Upper bounds on the number of the scattering poles

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Classical Resonances

Let P be a compactly supported perturbation of the Laplacian, $n \geq 2$, odd, such that $P = -\Delta$ outside $B(0, R)$. Assume the "black-box" conditions.

Typical example:

$$P = -\Delta_g + V(x)$$

with Dirichlet or Neumann B.C. on the boundary of the obstacle $\mathcal{O} \subset B(0, R)$ (can be empty).

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Semi-classical Resonances

$P(h) = -h^2\Delta$ outside $B(0, R)$. S.C. Resonances: poles of

$$\chi(P(h) - z)^{-1}\chi$$

from $\Im z > 0$ to a neighborhood of $E_0 > 0$ in $\mathbf{C}$. Scattering matrix $S(z)$.

Reference Operator $P^{\#}$ (or $P^{\#}(h)$) on the "perturbed torus" containing $B(0,R)$.

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Melrose, Sjöstrand, Zworski, Vodev, etc.:

$$\begin{aligned} N(r) &\le C_n(r+1)^n, && \text{if } n^{\#} = n, \\ N(r) &\le C\Phi(Cr), && \text{otherwise,} \end{aligned}$$

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In S.C. case: $N(\Omega) = O(h^{-n^{\#}})$, where Ω is a small neighborhood of $E_0 > 0$.

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- Sharp estimates in various neighborhoods of $\mathbf{R}$?

$O(|\lambda|^{-N})$ **(respectively $O(h^N)$) close to the real axis**

If $P = -\Delta_g + V$ in $\mathbf{R}^n$, then for some $N \gg 1$, [S]

$$\#\mathrm{Res}(P) \cap \left\{1 \leq |\lambda|;\ -\Im\lambda \leq C|\lambda|^{-N}\right\} \leq C_n r^n(1 + o(1))$$

with

$$C_n = 2(2\pi)^{-n}\mathrm{vol}(\mathcal{T}),$$

$\mathcal{T}$ being the *trapped set* in $\{p_0(x,\xi) \leq 1\}$. In a presence of an obstacle, $N = \infty$.

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In the S.C. case,

$$\#\mathrm{Res}(P(h)) \cap \{0 < a \leq \Re z \leq b;\ 0 \leq -\Im z \leq Ch^{-N}\} \leq C_n h^{-n}(1+o(1)),$$

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Those estimates are sharp in some cases (potential barrier).

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"Close" to $\mathbf{R}$, we have (semi-classical max principle)

$$\frac{1}{\mathrm{dist}(\lambda^2, \mathrm{spec}(P_\theta))} \leq \|(P_\theta - \lambda^2)^{-1}\| \leq \frac{|\lambda|^M}{\mathrm{dist}(\lambda^2, \mathrm{spec}(P_\theta))}$$

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with $M = M(n)$... but there are still difficulties connected to the fact that the resonant states are not asymptotically orthogonal, may cluster, etc.

$O(1)$ (respectively $O(h)$) close to the real axis

SJÖSTRAND: hyperbolic flow, additional assumptions:

$$\mathrm{Res}(P(h)) \cap \Big\{[E_1, E_2] + i[-\delta, 0]\Big\} \leq C h^{-n} \delta^{\mu/2-\epsilon},$$

where $Ch \leq \delta \leq 1/C$ and μ is the Minkowski codimension of $\mathcal{T}$. If $\delta \sim h$, we have

$$O(h^{-d/2-\epsilon})$$

resonances with $d = n - \mu$ (the Minkowski dimension of $\mathcal{T}$).

Resonances in sectors $0 \leq -\arg \lambda \ll 1$

SJÖSTRAND, ZWORSKI: Define $P_\epsilon^\#$ in an ϵ-neighborhood K_ϵ of the convex hull of the "black box" with Dirichlet B.C. Then

$$\#\mathrm{Res}(P) \cap \{-\theta < \arg \lambda \leq 0\} \ \leq \ 2(1 + C\epsilon)N_\epsilon^\#(r) + C\epsilon r^n,$$

where $\epsilon = \theta^{2/7}$ and $N_\epsilon^\#(r)$ satisfies reasonable conditions.

