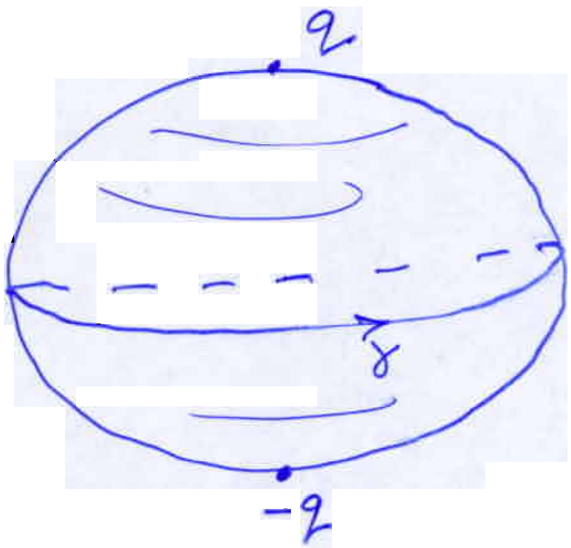
L^p-bounds for eigenfunctions (integrable case) 1).
(joint w. S. Zelditch)

Examples (i):



$(S^2, \text{can.})$

$\lambda_n = n(n+1)$

$\text{mult}(\lambda_n) = 2n+1.$

Zonal harmonics : $|\varphi_n(\pm q)| \sim \sqrt{n}$. $\left(\|\varphi_n\|_{L^2} = 1\right)$

(ii) $(\mathbb{T}^2, \text{can.})$

$$\varphi_n(x_1, x_2) = e^{i[n_1 x_1 + n_2 x_2]}$$

$$|\varphi_n(x)| = 1.$$

- $\varphi_n(x)$ uniformly bounded.

Define
$$N(\lambda, x, x) = \sum_{\lambda_j \leq \lambda} |\varphi_{\lambda_j}(x)|^2$$

with $\int_M |\varphi_{\lambda_j}(x)|^2 \, d\mathrm{vol}(x) = 1$ and $-\Delta \varphi_{\lambda_j} = \lambda_j \varphi_{\lambda_j}$

$\langle \varphi_{\lambda_i}, \varphi_{\lambda_j} \rangle = \delta_{ij}$.

Theorem [Avakumović–Hörmander]

$$N(\lambda; x, x) = c_n \, \mathrm{vol}(S_x^* M) \cdot \lambda^{n/2} + O(\lambda^{\frac{n-1}{2}}) \quad \text{as } \lambda \to \infty$$

$$\Rightarrow \quad \|\varphi_{\lambda_j}\|_{L^\infty} = O(\lambda_j^{\frac{n-1}{4}}). \quad (*)$$

Note: The bound $(*)$ is sharp (consider Ex. 1 of $(S^2, \text{can.})$).

Questions: ① When is the bounde $(*)$ attained? (what kind of geometry).

② When are all eigenfunctions uniformly bounded? (Ex. 2 of flat torus).

Quantum Integrable Case (QCI)

- $P_1^\hbar = \hbar\sqrt{-\Delta}$, $P_2^\hbar, \dots, P_n^\hbar$ self-adjoint, jointly-elliptic $\hbar$-ΨDO's.

- $[P_i^\hbar, P_j^\hbar] = 0 \quad \forall i,j = 1,\dots,n$

- $P_i^\hbar$ has principal symbol $p_i(x,\xi)$.
  $(p_1(x,\xi) = |\xi|_g)$.

- Assume that $dp_1 \wedge \dots \wedge dp_n \neq 0$ on a dense open set $\Omega \subset T^*M$.

Joint Flow: Given $t \in \mathbb{R}^n$; $t = (t_1, \dots, t_n)$

$$\Phi_t : \dot{T}^*M \longrightarrow \dot{T}^*M$$

$$\Phi_t(x,\xi) = \exp t_1 X_{p_1} \circ \dots \circ \exp t_n X_{p_n}(x,\xi)$$

Moment map:

$$P(x,\xi) = (p_1(x,\xi), \dots, p_n(x,\xi)).$$

Liouville-Arnold Thm.: For "most" values $b \in \mathcal{P}(\mathring{T}^*M)$, the set $\mathcal{P}^{-1}(b)$ consists of a union of Lagrangian tori. Near each torus, Λ, one has symplectic coordinates $(\theta_1, \dots, \theta_n; I_1, \dots, I_n)$ (action-angle) variables such that $I_1 = \dots = I_n = 0$ on Λ and the flow is quasiperiodic along Λ:

$$\frac{d\theta_j}{dt} = F_j(I) \qquad ; \quad j = 1, \dots, n.$$
$$\frac{dI_j}{dt} = 0$$

Question 2 (Integrable)

- Suppose (Δ_g) has all uniformly bounded eigenfunctions and Δ_g is QCI.

