MSRI : May 8, '03 P. Deift :

Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space.

Joint work with X. Zhou

I will be addressing the question of the $\boxed{\text{long-time behavior}}$ of solutions $q(t) = q(x,t)$ of the defocusing NLS equation on the line

(1) $$i q_t + q_{xx} - 2|q|^2 q = 0$$

$$q(x, t=0) = q_0(x) \to 0 \quad \text{as } |x| \to \infty.$$

I am particularly interested in the case where q_0 has $\boxed{\text{"minimal"}}$ smoothness and decay.

Outline of the talk:

$\boxed{\text{A.}}$ What has been done $\boxed{\text{up till now}}$.
Why we are interested in "minimal" smoothness/decay
Use the $\boxed{\text{oscillations}}$

$\boxed{\text{B.}}$ Statement of the result

$\boxed{\text{C.}}$ Some $\boxed{\text{key ideas}}$ from the $\boxed{\text{proof}}$

$\boxed{\text{D.}}$ $\boxed{\text{Extended}}$ result

A: Results

~~Zakharov - Manakov (76) deduced the asymptotic form~~

Recall that the NLS equation is equivalent to an $\boxed{\text{Bispectral deformation}}$ of the operator

$$L(q) = -i\sigma\left(\partial_x - \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix}\right), \qquad \sigma = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Scattering / inv. scatt. theory establishes a $\boxed{\text{bijection}}$

$$r = \mathcal{R}(q)$$

$$q = q(x) \to r = r(z) = \text{reflection coefficient}, \quad \|r\|_\infty < 1$$

and the key fact $\boxed{(\text{Zakharov / Shabat '72})}$ is that if $q = q(t)$ (r(t) = $\mathcal{R}(q(t))$ evolves simply) solves NLS then ~~$r(t)(z) = e^{-itz^2} r(t=0)(z)$~~

$$\boxed{r(t)(z) = e^{-itz^2} \cancel{r(t=0)(z)}\, r_0(z)}$$

where $r_0 = \mathcal{R}(q(t=0))$

In $\boxed{1976}$, Zakharov & Manakov derived the asymptotic form as $t \to \infty$ in terms of $r = \mathcal{R}(q_0)$

$$q(x,t) \sim q_{as}(x,t)$$

$$= t^{-1/2}\alpha(z_0)\, e^{ix^2/4t - i\nu(z_0)\log t}$$

where $z_0 = x/2t$, $\nu(z) = -\frac{1}{2\pi}\log(1 - |r(z)|^2)$

$x \sim \mathcal{R}(q_0)$

can ~~obtain~~

$$|\alpha(z)| = (\nu(z)/2)^{\frac{1}{2}}$$

$$\arg \alpha(z) = \frac{1}{\pi} \int_{-\infty}^{z} \log(z-s)\, d\log(1-|r(s)|^2)$$
$$+ \frac{\pi}{4} + \arg \Gamma(i\nu(z)) + \arg r(z)$$

Notice that the form of the solution $\boxed{\text{differs}}$ from the form of the solution of the $\boxed{\text{linear}}$ ~~free~~ Schrodinger equation

$$iq_t^F + q_{xx}^F = 0$$

for which

$$q^F(x,t) \sim \frac{1}{\sqrt{t}}\, \alpha^F(z_0)\, e^{ix^2/4t}$$

Solutions of NLS $\boxed{\text{never become free}}$.

Zakharov / Manakov's derivation was $\boxed{\text{not}}$ rigorous. In particular they provide $\boxed{\text{no estimates}}$ for the error. In 1993, using the steepest-descent method for Riemann-Hilbert problems (more later) $\boxed{\text{Deift - Its - Zhou}}$ showed that

$$\boxed{q(x,t) = q_{as}(x,t) + O\left(\frac{\log t}{t}\right)} \quad \text{as } t \to \infty$$

uniform in x

In their proof they needed to assume that q_0 had some $\boxed{\text{high orders of decay \& smoothness}}$