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ZERZERI: K_ϵ might be an ϵ-neighborhood of the "black box" (no convexity) but then we have to add $2(2\pi)^{-n}\times$ the measure of the "trapped rays" $\times r^n$ (not the same as $\mathcal{T}$ above).

Resonances in larger sectors $0 \leq -\arg\lambda < \pi/2 - \delta$ $(-\Im z = O(1))$

SJÖSTRAND, PETKOV-ZWORSKI: "bottle theorem"

$$N^{\#}((1-\epsilon)r) - E_{-}(r) \leq N_{\delta}(r) \leq N^{\#}((1+\epsilon)r) + E_{+}(r),$$

where

$$0 \leq E_{\pm}(r) \leq C_1 \epsilon r^{n^{\#}} + C_2(R_0, \epsilon) r^n + \ldots.$$

$C_{1,2}$ independent of P! In particular, if P depends on a parameter so that $N^{\#}(r)/r^n \gg 1$ (and $n^{\#} = n$), then we have almost an asymptotic formula! If $n^{\#} > n$, i.e., if $N^{\#}(r)$ dominates, (hypoelliptic operators), then we have true asymptotics (VODEV, SJÖSTRAND, PETKOV-ZWORSKI).

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Seimi-classical version: Petkov-Zworski

Resonances in a disk $|\lambda| \leq r$

$$P = -\Delta + V(r), \quad \operatorname{supp} V \subset [0, R]$$

ZWORSKI:

$$N(r) \leq C_n R^n r^n, \quad \text{for } r \gg 1,$$

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Black box in $B(0,R)$, $n^{\#} = n$ odd. Define the continuous functions

$$M(r) = \int_0^r \frac{N(t)}{t} dt, \quad M^{\#}(r) = \int_0^r \frac{N^{\#}(t)}{t} dt.$$

Assume also that $N^{\#}(r+1) - N^{\#}(r) = O(r^{n-\epsilon})$, with some $\epsilon \in (0,1]$.

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Theorem 1 *There exists $C_n > 0$ depending on the dimension only, such that*

$$M(r) \leq 2M^\#(r) + C_n R^n r^n + o(r^n). \tag{1}$$

As a consequence, for any $0 < \varepsilon \leq 1$,

$$N(r) \leq 2(1+\varepsilon)N^\#((1+\varepsilon)r) + C_n R^n r^n/\varepsilon + o_\varepsilon(r^n). \tag{2}$$

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If we know more about the regularity of $N(r)$, then (1) is true for $N(r)$ and $N^\#(r)$.

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Proof: Integrate along the semi-disk $[-r, r] \cup r\exp(i[0,\pi])$ using $s'/s = 2\pi i\sigma'$.

$$\begin{aligned} N(t) &= \frac{1}{2\pi i}\oint \frac{s'(z)}{s(z)} dz = \Im \frac{1}{2\pi}\oint \frac{s'(z)}{s(z)} dz \\ &= \int_{-t}^{t} \sigma'(z)\, dz + \frac{1}{2\pi}\int_0^\pi t\frac{d}{dt}\log|s(te^{i\theta})| d\theta \\ &= 2\sigma(t) + \frac{1}{2\pi}\int_0^\pi t\frac{d}{dt}\log|s(te^{i\theta})| d\theta. \end{aligned}$$

Divide by t and integrate to get the lemma.

