High frequency resolvent estimates for the Laplace–Beltrami operator on unbounded Riemannian manifolds and applications. Georgi Vodev

(joint work with F. Cardoso)

Outline

① Describe the class of complete, non-compact Riemannian manifolds the results apply to. Examples.

② High frequency resolvent estimates on weighted spaces. Limiting Absorption Principle. Hölder continuity of the resolvent.

③. Applications to the local energy decay of solutions to the corresponding wave equation.

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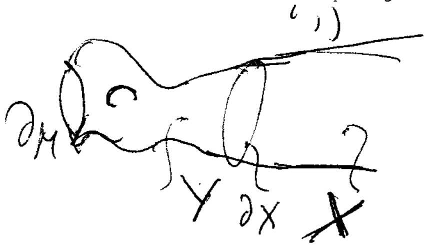
① (M, g) – complete, connexe, non-compact Riemann. manifold, $\dim M = n \geq 2$, with compact boundary ∂M, C^∞.

$g \in C^\infty(\overline{M})$; $M = Y \cup X$, Y – compact, $\partial Y = \partial M \cup \partial X$.

$X = [r_0, +\infty) \times S$, $r_0 \gg 1$, S – compact Riem. manifold without boundary, $\dim S = n-1$.

$g|_X = dr^2 + \sigma(r)$, $\sigma(r)$ – family of Riem. metrics on S depending smoothly on r, $r \in [r_0, +\infty)$.

$\sigma(r) = \sum_{i,j} g_{ij}(r, \theta)\, d\theta_i\, d\theta_j$, $\theta \in S$, $g_{ij} \in C^\infty$



$X_r = [r, +\infty) \times S$, $\forall r \geq r_0$.

$\partial X_r \cong (S, \sigma(r))$ – cross section

$\Delta_{\partial X_r} = -p^{-1} \sum_{i,j} \partial_{\theta_i}\left(p\, g^{ij} \partial_{\theta_j}\right)$,

(g^{ij}) is inverse to (g_{ij}), $p = (\det g_{ij})^{1/2}$

Let Δ_g denote the (positif) Laplace-Beltrami operator on (M, g).

$\Delta_X := \Delta_g|_X = -p^{-1}\partial_r (p\, \partial_r) + \Delta_{\partial X_r}$

$= -\partial_r^2 - \frac{p'}{p}\partial_r + \Delta_{\partial X_r}$.

$\tilde{\Delta}_X := p^{1/2} \Delta_X p^{-1/2} = -\partial_\tau^2 + \Lambda_\tau + q(\tau,\theta)$, where

$\Lambda_\tau = -\sum_{i,j} \partial_{\theta_i}(g^{ij}\partial_{\theta_j})$ and q is an effectif potential.

~~Denote~~ by $h(\tau,\theta,\xi)$ the principal symbol of Λ_τ, i.e.

$$h(\tau,\theta,\xi) = \sum_{i,j} g^{ij}(\tau,\theta,\xi)\, \xi_i \xi_j .$$

Assumptions: $\forall \tau \geq \tau_0$, $\forall \theta \in S$:

(1) $|q(\tau,\theta)| \leq C$, $\frac{\partial q}{\partial \tau}(\tau,\theta) \leq C\tau^{-1-\delta_0}$, $C, \delta_0 > 0$.

(2) $-\frac{\partial h}{\partial \tau}(\tau,\theta,\xi) \geq \frac{C}{\tau} h(\tau,\theta,\xi)$, $C > 0$.

the second fundamental form of ∂X_τ.

(τ,θ) - polar coordinates.

Ex. 1. $\mathbb{R}^n$: $q = \frac{(n-1)(n-3)}{4}\tau^{-2}$, $h = \tau^{-2} h_0(\theta,\xi)$

(τ,θ) the polar coordinates $S = \{x \in \mathbb{R}^n : |x| = 1\}$, h_0 - the principal symbol of L-B. op. on S.

Ex. 2. $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, $\mathcal{O}$ - compact domain in C^∞-smooth boundary.
$\rightsquigarrow$ convexe. Let g be a Riem. metric on Ω

$$g = \sum g_{ij}(x)\, dx_i dx_j, \quad g_{ij} \in C^\infty(\overline{\Omega}).$$

$\forall \alpha$: $\left|\partial_x^\alpha \left(g_{ij}(x) - \delta_{ij}\right)\right| \leq C_\alpha |x|^{-\delta_0 - |\alpha|}$, $|x| \gg 1$.

where $\delta_0 > 0$ Kronecker's symbol

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(2) ~~Study~~ Properties of the Resolvent.

Let G be the self-adjoint realization of Δ_g on the Hilbert space $\mathcal{H} = L^2(M, d\mathrm{Vol}_g)$ with boundary conditions $Bu = 0$ on ∂M, B - Dirichlet or Neumann.

Let G_0 be the self-adjoint realization of Δ_g on the Hilbert space $\mathcal{H}_0 = L^2(X, d\mathrm{Vol}_g)$ with Dirichlet B. cond. on ∂X.

G is a compactly supported perturbation of G_0.

