Pseudospectra and Applications

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Highly non-self-adjoint matrices and differential operators may have approximate eigenvalues which are not close to the spectrum of the operators. We will describe the relevance of semi-classical analysis to such 'pseudospectral' phenomena.

Bases

We say that $\{v_n\}$ forms a basis if every $f \in \mathcal{H}$ has a norm convergent expansion

$$f = \sum_{n=1}^{\infty} P_n f = \sum_{n=1}^{\infty} \langle f, \widehat{v}_n \rangle v_n. \qquad (1)$$

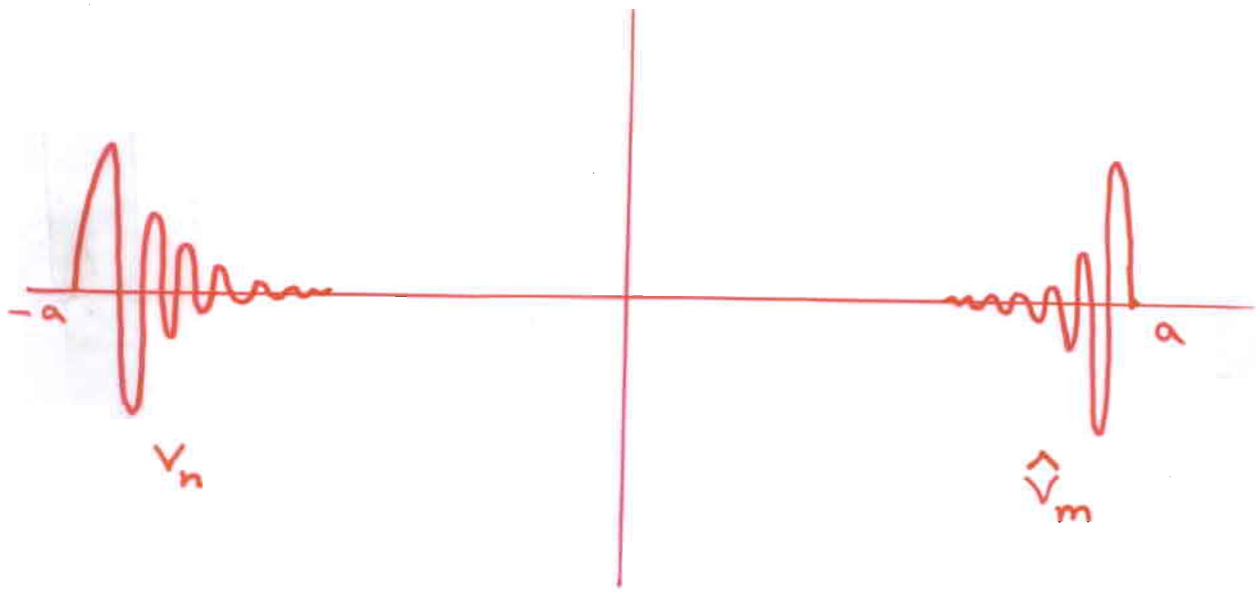
By the uniform boundedness theorem this implies that there exists a constant k such that

$$||P_n|| = ||v_n|| \, ||\widehat{v}_n|| \leq k$$

for all n.

$$Hf = f'' + f'$$

on $L^2(-a,a)$, $f(-a) = f(a) = 0$



Abel-Type Expansions

Even if $\{v_n\}$ do not form a basis it has been shown by Lidskii, Agranovich, Katznelson and others that for nsa elliptic differential operators one often has Abel-type convergence.

This involves some adaptation of the formula

$$f = \lim_{t \to 0+} \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n^{\beta} t} \langle f, \widehat{v}_n \rangle v_n$$

where $\beta > 0$, depending on the particular features of the operator involved.

The set of eigenvalues of A is invariant under similarity transformations of the form $A \to TAT^{-1}$. Another such invariant is the asymptotic behaviour of $||P_n||$ as $n \to \infty$, where the eigenvalues are suitably ordered.