To estimate the scattering phase, we apply results by T. Christiansen to get

$$M(r) = 2\left[M^{\#}(r) - \tau_n \frac{r^n}{n}\right] + \frac{1}{2\pi}\int_0^{\pi} \log|s(re^{i\theta})|\, d\theta + O(r^{n-\epsilon}),$$

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To estimate the scattering determinant $s(\lambda)$, we proceed in the usual way. The scattering operator $S(\lambda)$ is given by

$$S(\lambda) = I + c_n\lambda^{n-2}\mathbf{E}_-(\lambda)[\Delta,\chi_2]R(\lambda)[\Delta,\chi_1]\,{}^t\mathbf{E}_+(\lambda), \tag{3}$$

where

$$[\mathbf{E}_\pm(\lambda)f](\omega) = \int e^{\pm i\lambda\omega\cdot x} f(x)\,dx = \hat{f}(\mp\lambda\omega), \quad \omega \in S^{n-1},$$

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We estimate the characteristic values of the scattering amplitude which reduces to an estimate of

$$\Delta_\theta^m e^{i\lambda x\cdot\theta}, \quad |x| \le R.$$

We need to work here in a sector $0 < \delta \le \arg\lambda \le \pi - \delta$. Using a standard argument, to cover the missing sectors $0 \le \arg\lambda \le \delta$ and $\pi - \delta \le \arg\lambda \le \pi$, we use the fact that $|s| = 1$ on $\mathbf{R}$ and the Phragmen-Lindelöff principle.

Modified estimate:

$$M(r) \leq 2\left[M^{\#}(r) - \tau_n \frac{r^n}{n}\right] + \frac{1}{n} C_n R^n r^n + o(r^n).$$

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Here

$$M^{\#}(r) - \tau_n \frac{r^n}{n} = \begin{cases} 0, & \text{if } P = -\Delta + V, \\ -\frac{1}{n}(2\pi)^{-n} \int_{x \in \mathcal{O},\, |\xi| \leq 1} dx\, d\xi & \text{in obstacle scattering,} \\ \frac{1}{n}(2\pi)^{-n} \int_{|\xi| \leq 1} \left(|g|^{1/2} - 1\right) dx\, d\xi & \text{for } P = -\Delta_g. \end{cases}$$

Modified modified estimate: Let $P^{\#}$ be equal to P in $B(0, R)$ with Dirichlet B.C.

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Then

$$nM(r) \leq 2nM^{\#}(r) + C_n' R^n r^n + o(r^n),$$

where $C_n' = C_n - (2\pi)^{-n}\mathrm{vol}(B(0,1))^2$, and

$$nM^{\#}(r) = \begin{cases} (2\pi)^{-n} \int_{|x|\leq R, |\xi|\leq 1} dx\, d\xi, & \text{if } P = -\Delta + V, \\ (2\pi)^{-n} \int_{x\in B(0,R)\setminus\mathcal{O},\, |\xi|\leq 1} dx\, d\xi & \text{in obstacle scattering,} \\ (2\pi)^{-n} \int_{|x|\leq R, |\xi|\leq 1} |g|^{1/2} dx\, d\xi & \text{for } P = -\Delta_g. \end{cases}$$

