Quantum Resonances for Nonanalytic Potentials

A. Martinez (Berkeley, May 7, 2003)

From a series of papers (with various co-authors) on the resonances for the (2-body) Schrödinger operator

$$P = -h^2\Delta + V(x) \qquad (x \in \mathbb{R}^n).$$

Originally, the purpose was just to obtain a kind of simplified version of Helffer-Sjöstrand's theory of resonance, so that one can take advantage of their main idea without using complicated tools such as Fourier integral operators calculus with complex phase. Afterwards, these techniques also provides many generalizations.

1) Main Idea of Helffer-Sjöstrand's Theory (1986):

Replace the real phase-space $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ by a complex I-Lagrangian manifold of the form

$$\Lambda_{tG} = \{(x + it\partial_\xi G,\ \xi - it\partial_x G)\ ;\ (x,\xi) \in \mathbb{R}^{2n}\}$$

where $t>0$ small, $G : \mathbb{R}^{2n} \to \mathbb{R}$ is C^∞ and satisfies for some fixed energy level $E>0$:

(1.1) $\qquad H_p G > 0 \quad \text{near } p^{-1}(E) \cap \{|x| >> 1\}.$

(Here $p(x,\xi) = \xi^2 + V(x)$ is the symbol of P).

In such a way, the restriction of p on Λ_{tG} satisfies:

$$p|_{\Lambda_{tG}}(x+it\partial_\xi G,\ \xi - it\partial_x G) = p(x,\xi) - itH_pG + \mathcal{O}(t^2)$$

and thus:

$$p|_{\Lambda_{tG}} - E \text{ is elliptic } ~~for~~ |x| >> 1.$$

Thanks to this property, the corresponding operator (constructed by means of FIO's with complex phase) has discrete spectrum near E and its eigenvalues are the resonances of P.

At least 2 advantages of this theory:

(i) It allows more general behaviour of V at infinity than the usual dilation-analytic theory (Aguilar-Combes, Hunziker, ...)

(ii) There is a complete freedom on the choice of G within any arbitrarily large compact region of $T^*\mathbb{R}^n$.

Actually, this is this last advantage (ii) that permits, thanks to convenient choices of G adapted to the underlying classical geometry, to obtain very precise results on the location of resonances, in relation with the classical dynamics (as $h \to 0$).

2) <u>The globally analytic case</u>

In a paper with A-Lahmar-Benbernou (2001), we have given a simplified version of He-Sj's theory in the case where V extends holomorphically in a whole complex sector of the form $S_\delta := \{x \in \mathbb{C}^n ;\ |\mathrm{Im}\, x| < \delta \langle \mathrm{Re}\, x \rangle\}$ and $V \to 0$ as $|x| \to \infty$, $x \in S_\delta$.

After that, a remark by J.F. Bony and L. Michel (concerning another case) has made me realize that it can be done in an even simpler way. Indeed, everything rests on a general and easy a priori estimate on $\mathbb{R}^{2n}$, first proved by me (1994), then simplified by S. Nakamura (1995) and that can be found in my book (Springer, 2002). It is the following one:

Denote by T the Bargmann-transform defined by:

(2.1) $$Tu(x,\xi;h) = c\int_{\mathbb{R}^n} e^{i(x-y)\xi/h - (x-y)^2/2h} u(y)\,dy$$

where $c \sim h^{-\frac{3n}{4}}$ is a normalization factor that makes T an isometry: $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$. One has:

<u>Th 1</u>: Let $p(x,\xi)$ be a symbol (say, bounded) holomorphic in a strip $\{|\mathrm{Im}(x,\xi)| < a\}$ $(a>0)$ and let $\psi \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$ such that $\forall\alpha$, $\partial^\alpha\psi = \mathcal{O}(1)$ uniformly and $\sup|\nabla\psi| < a$. Then, for all $u, v \in C_0^\infty(\mathbb{R}^n)$:

$$\langle e^{\psi/h} T\,\mathrm{Op}_h^W(p)u, e^{\psi/h}Tv\rangle_{L^2(\mathbb{R}^{2n})} = \langle (q(x,\xi;h) + R(h))e^{\psi/h}Tu, e^{\psi/h}Tv\rangle_{L^2(\mathbb{R}^{2n})}$$

where Op_h^W stands for the semiclassical Weyl quantization of symbols, $q(x,\xi;h) \sim q_0(x,\xi) + hq_1 + \dots$ is a symbol with

$$q_0(x,\xi) = p(x - \partial_x\psi - i\partial_\xi\psi,\ \xi - \partial_\xi\psi + i\partial_x\psi)$$

and $R(h) = \mathcal{O}(h^\infty) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$.