We say that the spectral projections are polynomially bounded if there exist constants c, α such that

$$||P_n|| \leq cn^{\alpha}$$

for all n.

Approximate Eigenvalues

One needs to be careful not to suppose that if $Af \simeq zf$ then z is close to the spectrum of A. One always has a bound of the type

$$\|(zI - A)^{-1}\| \geq \text{dist}(z, \text{Spec}(A))^{-1} \qquad (2)$$

The existence of a reverse inequality of the form

$$\|(zI - A)^{-1}\| \leq k \, \text{dist}(z, \text{Spec}(A))^{-1} \qquad (3)$$

is a definite assumption.

Lemma 1 *If (3) and*

$$\|Af - zf\| < \varepsilon \|f\|$$

for some non-zero $f \in \text{Dom}(A)$ *then*

$$\text{dist}(z, \text{Spec}(A)) < k\varepsilon.$$

Pseudospectrum

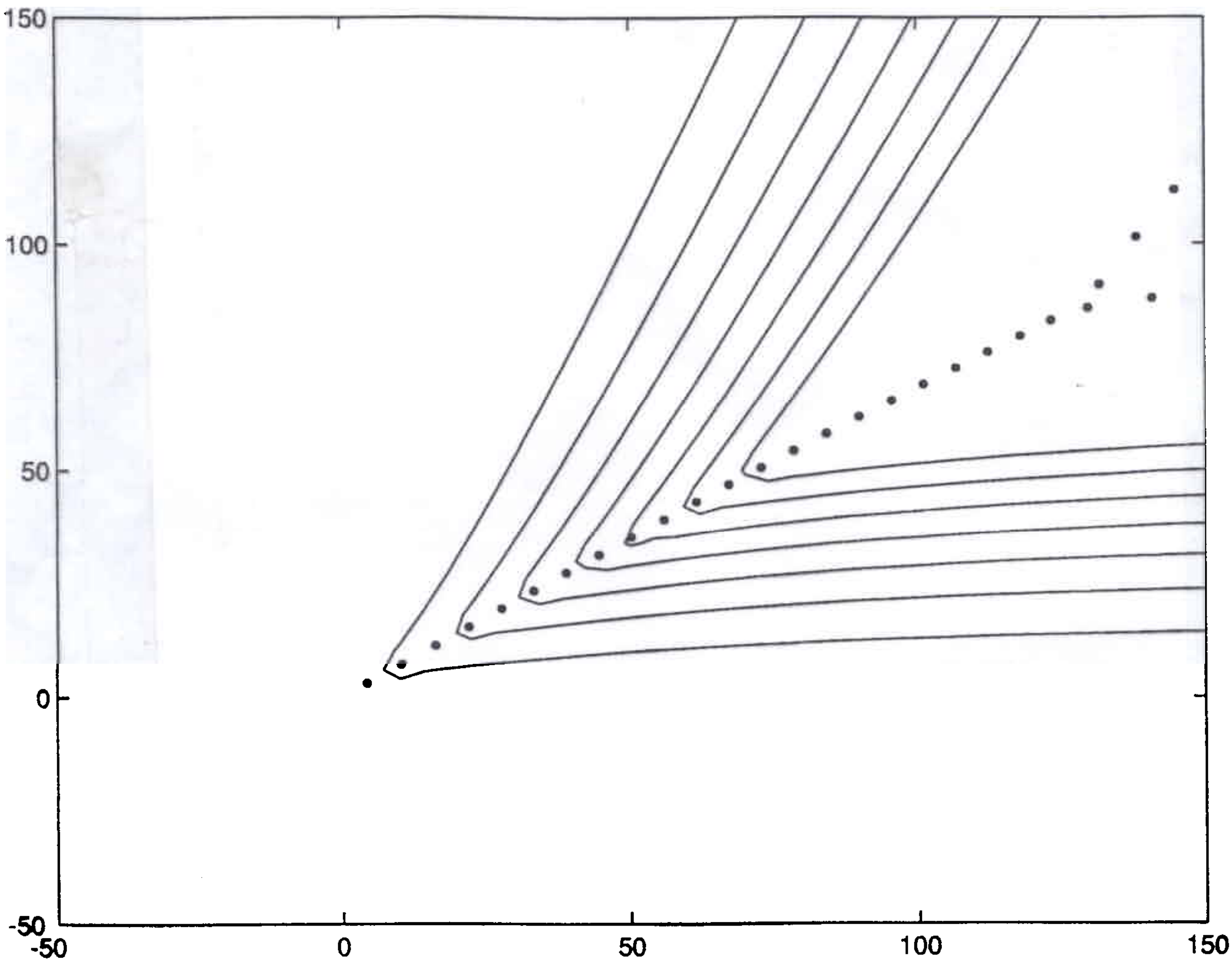
We define

$$\mathrm{Spec}_\varepsilon(A) = \{z \in \mathbf{C} : \|(z-A)^{-1}\| > \varepsilon^{-1}.$$

Theorem 2 *Given* $\varepsilon > 0$ *one has*

$$\mathrm{Spec}_\varepsilon(A) = \mathrm{cl}\left\{ \bigcup_{\|B\| \le \varepsilon} \mathrm{Spec}(A+B) \right\}$$

where cl *denotes the closure.*


150
100
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The Harmonic Oscillator

We describe the harmonic oscillator with a complex coupling constant

$$Hf(x) = -f''(x) + cx^2 f(x)$$