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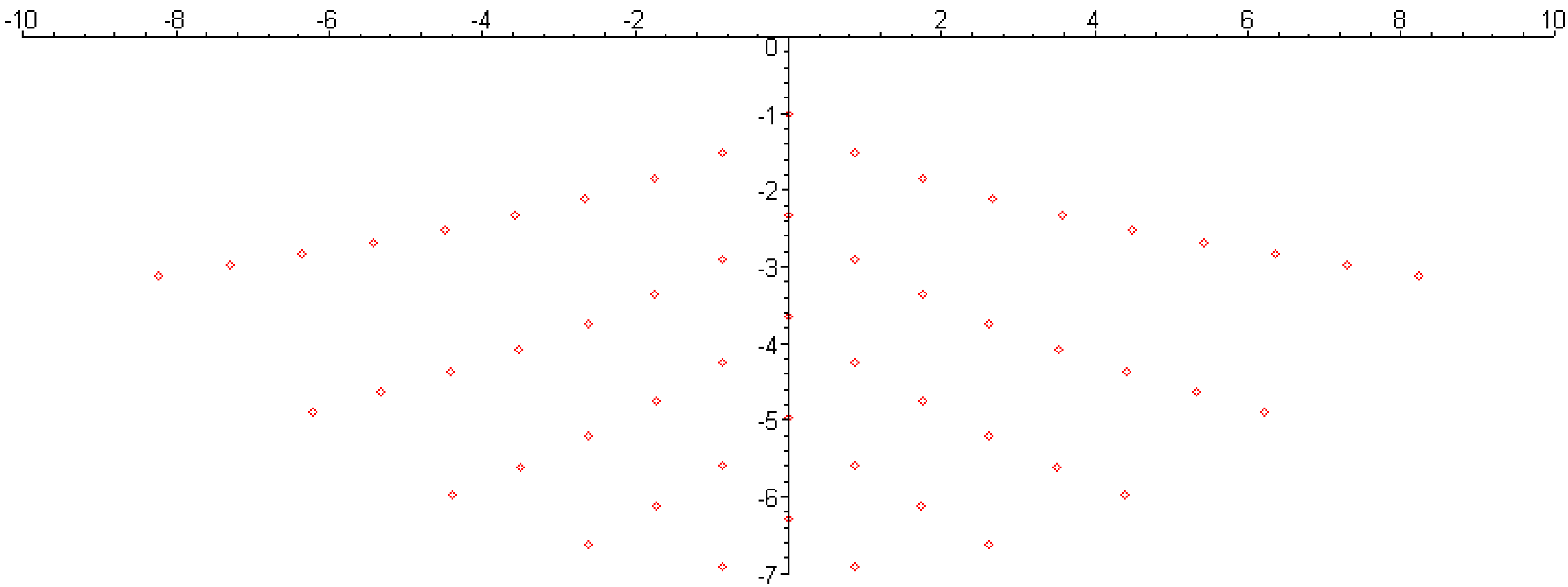
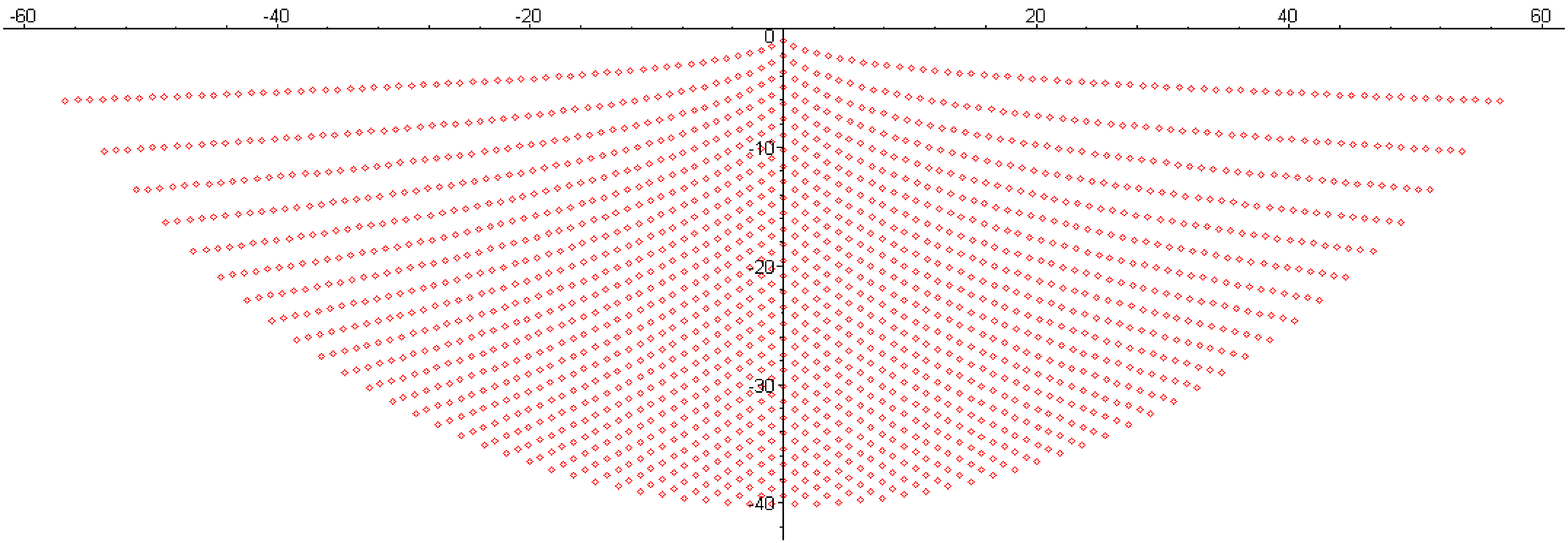
$$C_n' = ?$$

