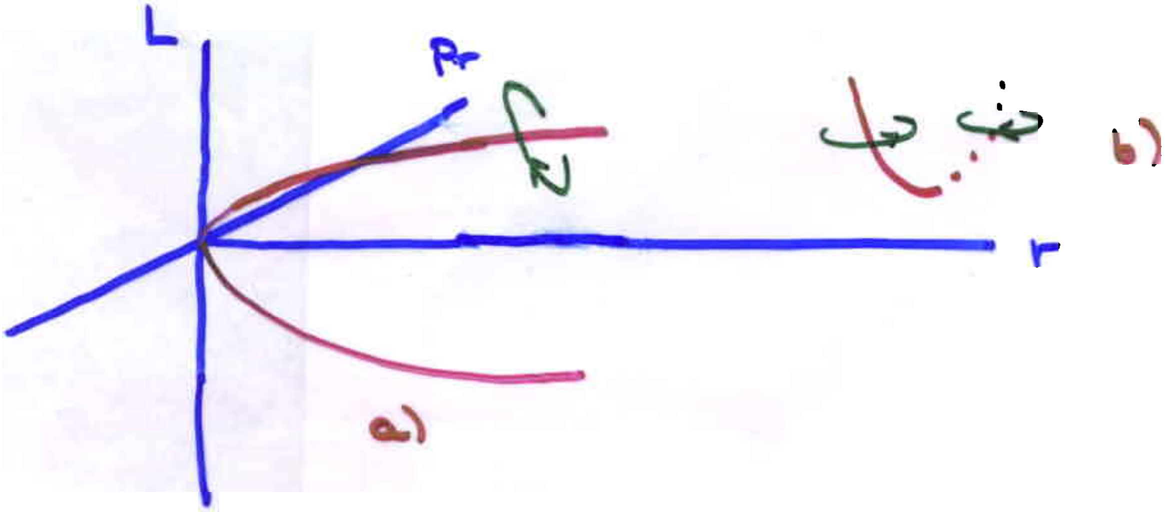
Example - Circular symmetry

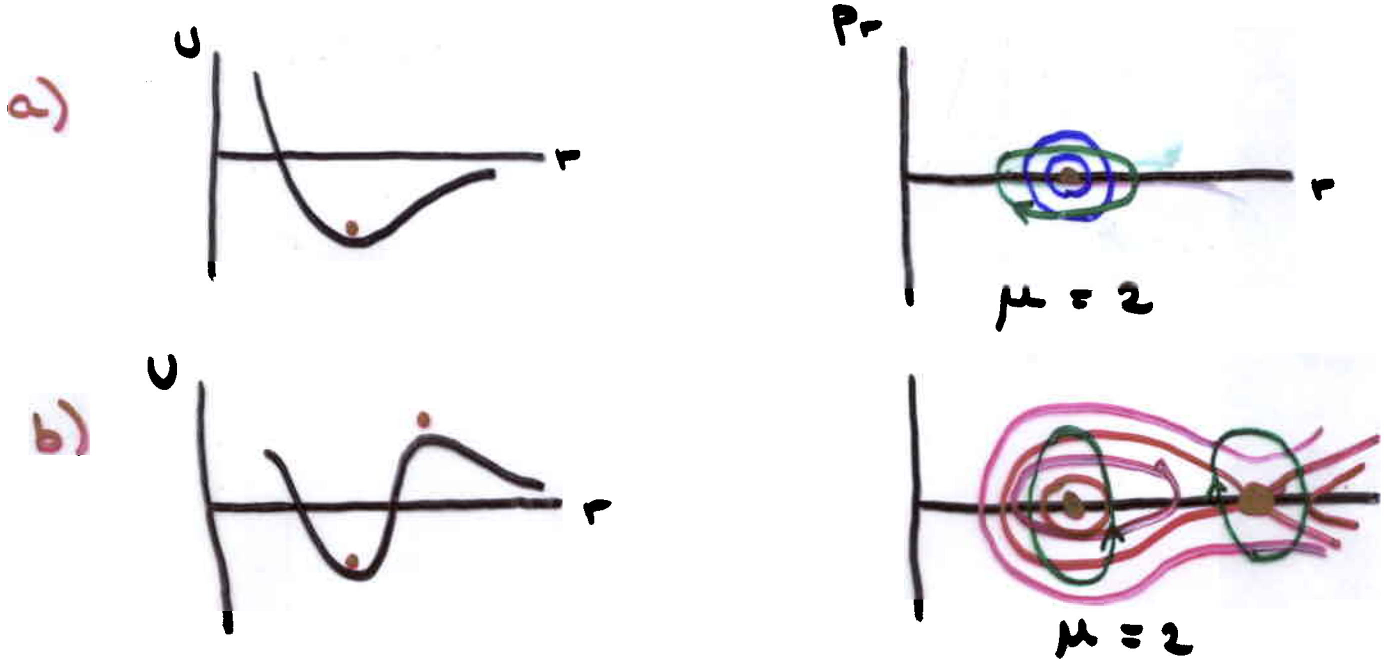
$$F_1 = H = \frac{p_r^2}{2} + U(r, L)$$

$$F_2 = L$$

$$\Delta = \left\{ \begin{matrix} \text{circular} \\ \text{orbits} \end{matrix} \right\}, \text{ ie } p_r = 0, \ \frac{\partial U}{\partial r} = 0.$$

$$\operatorname{Tr} K^2 = -\frac{\partial^2 U}{\partial r^2}.$$





change in orientation compensates
change in stability

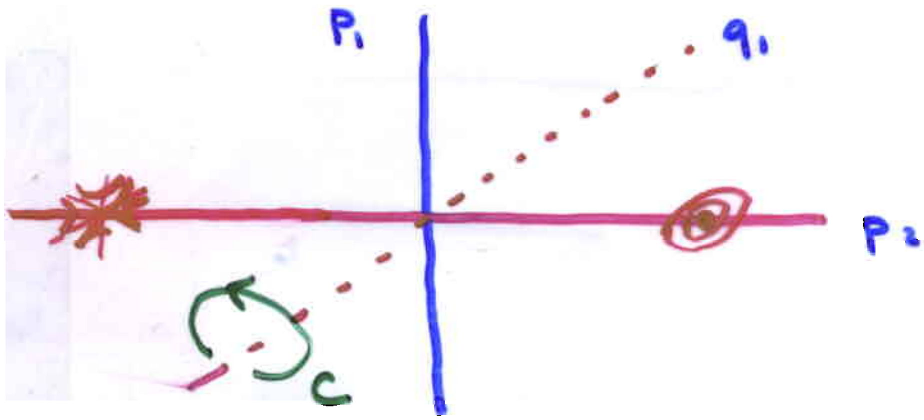