In $\boxed{\text{recent work}}$ of Deift and Zhou on $\boxed{\text{perturbations}}$ of NLS

(2)
$$\left.\begin{aligned} & i q^\varepsilon_t + q^\varepsilon_{xx} - 2|q^\varepsilon|^2 q^\varepsilon - \varepsilon |q|^\ell q = 0 \\ & q^\varepsilon(x, t=0) = q_0 \end{aligned}\right\}$$

$(\ell > 2)$, ~~that all one needs to know~~ the authors show that as $t \to \infty$ solution $q^\varepsilon(t)$ look like solutions $q(t)$ of NLS

$$\boxed{q^\varepsilon(t; q_0) \sim q(t; \tilde{q}_0)}$$

for some $\tilde{q}_0 = \tilde{q}_0(q_0)$. As the authors consider solutions $q(t)$ in the $\boxed{\text{weighted Sobolev space}}$ ~~Banach space~~

$$\boxed{H^{1,1} = \{q \in L^2 : xq,\ q' \in L^2\}}$$

it becomes $\boxed{\text{important}}$ to know how to evaluate the asymptotics of NLS with $\boxed{\text{initial data in } H^{1,1}}$. Of course this problem is also of independent interest.

Whereas the analysis in D-I-Z uses $\boxed{\text{absolute estimates}}$ we now have to use $\boxed{\text{cancellations from oscillations}}$.

(B) To see $\boxed{\text{what we might hope for}}$, note that if we consider the solution q^F of the linear Schrod. equation with $q_0 \in H^{1,1}$, then a simple stat. phase calc. shows that

$$q^F(x,t) = \frac{\hat{q}^F(z_0)}{t^{1/2}} e^{ix^2/4t} + O\left(\boxed{\frac{1}{t^{3/4}}}\right)$$

Our result is the following:

Thm (Zhou + D)

Let $q(t)$, $t \geq 0$, solve (1) with $q_0 = q(t=0) \in H^{1,1}$. Fix $0 < \kappa < \frac{1}{4}$. Then as $t \to \infty$

$$q(x,t) = q_{as}(x,t) + O\left(\frac{1}{t^{\frac{1}{2}+\kappa}}\right)$$

where q_{as} is given in terms of $r = \mathcal{R}(q_0)$ as above. The error term $O\left(\frac{1}{t^{\frac{1}{2}+\kappa}}\right)$ is uniform $\forall x \in \mathbb{R}$.

So the result is $\boxed{\text{optimal}}$ in the sense that the error term is arbitrarily close to that in the free case.

(c) Some key ideas of the proof:

The NLS equation with initial data q_0 is solved via the following $\boxed{\text{Riemann - Hilbert Problem (RHP)}}$:

Let $r = \mathcal{R}(q_0)$ = reflection coefficient for q_0 (and fix x, t): let

$$m = m(z) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

satisfy

- $m = m(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
- $m_+(z) = m_-(z)\, v_\theta(z), \quad z \in \mathbb{R}$

where $m_\pm(z) = \lim_{\varepsilon \downarrow 0} m(z \pm i\varepsilon)$

$$v_\theta(z) = \begin{pmatrix} 1 - |r(z)|^2 & r e^{i\theta} \\ -\bar{r} e^{-i\theta} & 1 \end{pmatrix}, \quad \theta = xz - tz^2$$

- $m(z) \to I$ as $z \to \infty$

(7)

Then if ~~expand~~ we expand.

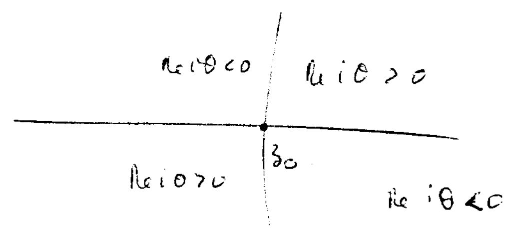
$$m = I + \frac{m_1(x,t)}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty,$$

$$q(x,t) = -i\,(m_1(x,t))_{12}$$

So, in order to evaluate $q(x,t)$ as $t \to \infty$, we evaluate the solution of this ~~oscillatory~~ RHP with highly oscillatory parameters $re^{i\theta} = re^{i(xz - tz^2)}$ as $t \to \infty$

For scalar integrals, (which arise, e.g., in the solution of linear NLS) such problems are analyzed using the (stationary phase method)? There is an analog of this method for the non-commutative RHP as above due to (Zhou + D), and this is what we use. This method ~~[illegible]~~ has been developed much further and now has applications to many diff. parts of mathematics & physics.