Resonances of the unit sphere, Dirichlet B.C.

$n = 3$, resonances are the zeros of $H^{(1)}_{k+1/2}(\lambda)$, $k = 0, 1, \ldots$. Multiplicity $= 2k + 1$.



More of them:



OLVER: Asymptotically, the sphere resonances are given by

$$\lambda_{k,s} \sim \nu\zeta^{-1}\left(-\nu^{-\frac{2}{3}}e^{\frac{i\pi}{3}}a_s\right), \qquad s = 1, 2, \ldots, k; \quad k = 1, 2, \ldots,$$

where $\nu = k + \frac{1}{2}$, and ... $< a_2 < a_2 < a_1 < 0$ are the zeros of $\mathrm{Ai}(s)$, and

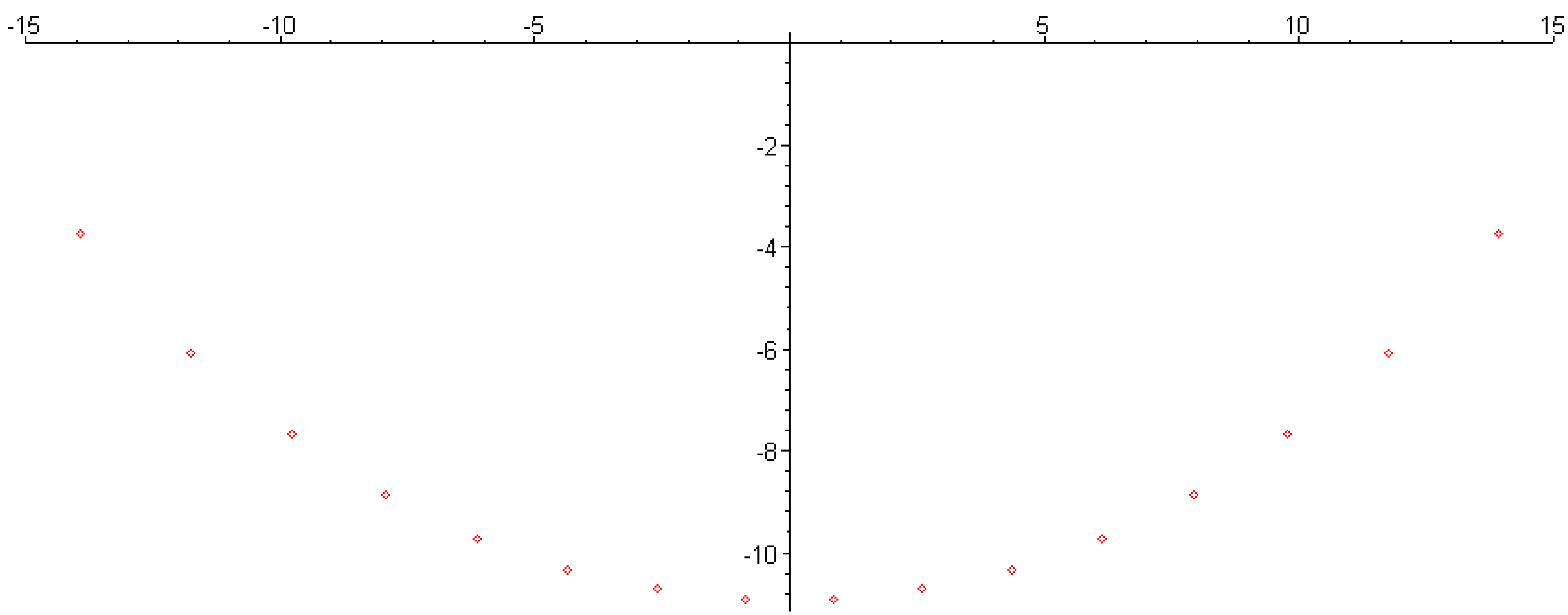
$$\frac{2}{3}\zeta^{\frac{3}{2}}(z) = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2}$$