Theorem 1 [T-Zelditch]

Under these assumptions, (M, g) isometric to a flat manifold.

Idea of Proof:

Step 1 (Topology of M):

- $A(x;\hbar D_x)$ bounded $\hbar$-ΨDO. Then for some subsequence of joint eigenfunctions, ϕ_μ, of $P_1^\hbar, \dots, P_n^\hbar$:

$$\langle A(x;\hbar D_x)\cdot\varphi_\mu, \varphi_\mu\rangle = \int_\Lambda a(x,\xi)\, d\mu_L + O(\hbar)$$

Put $A(x;\hbar D_x) = V(x) \in C^\infty(M)$

$$\Rightarrow \quad \int_M V(x)\, |\varphi_\mu(x)|^2\, d\mathrm{vol}(x) = \int_\Lambda \pi^* V \cdot d\mu_L + O(\hbar)$$

$$\Rightarrow \quad \pi|_\Lambda : \Lambda \longrightarrow M \quad \text{is regular,}$$

- This determines topology of (M,g).

Step 2 (Geometry of M):

- (Mañé) C^0 Lagrangian foliation of T^*M
  $\Rightarrow$ (M,g) has no conjugate points.

- (Burago – Ivanov) (T^n, g) has no conjugate pts.
  $\Rightarrow$ (T^n, g) is <u>flat</u>.

Main Point :

- To go from Step 1 $\Rightarrow$ Step 2
  (Topology of M) $\Rightarrow$ (Geometry of M)

one shows that under the uniform bdd. assumption on eigenfunctions there cannot exist compact, singular (dim. < n) orbits of the $\mathbb{R}^n$-action, Φ_t.

$\Rightarrow$ can apply Mañé $\Rightarrow$ apply Burago-Ivanov.

- There is a related result of independent interest :

Theorem [T-Zelditch] : Let (M, g) have integrable geodesic flow. Then, unless (M, g) is a flat torus, the $\mathbb{R}^n$-action Φ_t must have a singular orbit of dimension $< n$.

<u>Main Point</u> : One of the main steps in the proof of Theorem 1 involves showing that as soon as there are compact, singular (dimension $< n$) orbits, there exists subsequence of joint eigenfunctions that blow-up along these orbits.

<u>Problem</u> : Quantify the blow-up along a singular orbit, Γ.

Birkhoff Normal Form

- $c \in \mathbb{R}^n$ a singular value of moment map P.
- $\nu \in T^*M$ and $\Gamma^k \subset P^{-1}(c)$ k-dimensional compact orbit through ν.

- Let $K = \bigcap_{i=1}^{n} \ker dp_i(\nu)$, $L = \text{span}\{X_{p_1}(\nu), \dots, X_{p_n}(\nu)\}$.

<u>Definition</u> : The orbit Γ^k is non-degenerate (Eliasson) if $\langle d_\nu^2 p_1, \dots, d_\nu^2 p_n \rangle$ is of rank k a Cartan subalgebra of $S^2(K/L, \omega_\nu)^*$.

$Q(2m)$ = Lie algebra of quadratic forms on $\mathbb{R}^{2m}$

- <u>Action variables</u> :
  (i) Real hyperbolic $I_i^{h}(x,\xi) = x_i \xi_i$
  (ii) Elliptic $I_i^{e} = x_i^2 + \xi_i^2$
  (iii) Loxodromic $I_i^{ch} = (x_i \xi_{i+1} - x_{i+1}\xi_i,\ x_i\xi_i + x_{i+1}\xi_{i+1})$.

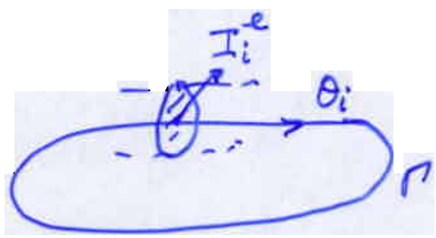
If Γ^k is non-degenerate, by a result of Eliasson, there exists a local normal form for the p_i's near Γ^k: when Γ^k is stable, given a small open neighbourhood $\Omega^k \supset \Gamma^k$, there exists a canonical map

$$\kappa : \Omega_k \longrightarrow B_{n-k}(0) \times \mathbb{T}^k \times \mathbb{R}^k$$

and C^∞ functions f_i such that $\forall i = 1, \dots, n$:

$$p_i \circ \kappa^{-1} - c_i = f_i(I_1^e, \dots, I_{n-k}^e, I_1, \dots, I_k) \quad (*).$$

• The general case is similar.