The method is basically $\boxed{\text{applied as follows}}$: the $\boxed{\text{key idea}}$ is to take advantage of the signature table for $\operatorname{Re} i\theta$



$z_0 = \frac{x}{4t}$

= stationary phase pt. for θ

$= xz - tz^2$

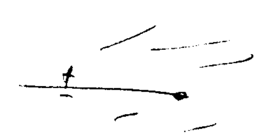
This is done by noting that v_θ has an $\boxed{\text{upper/lower}}$ and a $\boxed{\text{lower/upper factorization}}$

(3)
$$v_\theta = \begin{pmatrix} 1-|r|^2 & re^{i\theta} \\ -\bar{r}e^{-i\theta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & re^{i\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\bar{r}e^{-i\theta} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{-\bar{r}e^{-i\theta}}{1-|r|^2} & 1 \end{pmatrix}\begin{pmatrix} 1-|r|^2 & 0 \\ 0 & (1-|r|^2)^{-1} \end{pmatrix}\begin{pmatrix} 1 & \frac{re^{i\theta}}{1-|r|^2} \\ 0 & 1 \end{pmatrix}$$

The diagonal term can be $\boxed{\text{removed by}}$ $\boxed{\text{conjugating}}$ v_θ by the solution $\delta = \delta(z)$ of the following $\boxed{\text{scalar RHP}}$

- $\delta(z)$ analytic in $\mathbb{C} \setminus (-\infty, z_0]$
- $\delta_+(z) = \delta_-(z)\,(1-|r(z)|^2)$, $-\infty < z < z_0$
- $\delta(z) \to 1$ as $z \to \infty$



Then if we set

$$\boxed{\check v_\theta = \delta_-^{\sigma_3}\, v_\theta\, \delta_+^{-\sigma_3}}$$

$\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\delta_\pm^{\sigma_3} = \begin{pmatrix}\delta_\pm & 0\\ 0 & \delta_\pm^{-1}\end{pmatrix}$

$\delta_+ = \delta_-$ for $z > z_0$.

we obtain for $z > z_0$

$$\check v_\theta = \begin{pmatrix}1 & re^{i\theta}\delta^2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\bar r e^{-i\theta}\delta^{-2} & 1\end{pmatrix}$$

and for $z < z_0$

$$= \begin{pmatrix}1 & 0\\ \frac{-\bar r e^{-i\theta}\delta_-^{-2}}{1-|r|^2} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{r e^{i\theta}}{1-|r|^2}\delta_+^2\\ 0 & 1\end{pmatrix}.$$

and

$$\check m(z) \equiv m\,\delta^{-\sigma_3} = m\begin{pmatrix}\delta^{-1} & 0\\ 0 & \delta\end{pmatrix}$$