Olver: Asymptotically, the sphere resonances are given by

$$\lambda_{k,s} \sim \nu\zeta^{-1}\left(-\nu^{-\frac{2}{3}}e^{\frac{i\pi}{3}}a_s\right), \qquad s = 1, 2, \ldots, k; \quad k = 1, 2, \ldots,$$

where $\nu = k + \frac{1}{2}$, and ... $< a_2 < a_2 < a_1 < 0$ are the zeros of $\mathrm{Ai}(s)$, and

$$\frac{2}{3}\zeta^{\frac{3}{2}}(z) = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2}$$



$\lambda_{k,s}$ for $k = 16$

Define $N_{S^2}(r) = \#\mathrm{Res}(P) \cap \{|\lambda| < r\}$. We have

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C_{S^2} can be expressed as a certain integral involving ζ^{-1}.

Resonances of $P = -\Delta + \mathbf{1}_{B(0,R)}$, $n = 3$

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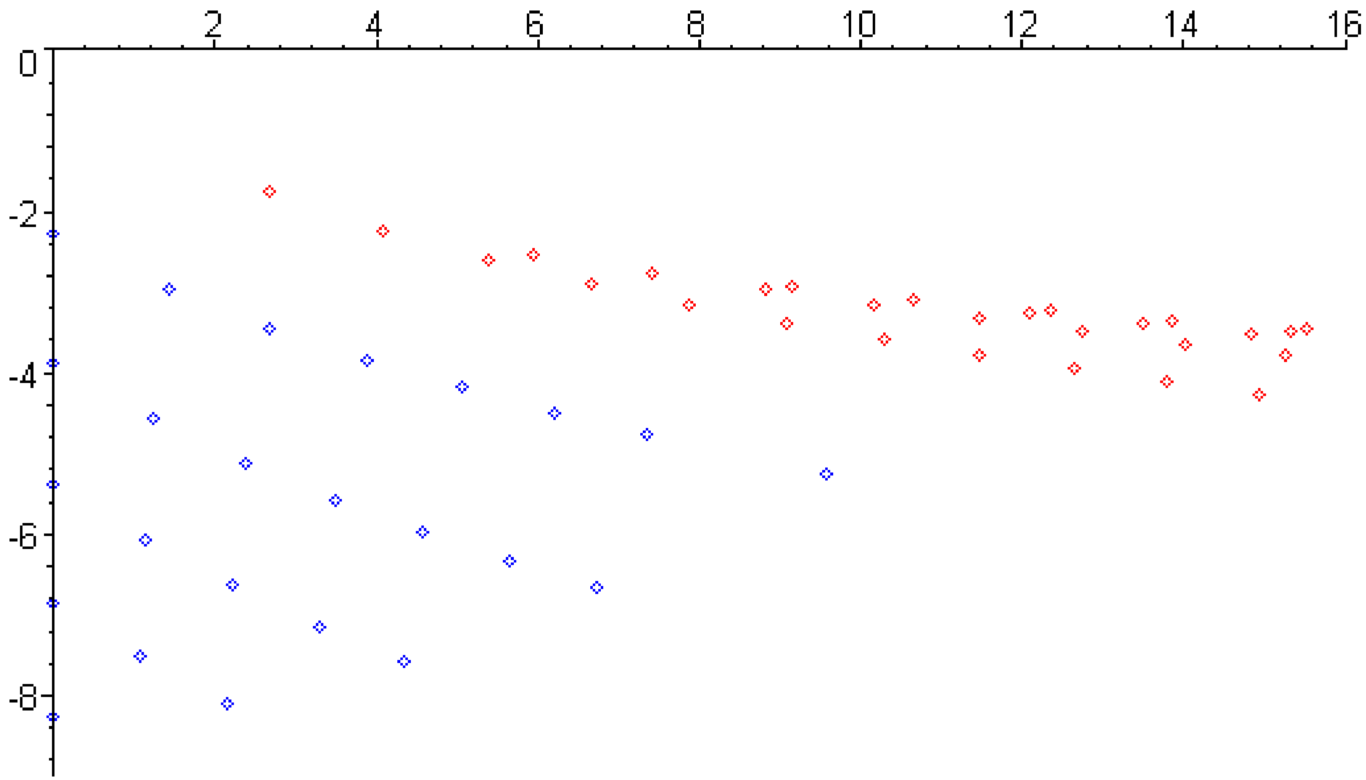