acting in $L^2(\mathbf{R})$. We assume that Im $(c) > 0$.

If one initially defines H on Schwartz space $\mathcal{S}$, then the closure has compact resolvent. One may prove by direct computation or analytic continuation from the real case that the operator has eigenvalues $\lambda_n = c^{1/2}(2n+1)$ where $n = 0, 1, \ldots$, the corresponding eigenfunctions being Hermite functions which all lie in $\mathcal{S}$ and form a complete set in $L^2(\mathbf{R})$.

Theorem 3 *If* $z = re^{i\theta}$ *where* $0 < \theta < \arg(c)$ *then*

$$\lim_{r \to \infty} \|(zI - H)^{-1}\| = \infty.$$

On the other hand if $\arg(c) < \theta < 2\pi$ *then the value of the limit is* 0.

Semi-Classical Limit

Consider the operator

$$H_h f(x) = -h^2 f''(x) + V(x) f(x)$$

where $h > 0$ and V is any smooth complex-valued potential on $\mathbf{R}$. We do not need to specify the full domain of H_h but only to suppose that it contains $C_c^\infty(\mathbf{R})$.

The following theorem shows that as $h \to 0$ the pseudospectrum of H_h expands to fill up the range of the complex classical Hamiltonian $H(p,q) = p^2 + V(q)$, p,q being restricted to taking real values.

Theorem 4 *Let* $z = p^2 + V(q)$ *for some* $p, q \in \mathbf{R}$ *and suppose that* Im $(V'(q)) \neq 0$. *For all* $m > 0$ *one has*