In the dilation-analytic case, we apply this result to the dilated operator $P_\theta = \frac{1}{(1+i\theta)^2}(-h^2\Delta) + V((1+i\theta)x)$ $(\theta>0)$ and to $\psi = -tG_1$ where $G_1 \in C_0^\infty(\mathbb{R}^{2n})$ has to be determined. Taking also $t = \theta$, a Taylor expansion with respect to t shows that we have in this case:

(2.2) $$q_0(x,\xi) = p(x,\xi) + t\nabla p\cdot\nabla G_1 - itH_p(x\cdot\xi + G_1) + \mathcal{O}(t^2)$$

Therefore, the original idea of Helffer-Sjöstrand can be applied under the condition that (1.1) is satisfied with $G = x\cdot\xi + G_1$, that is, with $G = x\cdot\xi$ since G_1 is compactly supported. (Also observe that $t\nabla_p \nabla G_1 = \mathcal{O}(t)$ for the same reason).

As an immediate consequence of (2.2), one recovers a result by Helffer-Sjöstrand saying that there is no resonances in a ~~h~~ h-independent neighborhood of any non-trapping energy level E. (Indeed, in that case one can construct $G_1 \in C_0^\infty$ such that $H_p(G_1 + x.\xi) > 0$ near $p^{-1}(E)$: see e.g. Gerard-Martinez 1988).

One can probably also recover, in a simpler way, the results of Helffer-Sjöstrand on the widths of the shape resonances.

3) <u>The distorsion analytic case</u>

Now we assume only that V is C^∞ and extends holomorphically near ∞ in a sector of the form $S_{\delta,R} = \{x \in \mathbb{C}^n ; |\mathrm{Re}\, x| > R, \ |\mathrm{Im}\, x| < \delta \langle \mathrm{Re}\, x\rangle\}$ ($\delta > 0$ small, $R > 0$ large).

Then, one can define the distorded operator $P_\theta = U_\theta P U_\theta^{-1}$

where

$$(3.1)\quad \begin{cases} U_\theta \varphi(x) = |\det dF_\theta(x)|^{1/2} \varphi(F_\theta(x)) \ ; \\ F_\theta(x) = (1 + i\theta \chi_R(x))x \ ; \\ \chi_R \in C^\infty(\mathbb{R}^n; \mathbb{R}) ; \ \mathrm{Supp}\, \chi_R \subset \{|x| > R\} \ ; \ \chi_R = 1 \text{ on } \{|x| > R+1\}. \end{cases}$$

In that case, we can apply a C^∞-version of Th 1, namely (see e.g. my book, exercises)

<u>Th2</u>. Same assumptions as Th1 except that now $p \in C^\infty(\mathbb{R}^{2n})$ is not necessarily analytic. Then, for any $t \in \mathbb{R}$ small enough, and for $u, v \in C_0^\infty(\mathbb{R}^n)$, one has,

$$\langle e^{t\psi/h} T\, Op_h^w(p) u, e^{t\psi/h} Tv \rangle_{L^2(\mathbb{R}^{2n})} = \langle (q(x,\xi;h) + R(t,h)) e^{t\psi/h} Tu, e^{t\psi/h} Tv \rangle_{L^2(\mathbb{R}^{2n})}$$

where $q \sim q_0 + hq_1 + h^2 q_2 + \dots$ as before,

$$q_0(x,\xi) = \tilde{p}(x - t\partial_x \psi - it\partial_\xi \psi, \xi - t\partial_\xi \psi + it\partial_x \psi)$$

where $\tilde{p}$ stands for an almost-analytic extension of p, and

$R(t,h) = \mathcal{O}(h^\infty) + \mathcal{O}(t^\infty e^{\delta t/h}) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$ (for some constant $\delta > 0$).

In particular, if we choose $t = Ch \log \frac{1}{h}$ ($C > 0$ arbitrarily large) we obtain as in the globally-analytic case (generalizing a result by Tang and Zworski, 2000):

Corollary (Ann. H. Poincaré 2002). If E is non trapping, there are no resonances of P in a (h-dependent) neighborhood of E of the form: $[E-\delta, E+\delta] - i[0, Ch\log\frac{1}{h}]$ ($\delta > 0$ constant, small enough; $C > 0$ constant, arbitrarily large).

(joint work with S. Fujiie and A. Lahmar-Benbernou)

One can also apply it in the case of shape resonances:

Assume $\{V \leq E\} = \{x_0\} \cup \ddot{O}^c$, with $\ddot{O}^c$ connected, $x_0 \notin \ddot{O}^c$, $V''(x_0) > 0$, E non-trapping on $T^*(\ddot{O}^c)$. (again with $V \in C^\infty(\mathbb{R}^{2n})$, analytic only for $|x| >> 1$).