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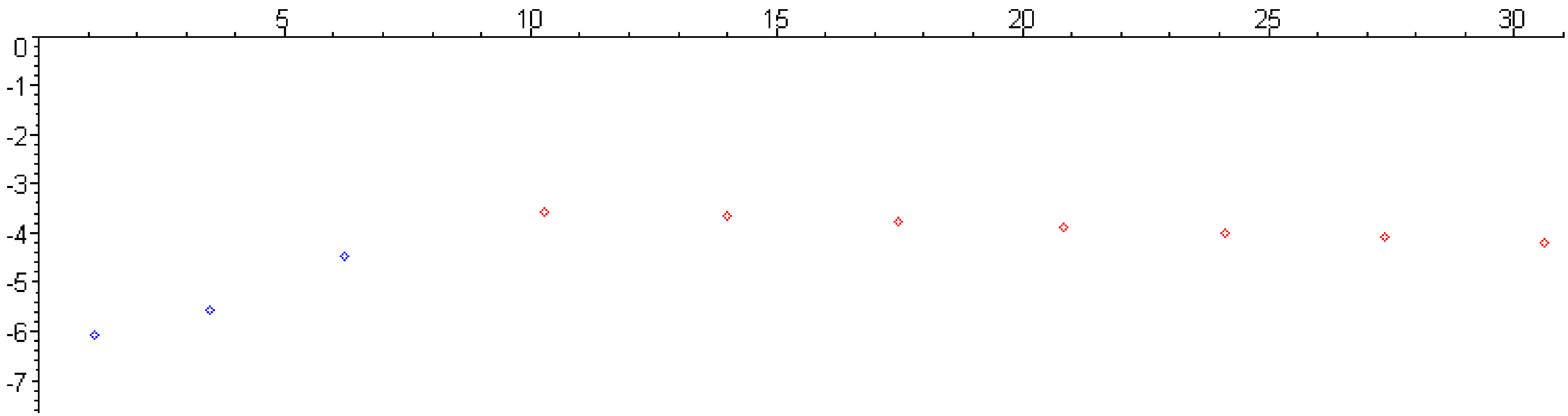