Nongeneric example

$$F_1 = \frac{p_1^2}{2} + p_2 \frac{q_1^2}{2}$$

$$F_2 = p_2$$

$$\Sigma = \{ p_1 = 0,\ p_2 q_1 = 0 \}$$

$$\mathrm{Tr}\, K^2 = -p_2$$



$p_1 = 0,\ q_1 = 0$ ($p_2 \neq 0$) is nondegenerate. Δ

$p_1 = 0,\ p_2 = 0$ is degenerate. $\Sigma - \Delta$.
(not symplectic)

$\mu(C) = 2$, but singularity formula does not apply.

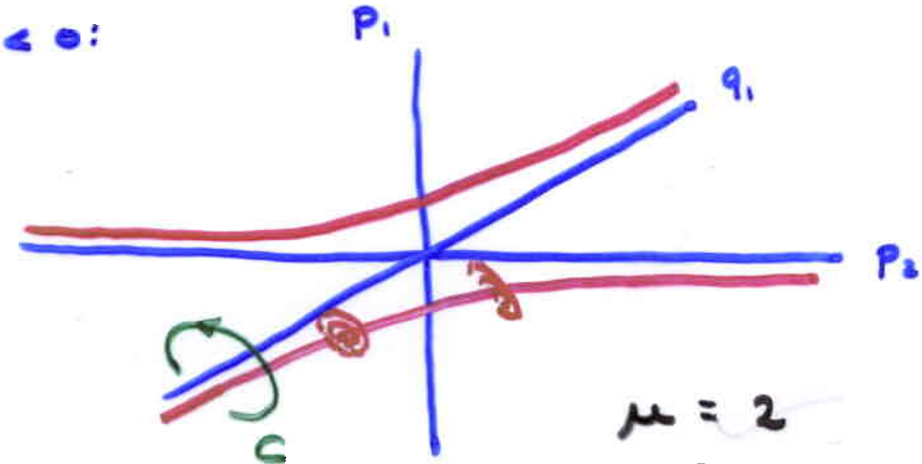
singularity in C is marginally stable
orientation not defined