$$\|(zI - H_h)^{-1}\| \geq h^{-m}$$

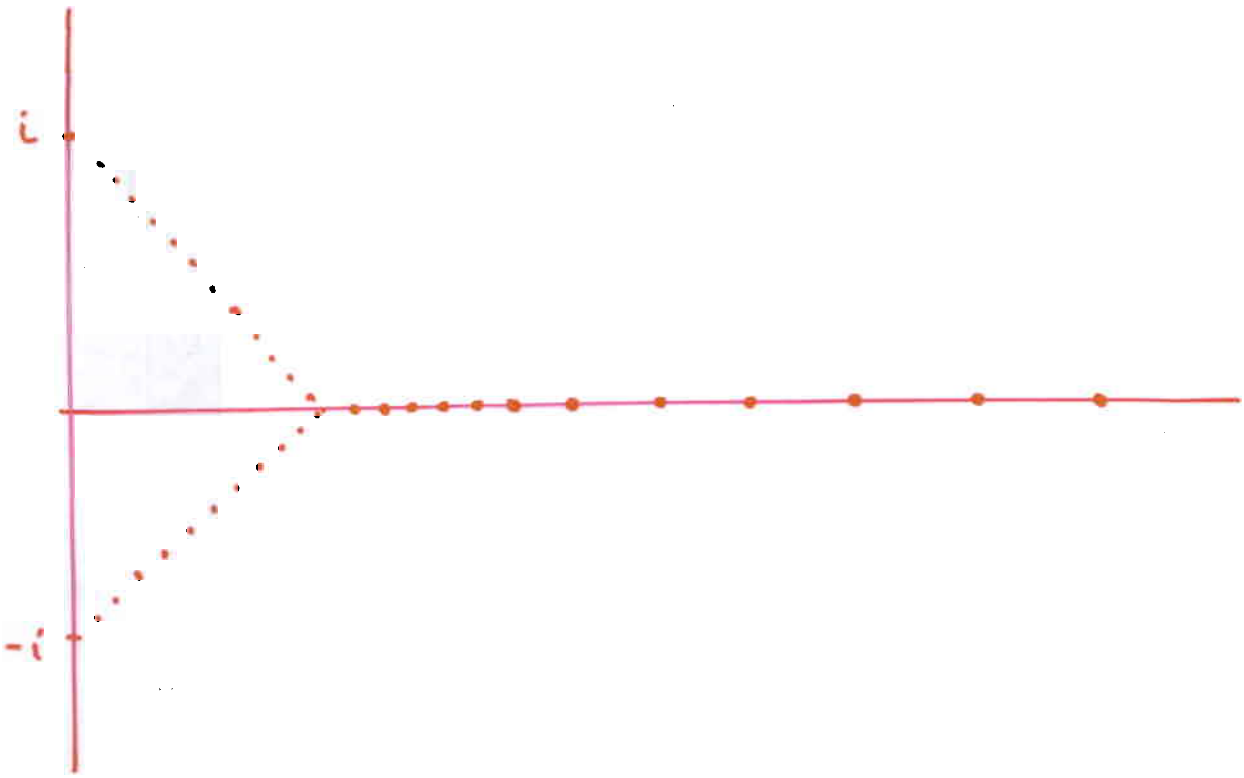
for all sufficiently small $h > 0$.