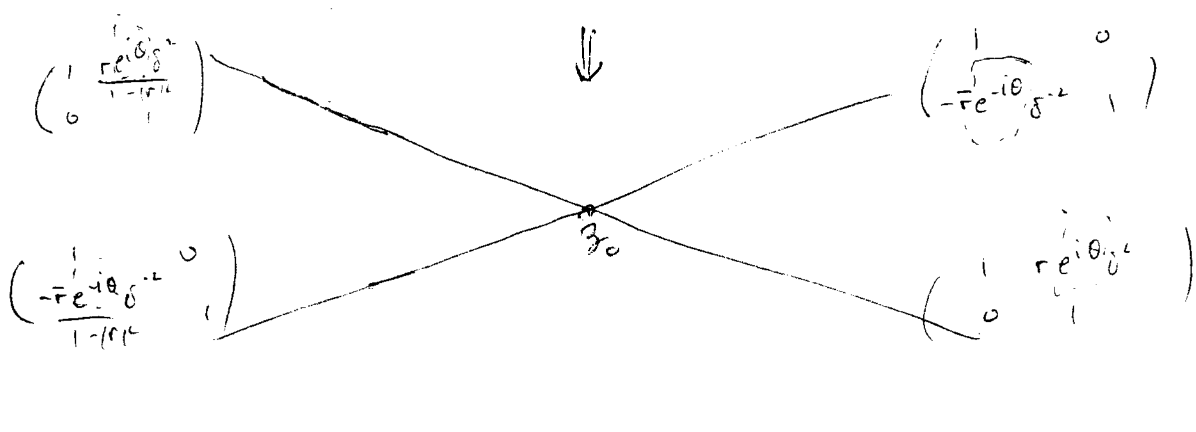
solves ~~[struck out]~~ a similar RHP to m but with

v_θ replaced by $\check v_\theta$

$$\check m_+ = \check m_-\begin{pmatrix}1 & 0\\ \frac{-\bar r e^{-i\theta}\delta_-^{-2}}{1-|r|^2} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{re^{i\theta}\delta_+^2}{1-|r|^2}\\ 0 & 1\end{pmatrix} \qquad \check m_+ = \check m_-\begin{pmatrix}1 & re^{i\theta}\delta^2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\bar r e^{-i\theta}\delta^{-2} & 1\end{pmatrix}$$

z_0

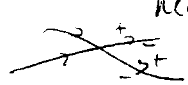
$\Downarrow$



Noticing that ~~[illegible]~~ the exponential factors are all exponentially decreasing in their respective sectors, we see that the RHP localizes to a ~~nbd~~ nbhood of z_0 & it turns out that this local RHP can be solved explicitly giving the ZM solution + error terms.

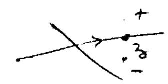
The difficulties clearly arise from the fact that r, F, $F/(1-|r|^2)$, $\frac{F}{1-|r|}$ are not analytic & so they have to be approximated appropriately and the real point of this talk is that these kind of approximations can be done effectively if we just assume that $q_0 \in H^{1,1}$, or equivalently by the scattering map, $r \in H^{1,1}$. But it is a delicate business!

I want to give some idea of the kind of estimates that we need. Recall that RHP's on a contour Σ



are ($\Sigma = \mathbb{R}$ or with ✗ in the above cases) equivalent to a set of singular integral equations on Σ. The singular operators in these equations are

The $\boxed{\text{Cauchy operator}}$

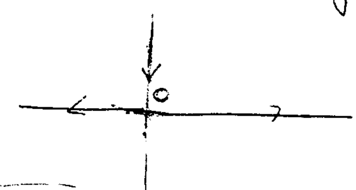


$$(C_\Sigma^\pm f)(z) = \lim_{z' \to z\pm} \int_\Sigma \frac{f(s)}{s - z'} \frac{ds}{2\pi i}$$

$C_\Sigma^\pm$ have the important property that $C_\Sigma^+ - C_\Sigma^- = 1$.

$\boxed{\text{One}}$ of the $\boxed{\text{key estimates}}$ is the following: let

— Γ be the contour



Suppose $f \in H^{1,0}$ on $(0,\infty)$, & $f \equiv 0$ on $\Gamma \setminus (0,\infty)$. Then for any $2 \le p < \infty$ & $\forall t \ge 0$