Unfolding

$$F_1 = \frac{p_1^2}{2} + p_2 \frac{q_1^2}{2} - \varepsilon q_1, \quad F_2 = p_2$$

$$\Sigma_\varepsilon = \{ p_1 = 0, \; q_1 p_2 = \varepsilon \}, \text{ nondegenerate}$$

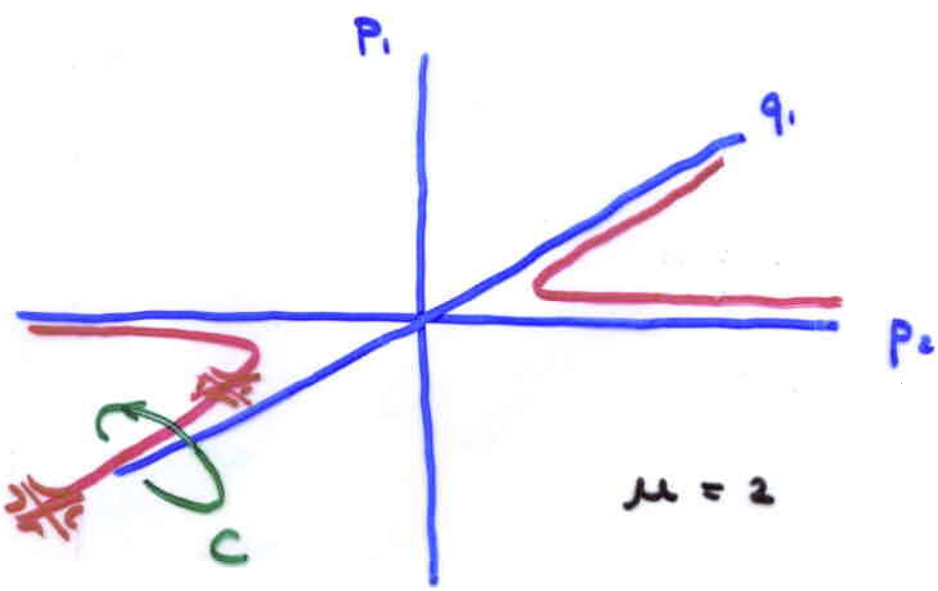
$\varepsilon < 0$:



$\mu = 2$

$\sigma = 1$, stable

$\varepsilon > 0$:



$\mu = 2$

$\sigma = -1$, unstable

Periodic Toda Chain

A nonseparable system.

$$H = \sum_{j=1}^{n} \left[\frac{1}{2} p_j^2 + e^{(q_j - q_{j+1})} \right], \quad q_{n+1} \equiv q_1$$

Let $a_j = e^{(q_j - q_{j+1})}$.

$$L(\underline{z}) = \begin{pmatrix} p_1 & \sqrt{a_1} & 0 & \cdots & 0 & \sqrt{a_n} \\ \sqrt{a_1} & p_2 & \sqrt{a_2} & & & 0 \\ 0 & \sqrt{a_2} & p_3 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \sqrt{a_n} & 0 & 0 & \cdots & & p_n \end{pmatrix}, \quad \text{symmetric}$$

$$M(\underline{z}) = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{a_1} & 0 & \cdots & 0 & -\sqrt{a_n} \\ -\sqrt{a_1} & 0 & \sqrt{a_2} & \cdots & & 0 \\ 0 & -\sqrt{a_2} & 0 & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \sqrt{a_n} & 0 & 0 & & & 0 \end{pmatrix}, \quad \text{antisymmetric}$$

Hamilton's equations imply

$$\boxed{\dot{L} = [L, M]}$$

Integrability

$$\dot{L} = [L, M] \Rightarrow L(t) = R(t)\, L(0)\, R^T(t)$$

orthogonal

$\Rightarrow$ eigenvalues $\lambda_j(z)$ are constants of motion

In fact,

$$\boxed{\{\lambda_j, \lambda_k\} = 0, \quad \text{integrability}}$$

Flaschka
Hénon 1974

Let

$$\boxed{F_j = \frac{1}{j} \operatorname{Tr} L^j} = \frac{1}{j}\left(\lambda_1^j + \cdots + \lambda_n^j\right)$$

The F_j's constitute n constants of the motion in involution.