SHKALIKOV

$$Hf = -h^2 f'' + iVf$$

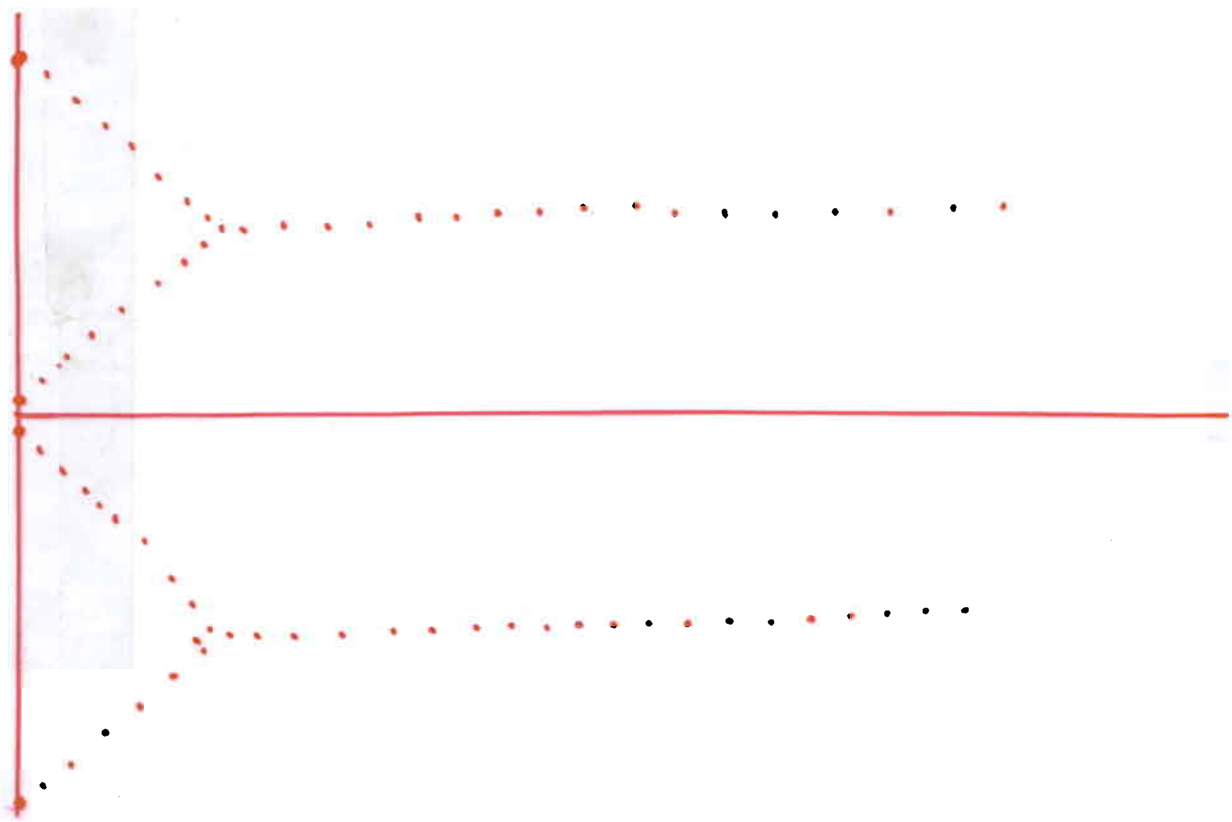
in $L^2(-1,1)$ $\qquad f(-1) = f(1) = 0$

$$V(x) = x$$



REDPARTH

$$V(x) = \begin{cases} x + \delta & \text{if } x > 0 \\ x - \delta & \text{if } x < 0 \end{cases}$$



Redparth's Theorem

Now assume that $H_h = -h^2\Delta + V$ on $L^2(\Omega)$, where $h > 0$, V is a complex-valued continuous function on the closure of the bounded region $\Omega \subseteq \mathbf{R}^n$ and we impose Dirichlet boundary conditions.

Theorem 5 *Put*

$$\Phi = \overline{\mathrm{Ran}(V)} + [0, \infty)$$

If $\lambda \in \Phi$ *then*

$$\|(\lambda I - H_h)^{-1}\| \to \infty$$

as $h \to 0$. *If, however,* $\lambda \notin \Phi$ *then*

$$\limsup_{h \to 0} \|(\lambda I - H_h)^{-1}\| \le \mathrm{dist}(\lambda, \Phi)^{-1}.$$

Quasi-Orthogonal Polynomials

We consider polynomials p_n which are orthogonal with respect to a complex weight σ on $[0, \infty)$ in the following sense. We suppose that p_n is of degree n and

$$\int_0^\infty p_m(x)p_n(x)\sigma(x)^2 \, \mathrm{d}x = \delta_{m,n}$$

for all non-negative integers m, n.