(4) $$\left\| C_\Gamma^\pm \, \delta^{\pm 2} f \, e^{\mp i t \diamond^2} \right\|_{L^p(\Gamma)} \le \frac{c}{(1+t)^{1/2p}} \, \frac{\|f\|_{H^{1,0}(0,\infty)}}{1-\rho}$$

$$\rho := \sup_{s \in \mathbb{R}} |r(s)|$$

What is $\boxed{\text{subtle}}$ about estimates is that on $\mathbb{R}_+$

$$\underbrace{C_\Gamma^+ \, \delta^2 f e^{-it\diamond^2}}_{\text{decays in } t} - C_\Gamma^- \, \delta^2 f e^{-it\diamond^2} = \delta^2 f e^{itz^2}$$

clearly does not decay

$\therefore$ does not decay

So the signs have to match $C^+ \cdots e^-$ or $C^- \cdots e^+$ to make the estimate work

Insert from p 12+, 12++

There are $\boxed{\text{many other places}}$ in the proof where a delicate cancellation is needed, but I do not have the time to describe them.

(D) Extended results

It turns out that the estimates in the theorem only depend on the $H^{0,1}$ norm of q_0,

$$\|q_0\|^2_{H^{0,1}} \equiv \int (1+x^2)|q_0(x)|^2 < \infty .$$

But ~~it is not clear~~ what do we mean by a solution of NLS with initial data in $\boxed{H^{0,1}}$? It turns out that ~~it is easy~~ using Strichartz-type estimates we can show the following:

Let $q_0 \in L^2$. Then $\exists$ a unique solution of NLS with $q \in C(\mathbb{R}_+, L^2(dx)) \cap (L^\infty(dx) \otimes L^4_{loc}(dt))$

$$\left(\int_0^T \|q(\cdot,t)\|^4_{L^\infty(dx)} dt \right)^{\frac{1}{4}} \text{ for all } T<$$

and $q(t=0) = q_0$.

Moreover, if $q_0 \in H^{0,1}$, then the solution $q(t)$

Insert on p. 12

Another type of cancellation that is needed can be seen from the following:

(corresponding to the U/L factorization in (3) above at $t=0$ ($\theta = xz - tz^2 \big|_{t=0} = xz$)

$$v_x = \begin{pmatrix} 1 & re^{izx} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{r}e^{-izx} & 1 \end{pmatrix} = (v_x^-)^{-1} v_x^+$$

$$v_x^+ - I = \underbrace{\begin{pmatrix} 0 & 0 \\ -\bar{r}e^{-izx} & 0 \end{pmatrix}}_{\text{strictly lower triangular}}, \quad v_x^- - I = \underbrace{\begin{pmatrix} 0 & -re^{izx} \\ 0 & 0 \end{pmatrix}}_{\text{strictly upper triangul}}$$

one has analogues to (4) above

(5) $$\| C^+ (I - v_x^-) \|_{L^2}, \quad \| C^- (I - v_x^+) \|_{L^2} \le \frac{1}{(1+x^2)^{1/2}} \| r \|_{H^1}$$

for $x \le 0$,

In analyzing the inverse problem one needs to show along the way, in particular that for $x \le 0$

(6) $$\left| \int_{\mathbb{R}} \left[C^+(I - v_x^-) + C^-(v_x^+ - I) \right](z) \, (v_x^+ - v_x^-)(z) \, dz \right| \le \frac{c}{1+x^2}$$

But using the fact that

- $v_x^+ - I$, $v_x^- - I$ are strictly lower / upper triangular, resp

The integral can be rewritten in the form

$$\int_{\mathbb{R}} \left[\left(C^+ (I - v_x^-) \right)(v_x^+ - I) + \left(C^- (v_x^+ - I) \right)(I - v_x^-) \right] dz$$

and using the facts that

- $C^+ - C^- = 1$ and $C^+ C^- = C^- C^+ = 0$

the integral can further be rewritten as

$$\int_{\mathbb{R}} \left[-\left(C^+ (I - v_x^-) \right)\left(C^- (v_x^+ - I) \right) + \left(C^- (v_x^+ - I) \right)\left(C^+ (I - v_x^-) \right) \right]$$

and the desired estimate (6) now follows from (5).

So the estimates need things to be done just right!

constructed via the RHP, is the (unique) Strichartz solution and moreover, have the asymptotic form given in Theorem 1 ~~[illegible]~~ above.