• (θ_i, I_i) are canonically dual action-angle variables.

Quantum Birkhoff Normal Form (QBNF)

Colin de Verdière–Parisse, Helffer–Sjöstrand,

• By a result of Vũ-Ngọc, can find $\hbar$-FIO's, $F_\hbar$, with relation, κ, such that:

$$\text{(i)} \quad F_\hbar^* F_\hbar \underset{\Omega_k}{\equiv} Id$$

$$\text{(ii)} \quad F_\hbar^* \cdot P_i^\hbar \cdot F_\hbar - c_i \underset{\Omega_k}{\equiv} f_i(\hat{I}_1^e, \dots, \hat{I}_{n-k}^e, \hbar D_{\theta_1}, \dots, \hbar D_{\theta_k}; \hbar).$$

• Here, $f_i(x;\hbar) \sim f_i(x) + f_{i,1}(x)\cdot\hbar + f_{i,2}(x)\hbar^2 + \cdots$

$$\hat{I}_i^e = (\hbar D_{x_i})^2 + x_i^2.$$

• General case is similar.

Microlocal normal form for eigenfunctions

- **Model Eigenfunctions**:

$$\cdot\, u_{\hbar}(y;\lambda(\hbar)) = |\log \hbar|^{-1/2} \cdot \left[c_+(\hbar)\, Y(y)\, |y|^{-\frac{1}{2} + i\lambda(\hbar)/\hbar} + c_-(\hbar)\, Y(-y) \cdot |y|^{-\frac{1}{2} + i\lambda(\hbar)/\hbar} \right]$$

where $|c_+(\hbar)|^2 + |c_-(\hbar)|^2 = 1$ and $\lambda(\hbar) \in \mathbb{R}$.

$$\cdot\, u_e(y;n,\hbar) = \hbar^{-1/4} e^{-y^2/\hbar}\, \Phi_n(y/\sqrt{\hbar}) \quad ; n \in \mathbb{N}.$$

$$\cdot\, u_{ch}(r,\theta;t_1,t_2,\hbar) = |\log \hbar|^{-1} \cdot r^{(-1+it_1(\hbar))/\hbar} \cdot e^{it_2(\hbar)\cdot\theta} \quad ; t_j(\hbar) \in \mathbb{R}$$

$$\cdot\, u_{reg}(\theta;m) = e^{im\theta}$$

Let L = number of loxodromic subspaces

H = number of real-hyperbolic subspaces

E = number of elliptic subspaces.

Short Form Notation: Write

$$\psi_\hbar := u_e(y;n) \cdot u_h(y,\lambda(\hbar)) \cdot u_{ch}(y\,;\,t_1(\hbar),t_2(\hbar)) \cdot \prod_{j=1}^{k} e^{im_j\theta_j}$$

for

$$\prod_{j=1}^{H} u_h(y_j;\lambda_j(\hbar)) \otimes \prod_{j=H+1}^{H+L} u_{ch}(y_j, y_{j+1}; t_{1j}, t_{2j}) \otimes \prod_{j=H+L+1}^{H+L+E} u_e(y_j;n_j) \otimes \prod_{j=1}^{k} e^{im_j\theta_j}$$

Theorem [Colin de Verdière–Parisse, Vu-Ngọc, Helffer–Sjöstrand]

Given any joint eigenfunctions $\phi_\lambda(y;\hbar)$ of $P_1^\hbar, \dots, P_n^\hbar$ there exist quantum numbers $\lambda_j(\hbar), t_{1j}(\hbar), t_{2j}(\hbar), n_j, m_j$ such that:

$$\phi_\lambda(y;\hbar) \underset{\Omega_K}{=} c(\hbar) \cdot F_\hbar \Big(u_e(y;n) \times u_h(y;\lambda(\hbar)) \times u_{ch}(y;\vec{t}(\hbar)) \times \times \prod_{i=1}^{k} e^{i m_j \theta_j} \Big).$$

where $c(\hbar) \in \mathbb{C}(\hbar)$

Consider, for $c \in \mathbb{R}^n$ critical

$$\Sigma_c(\hbar) = \{ \mu(\hbar) = (\mu_1(\hbar), \dots, \mu_n(\hbar)) \in \mathrm{Spec}(P_1^\hbar, \dots, P_n^\hbar); \; |\mu_j(\hbar) - c_j| \leqq C\hbar \}.$$

$$V_c(\hbar) = \{ \varphi_\mu ; \|\varphi_\mu\|_{L^2} = 1 \text{ with } \mu(\hbar) \in \Sigma_c(\hbar) \}.$$