As an example consider the non-self-adjoint harmonic oscillator

$$(Hf)(x) = -f''(x) + z^4x^2f(x)$$

acting in $L^2(\mathbf{R})$ for some complex z. In this situation the relevant weight is

$$\sigma(x) = \mathrm{e}^{-z^2x^2/2}$$

$$H = z^2 \left(z^{-2} P^2 + z^2 Q^2 \right)$$

$$\varphi_n(x) = H_n(zx)\, e^{-z^2 x^2/2}$$

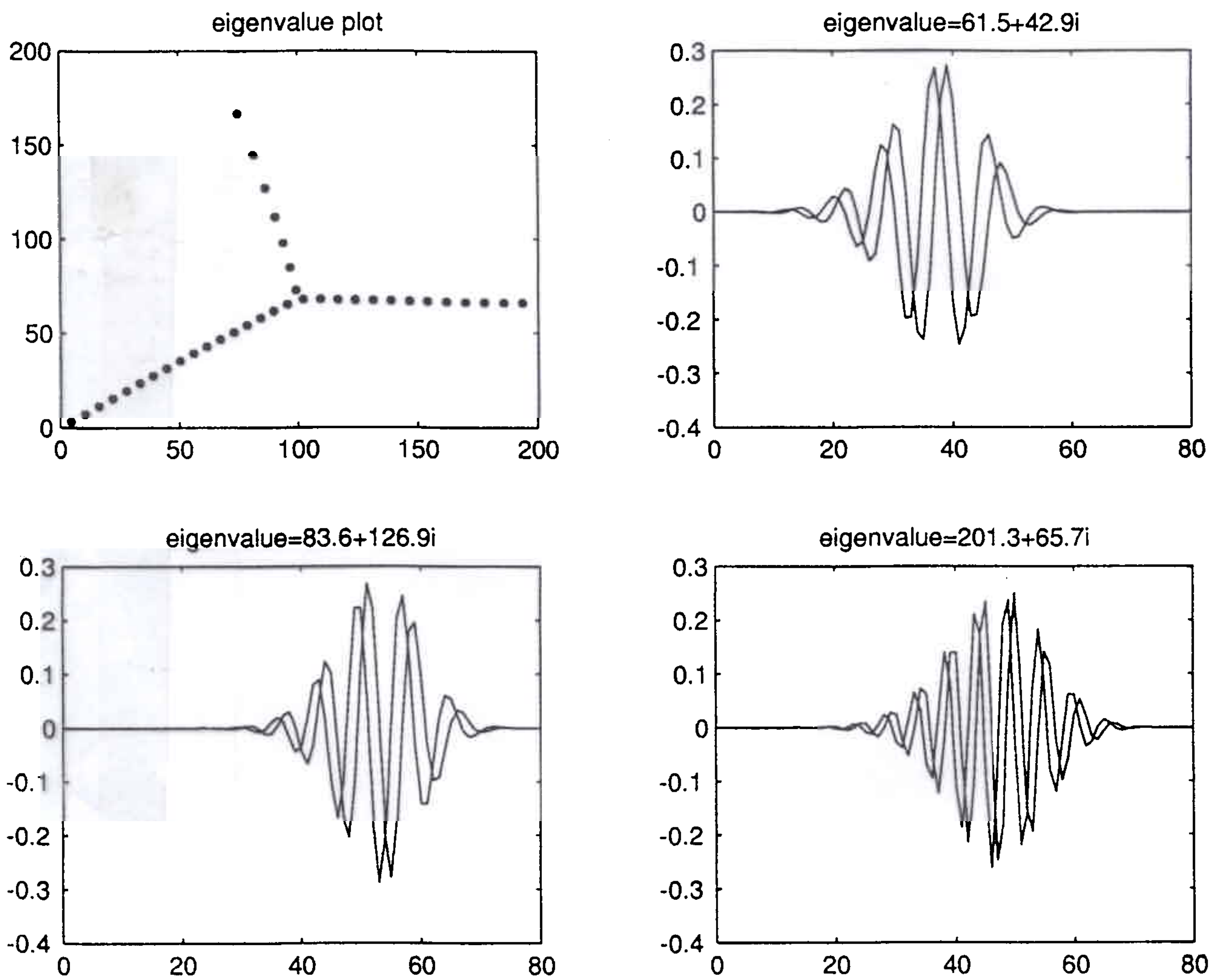
$$\lambda_n = z^2 (2n+1) \qquad n = 0, 1, 2, \ldots$$

Our goal is to obtain bounds on the quantities

$$N_n = \int_0^\infty |p_n(x)\sigma(x)|^2 \, \mathrm{d}x.$$

In the case of the harmonic oscillator. these are the norms of the spectral projections.