Then one can adapt to the C^∞ case the constructions of Helffer-Sjöstrand, and this permits to generalize their arguments. Denoting $S_0 = d_{V-E}(x_0, \ddot{O}^c)$ the Agmon distance between x_0 and $\ddot{O}^c$, and assuming there are only a finite numbers of smooth minimal geodesics (respectively to the Agmon metric $(V-E)_+ dx^2$) between x_0 and $\mathcal{O}''^c$, our result is

Th3 (S. Fujiie, A. Lahmar-Benbernou, A.M) (in progress)

Inside any domain of the form $[E, E+Ch]-i[0, Ch\log\frac{1}{h}]$ ($C>0$ arbitrarily large) the resonances of P are of the form:

$$\rho_j(h) \sim E + e_j h + \rho_{j,1} h^{3/2} + \dots \qquad (\text{mod } \mathcal{O}(h^\infty))$$

where $e_1 < e_2 \leq \dots$ are the eigenvalues of the harmonic oscillator $-\Delta + \frac{1}{2}\langle V''(x_0)x, x\rangle$. Moreover,

$$\forall j, \quad |\mathrm{Im}\,\rho_j(h)| = \mathcal{O}(h^{-N_k} e^{-2S_0/h})$$

for some $N_k > 0$ ($h \to 0_+$), and, for $j=1$:

$$\mathrm{Im}\,\rho_1(h) = -\alpha_1 h^{1/2} e^{-2S_0/h}\left(1 + \mathcal{O}\left(\left(\frac{1}{\log\frac{1}{h}}\right)^\infty\right)\right)$$

with $\alpha_1 > 0$ constant.

4) The C^∞ case (joint work with C. Cancelier and T. Ramond, in progress)

Now, only assume that $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ and, $\forall \alpha \in \mathbb{N}^n$,

(4.1) $\quad x^\alpha \partial^\alpha V(x) \longrightarrow 0 \quad (|x| \to +\infty)$

(actually, it is enough to have $V \in C^\infty$ outside a compact set, and $V \in L^1_{loc}$, Δ-compact elsewhere).

thanks to (4.1), it is possible to define an almost-analytic extension $\tilde{V}$ of V near infinity, in such a way that

$$\bar\partial \tilde V(x) = \mathcal{O}\left(\left(\frac{|\mathrm{Im}\, x|}{|\mathrm{Re}\, x|}\right)^\infty\right) \qquad (|\mathrm{Re}\, x| >> 1)$$

(Actually, there is a sistematic way to do this that re-gives V in the case where V is analytic).

Now, for $R > 0$ large enough, let χ_R be as in (3.1), and denote by $F_{R,\theta}(x) = (1 + i\theta\chi_R(x))x$ the corresponding distorsion field.

We set:

$$\tilde P_{R,\theta} := \left(dF_{R,\theta}(x)^{-1} \cdot hD_x\right)^2 + \tilde V(F_{R,\theta}(x))$$

(formally, it is the distorded Hamiltonian), and we define the resonances as the elements of the set:

$$\Gamma_\theta := \bigcap_{R' >> 1} \overline{\bigcup_{R > R'} \sigma_{pp}(\tilde P_{R,\theta})}$$

where $\sigma_{pp}(\tilde P_{R,\theta})$ stands for the pure point spectrum of $\tilde P_{R,\theta}$ (note that it essential spectrum is: $\sigma_{ess}(\tilde P_{R,\theta}) = (1+i\theta)^{-2}\mathbb{R}_+$)

For $\rho \in \Gamma_\theta$, we also define its multiplicity as follows:

$$N(\rho) := \lim_{\varepsilon \to 0_+} \limsup_{R \to +\infty} \#\left(B(\rho,\varepsilon) \cap \sigma_{pp}(\tilde P_{R,\theta})\right)$$

(counted according to the multiplicity)

where $B(\rho, \varepsilon)$ is the ball of radius ε centered at ρ.

Of course, these definitions may depend on θ and on the choice of χ_R. However, ~~~~ one can easily show that:

Prop 1: $\forall \theta > 0$ small,

$$\Gamma_\theta \cap \{\mathrm{Re}\,\rho < 0\} = \sigma_{pp}(P), \text{ with same multiplicity.}$$

Moreover, using again Th2, we have:

Prop 2: If $E > 0$ is non-trapping, then for any $C > 0$ and for $\theta = Ch\log\frac{1}{h}$, one has

$$\Gamma_\theta \cap \{\mathrm{Re}\,\rho \in [E-\delta, E+\delta]\} = \emptyset$$

The problem of the Shape resonances can probably be treated too, and should give the same result as in Th3. In particular, the corresponding resonances ρ_j should be independent of θ and χ_R up to $\mathcal{O}(h^\infty)$, while their widths are independent of θ and χ_R up to $\mathcal{O}(h^{-N_k}e^{-2S_0/h})$. Moreover, the width of ρ_1 should be independent of θ and χ_R modulo $\mathcal{O}\left(h^{1/2}\left(\frac{1}{\log\frac{1}{h}}\right)^\infty e^{-2S_0/h}\right)$.