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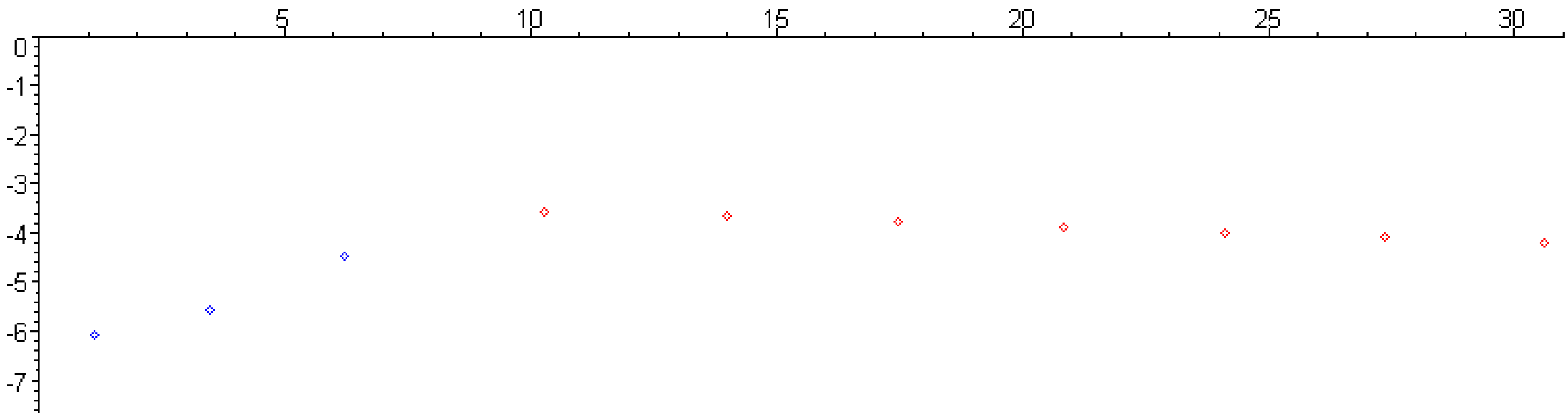

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It appears that

$$N(r) = C_B r^3 + C_{S^2} r^3 + o(r^3),$$

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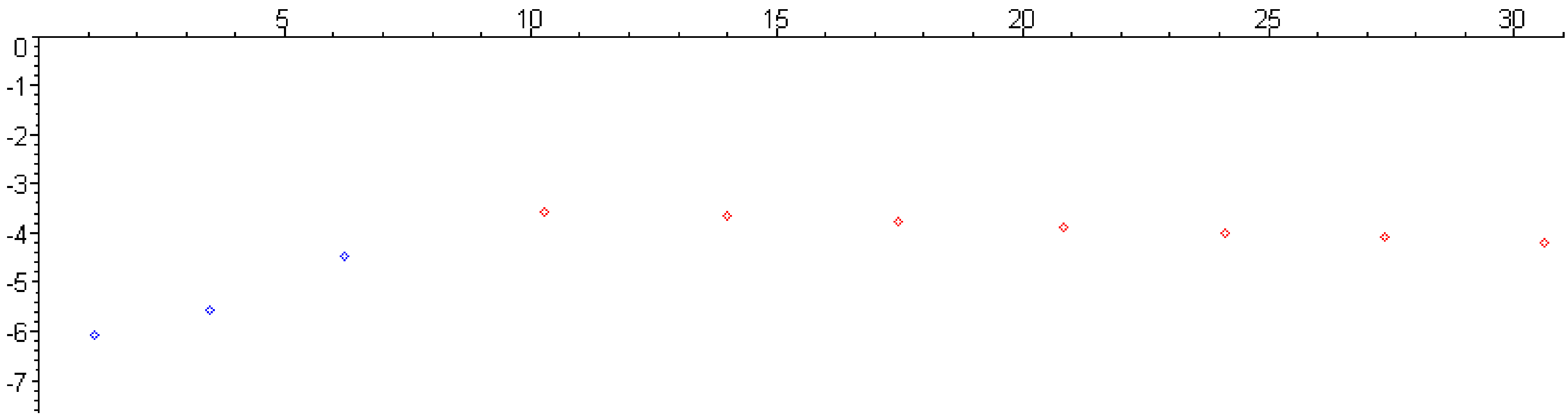

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So, at least in this case,

$$C_n' = C_{S^{n-1}}$$

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$$N(r) \overset{?}{\leq} N^{\#}(r) + C_{S^{n-1}} R^n r^n + o(r^n).$$