- We need the estimate:

Lemma 1: Let $\chi_1 \in C_0^\infty(B_{n-k}(0))$ with $\chi_1(y,\eta) = 1$ near $(y,\eta) = (0,0)$ and $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ with $\chi_2(I_1, \dots, I_k) = 1$ near $I_1 = \dots = I_k = 0$. Then, for $\mu \in \Sigma_c(\hbar)$, $0 \leqq \delta < \frac{1}{2}$

$$|c(\hbar)|^{-2} \cdot \langle Op_\hbar(\chi_1(\hbar^{-\delta} y, \hbar^{-\delta}\eta) \cdot \chi_2(\hbar^{-\delta} I)) \cdot \psi_\hbar, \psi_\hbar \rangle \geqq C(\delta) > 0.$$

By a local Weyl law argument it follows that for a subsequence of $\varphi_\mu \in V_c(\hbar)$, we have:

$$|c(\hbar)|^2 \geqq C \cdot |\log \hbar|^{-m}$$

for some $m \geqslant 0$.

Assume $\pi(\Gamma)$ is embedded.

$$\pi(\Gamma): \; x_{k+1} = \cdots = x_n = 0 \qquad x' = (x_{k+1}, \ldots, x_n)$$

$$\int_M |\varphi_\mu(x)|^2 \cdot \chi_1(\hbar^{-\delta} x_{k+1}, \ldots, \hbar^{-\delta} x_n) \, d\mathrm{vol}(x)$$

$$\gg \langle Op_\hbar [\chi_1(\hbar^{-\delta} x') \cdot \chi_2(\xi)] \varphi_\mu, \varphi_\mu \rangle$$

$$\gg \langle Op_\hbar [\chi_1'(\hbar^{-\delta} y, \hbar^{-\delta} \eta) \cdot \chi_2'(\hbar^{-\delta} I)] \psi_\hbar, \psi_\hbar \rangle$$

$$\times |c(\hbar)|^2.$$

But we prove,

$$|c(\hbar)|^2 \cdot \langle Op_\hbar [\chi_1(\hbar^{-\delta} y, \hbar^{-\delta} \eta) \cdot \chi_2(\hbar^{-\delta} I)] \psi_\hbar, \psi_\hbar \rangle$$

$$\gtrsim_\delta |c(\hbar)|^2 \gtrsim_\delta |\log \hbar|^{-m} \quad (0 \leq \delta < \tfrac{1}{2}).$$

some $m \geq 0$

• The upshot is that:

$$\left\{ \begin{array}{l} \|\varphi_\mu\|^2_{L^\infty} \cdot \int_M \chi(\hbar^{-\delta} x_{k+1}, \ldots, \hbar^{-\delta} x_N)\, d\mathrm{vol}(x) \\ \gtrsim_\delta |\log \hbar|^{-m} \qquad\qquad (\dagger) \end{array} \right\}$$

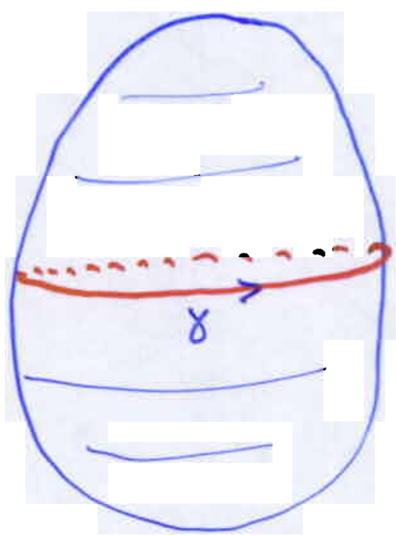
for a subsequence of $\varphi_\mu \in V_c(\hbar)$.

Theorem 2 [T–Zelditch]

Suppose (M,g) is compact with QCI Laplacian, Δ and that the Hamiltonian $\mathbb{R}^n$-action satisfies Eliasson's non-degeneracy condition. Then, unless (M,g) is a flat torus, this action must have a singular orbit of dimension $< n$. If the minimal dimension of the singular obits is ℓ, then for every $\varepsilon > 0$, there exists a subsequence of joint eigenfunctions φ_{μ_k} satisfying:

$$(*) \quad \begin{cases} \|\varphi_{\mu_k}\|_{L^\infty} \geq C(\varepsilon)\, \lambda_k^{\frac{n-\ell}{4}-\varepsilon} \\ \|\varphi_{\mu_k}\|_{L^\infty} \geq C(\varepsilon)\, \lambda_k^{\frac{(n-\ell)(p-2)}{4p}-\varepsilon}, \quad 2<p<\infty. \end{cases}$$

Example:



- Convex surface of revolution.