Where are the JF_j' vectors

linearly dependent?

Further work.

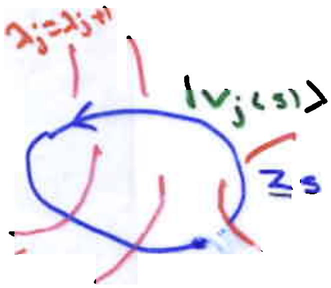
- Infinite-dimensional integrable systems

- Higher-dimensional Maslov classes

Geometric Phases

$\underline{L}(\underline{z})$, symmetric matrix parameterized by $\underline{z}$

$|v_j\rangle(\underline{z})$ $\lambda_j(\underline{z})$ normalized eigenvectors, eigenvalues



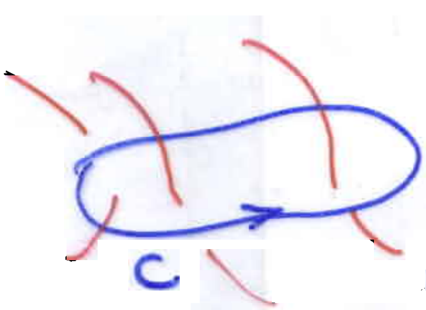
Choose sign of $|v_j(s)\rangle$ so that it is continuous.

If $\underline{z}_s$ encloses an odd number of simple degeneracies of λ_j, then

$$\boxed{|v_j(1)\rangle = -|v_j(0)\rangle}$$

Geometric phase $e^{i\delta_j}$

Berry, Berry + Wilkinson, Longuet-Higgins, Uhlenbeck



M Audin

$$\boxed{\mu(C) \equiv \{\# \text{ of } e^{i\delta_j} = -1\} \bmod 4}$$

it is almost. $(n=3)$

Signs of roots $\sqrt{a_j}$ in Lax pairs can be chosen arbitrarily.

$$L = \begin{pmatrix} p_1 & \sigma_1\sqrt{a_1} & 0 & \cdots & \sigma_n\sqrt{a_n} \\ \sigma_1\sqrt{a_1} & p_2 & \sigma_2\sqrt{a_2} & \cdots & 0 \\ \vdots & \sigma_2\sqrt{a_2} & \ddots & \ddots & \vdots \\ \sigma_n\sqrt{a_n} & 0 & \cdots & \cdots & p_n \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & \sigma_1\sqrt{a_1} & 0 & \cdots \\ -\sigma_1\sqrt{a_1} & & & \\ \vdots & & & \end{pmatrix}$$

Many choices are related by similarity transformations, $\begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}$

Two inequivalent choices: $\boxed{\sigma_n = \pm 1} \rightarrow L_\pm$

$\boxed{\text{$\Delta$ consists of points where precisely two eigenvalues of $L_\pm$ are degenerate}}$ $(n=3)$

Degeneracies

$$F_j' = \mathrm{Tr}\left(L^{j-1} L' \right)$$

eigenvector basis

$$= \sum_{k=1}^{n} \langle v_k | L^{j-1} L' | v_k \rangle$$

$$= \sum_{k=1}^{n} \lambda_k^{j-1} \langle v_k | L' | v_k \rangle .$$

$$\begin{pmatrix} \underline{F}_1' \\ \vdots \\ \underline{F}_n' \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} \langle v_1 | \underline{L}' | v_1 \rangle \\ \vdots \\ \langle v_n | \underline{L}' | v_n \rangle \end{pmatrix}$$

$$\downarrow$$

$$\det = \prod_{j>k} (\lambda_j - \lambda_k) .$$

> If the eigenvalues of L are degenerate at $\underline{z}_0$, then the vectors $J F_j'$ are linearly dependent at $\underline{z}_0$, ie $\underline{z}_0 \in \Gamma$.

The converse is <u>not</u> true, but ...