The asymptotic behaviour as $n \to \infty$ should be regarded as a question in semi-classical asymptotics.


eigenvalue plot
eigenvalue=61.5+42.9i
eigenvalue=83.6+126.9i
eigenvalue=201.3+65.7i

We assume that

$$|\sigma(\mathrm{e}^{i\theta}r)| \geq c_\theta \sigma(s_\theta r) \tag{4}$$

for all $|\theta| < \alpha$ and all $r > 0$, where $c_\theta > 0$ and $0 < s_\theta < 1$.

Theorem 6 *Under the assumption (4) we have*

$$N_{n,z} \geq c_\theta^2 s_\theta^{-2n-1}$$

provided $z = r\mathrm{e}^{i\theta}$ *and* $|\theta| < \alpha$.

Let

$$\sigma(z) = z^{\gamma/2} e^{-z^\beta}$$

where $\gamma > -1$ and $\beta > 0$. If $r > 0$ and $|\theta| < \pi/(2\beta)$ then

$$|\sigma(re^{i\theta})| = c_\theta \sigma(s_\theta r)$$

where $s_\theta = \{\cos(\theta\beta)\}^{1/\beta}$ and $c_\theta = s_\theta^{-\gamma/2}$.

After replacing $(0, \infty)$ by $(-\infty, \infty)$, the particular choice $\gamma = 0$ and $\beta = 2$ leads one to the study of the Hermite polynomials with a complex scaling, which is relevant to the non-self-adjoint harmonic oscillator. The choice $\beta = 1$ leads to the Laguerre polynomials L_n^γ.

Theorem 7 *If*

$$\sigma(x) = \exp\{-\sum_{j=1}^{n} c_j x^j\}$$

for all $x \in (0, \infty)$, *where* $c_j \in \mathbf{R}$ *for all* j *and* $c_n > 0$, *then* σ *satisfies (4) provided* $|\theta| < \pi/(2n)$.

When considering the Hermite polynomials we restrict attention to the case of even integers; the treatment of odd integers is very similar.

Theorem 9 *If* $\sigma(x) = \mathrm{e}^{-x^2/2}$ *and* $z = \mathrm{e}^{i\theta}$ *where* $|\theta| < \pi/4$*, and put* $s_\theta = (\cos(2\theta))^{1/2}$*. Then*

$$N_{2n} \leq \pi(n+1)^{1/2} 2^{4n+2} s_\theta^{-4n-1}.$$

for all non-negative integers n*.*

Theorem 10 *There exists an explicit* $t_z > 0$ *such that the spectral expansion*

$$\mathrm{e}^{-Ht} = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n t} P_n$$

of the nsa harmonic oscillator is norm convergent if $t > t_z$ *and is norm divergent if* $0 \le t < t_z$.

Numerical Results

The second column lists the constants $s_\theta^{-2} = \sec(2\theta)$ associated with the lower bound of Theorem 6. The the third column provides the numerical results. The fourth column lists the constants $4s_\theta^{-2} = 4\sec(2\theta)$ associated with the upper bound of Theorem 9.

θ	s_θ^{-2}	$\rho_{100}(\theta)$	$4s_\theta^{-2}$
0	1	1	4
0.025	1.012	1.165	4.050
0.05	1.051	1.369	4.206
0.1	1.236	1.953	4.945
0.15	1.701	3.062	6.806
0.20	3.236	6.282	12.945