The exist φ_μ with $\varphi_\mu \sim \lambda^{1/4}$ along γ.

Recurrent Points (Integrable)

Want to give a partial answer to Question 2 when (M, g) is integrable. ($\dim M = 2$).

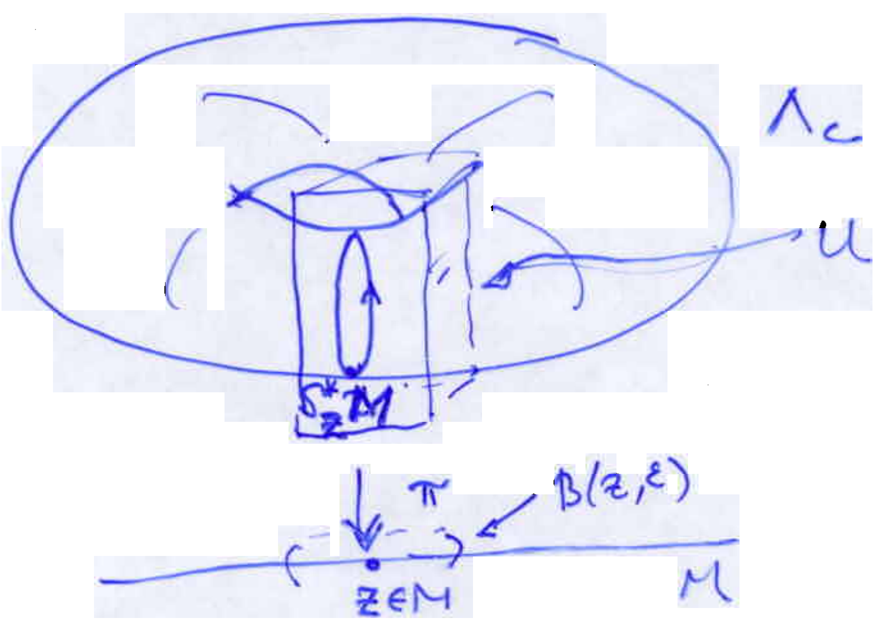
Setup: $p_1(x,\xi) = |\xi|_g$; $p_2(x,\xi)$

(i) $\{p_1, p_2\} = 0$

(ii) $p_1(x,\xi)$, $p_2(x,\xi)$ positive homogeneous of degree one.

(iii) $\Lambda_c = \{(x,\xi) \in \overset{o}{T}{}^*M \,;\, p_1(x,\xi) = 1,\ p_2(x,\xi) = c\}$ is a Lagrangian torus.

(iv) $S_z^* M \subset \Lambda_c$ for some $z \in M$.



<u>Definition</u>: Call $z \in M$ a blow-down point if (i) – (iv) are satisfied.

<u>Theorem 3</u> [T - Zelditch]

Assume $P_1^{\hbar} = \hbar\sqrt{-\Delta}$ is QCI on (M^2, g) and that $z \in M$ is a blow-down point. Then, there exists a subsequence of L^2- normalized Laplace eigenfunctions $\{\varphi_\mu\}$ s.t. $\forall \varepsilon > 0$

$$\|\varphi_\mu\|_{L^\infty} \geq C(\varepsilon)\, \hbar^{-\frac{1}{2}+\varepsilon}$$

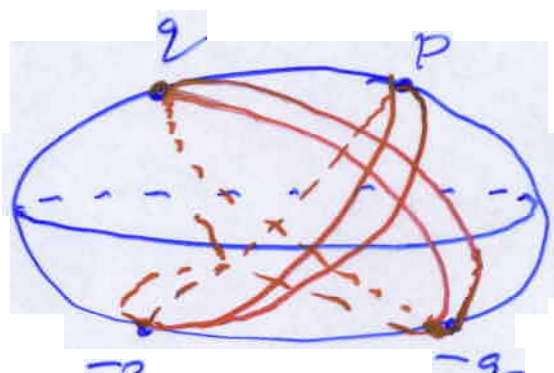
- Similar result for L^p; $2 < p < \infty$.

- This is a partial converse to a result of Sogge - Zelditch.

<u>Examples</u>

<u>Ex.1</u> : $(S^2, \text{can.})$

<u>Ex2</u> : Ellipsoid



Problems & Questions

(1) Replace t^δ scales by $t|\log t|$.

(2) In Thm. 3 (blowdown), what happens when $\dim M > 2$.