Spec (G): $\sigma_p(G)$, 0 comp α $\sigma_c(G)$

$(\alpha \geq 0)$

Given $s > 1/2$, chose a fcts $\chi_s \in C^\infty(\overline{M})$, $\chi_s = 1$ on Y and $\chi_s = \tau^{-s}$ on X_{τ_0+1}.

Thm 1. Under the assumptions (1) and (2):

a) $\| \tau^{-s}(G_0 - z \pm i\varepsilon)^{-1} \tau^{-s} \|_{\mathcal{L}(H_0)} \leq C z^{-1/2}$

for $z \geq c_0$, $0 < \varepsilon \leq 1$, with constants $C, c_0 > 0$ independent of ε and z.

b) $\| \chi_s (G - z \pm i\varepsilon)^{-1} \chi_s \|_{\mathcal{L}(H_0)} \leq C_1 e^{C_2 z^{1/2}}$

for $z \geq c_0'$, $0 < \varepsilon \leq 1$, with const $C_1, C_2, c_0' > 0$ indep of ε and z.

c) $\| \eta_a \chi_s (G - z \pm i\varepsilon)^{-1} \chi_s \eta_a \|_{\mathcal{L}(H)} \leq \tilde{C} z^{-1/2}$

for $z \geq c_0'$, $0 < \varepsilon \leq 1$, where $\eta_a = 0$ on $M \setminus X_a$

$\eta_a = 1$ on X_{a+1}

$a \gg \tau_0$ - constant.

Burq 1998, 2002 - proved (b) and (c) in case of Ex. 2, with anal. coeff. outside a compact with χ_s - compactly supported

Cutoff absorption principle: (CLAP)

$R_s^{\pm}(z) := \lim_{\varepsilon \to 0} \chi_s (G - z \pm i\varepsilon)^{-1} \chi_s : H \to H$ exists ? for $z \geq c_0$.

Prop 2. Suppose that

$R_{0,s}^{\pm}(z) := \lim_{\varepsilon \to 0} \tau^{-s}(G_0 - z \pm i\varepsilon)^{-1} \tau^{-s} : H_0 \to H_0$ exists for $z \geq \tilde{c}_0$.

Then, $R_s^{\pm}(z)$ exists for $z \geq \overline{c}_0 = \max\{\tilde{c}_0, c_0'\}$.

Moreover, if $R_{0,s}(z)$ is Hölder of order $0 < \mu \leq 1$, i.e.

(3) $\| R_{0,s}^{\pm}(z_2) - R_{0,s}^{\pm}(z_1) \|_{\mathcal{L}(H_0)} \leq A(z_1, z_2) |z_2 - z_1|^\mu$, then

for $z_2, z_1 \geq \tilde{c}_0$ with $A \leq e^{Const |z|^{1/2}}$, $z = \max\{z_1, z_2\}$

so is $R_s^{\pm}(z)$ ~~is Hölder of order μ~~, i.e.

(4) $\| R_s^{\pm}(z_2) - R_s^{\pm}(z_1) \|_{\mathcal{L}(H)} \leq B(z_1, z_2) |z_2 - z_1|^\mu$.

$B = e^{Const |z|^{1/2}}$, $z = \max\{z_1, z_2\}$

for $z_2, z_1 \geq \max\{\tilde{c}_0, c_0'\}$.

Asympt. Euclidean Spaces:

(1') $\left| \frac{\partial^k q}{\partial r^k} \right| \le C r^{-k-\delta_0}, \quad k = 0, 1,$

(2') $\left| \frac{\partial (r^2 h)}{\partial r} \right| \le C r^{1-\delta_0} h,$

$C, \delta_0 > 0$ consts.

(1') => (1); (2') => (2).

Prop. 3. Under (1') and (2'): and μ depending on g and δ.

(3) holds with $A(t_1, t_2) = \mathrm{Const}$.

=> (4) holds with $B(t_1, t_2) = e^{\mathrm{Const}|z|^{1/2}}$,

$z = \max\{t_1, t_2\}$.

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③ Wave equation:

$$\begin{cases} (\partial_t^2 + \Delta_g) u(t,x) = 0 \text{ in } \mathbb{R} \times M, \\ B u(t,x) = 0 \quad \text{on } \partial M, \\ u(0,x) = \varphi(G) f, \quad \partial_t u(0,x) = 0, \quad f \in H \end{cases}$$

$\varphi \in C^\infty(\mathbb{R})$: $\varphi = 0$ ... $\varphi = 1$

0 b $b e^{1}$ $b >> 1$ constant.

$u(t,x) = \cos(t\sqrt{G})\varphi(G) f.$

Thm 7. $\forall s > 1/2$

(4) => $\forall m > 0$.

(5) $\| \chi_s \cos(t\sqrt{G}) \varphi(G)(G+1)^{-m/2} \chi_s \|_{L^2 \to L^2} \le C_m (\log t)^{-m}, \quad t >> 1.$

In case of Exp 2. one can take $\varphi \equiv 1$

(6) (c) of Thm 1. =>

$$\int_0^\infty \| \eta_a \chi_s \cos(t\sqrt{G}) \varphi(G) \chi_s \eta_a f \|^2_{H^1(M)} dt$$

$$\le C \| f \|^2_{L^2(M)} \quad \forall f \in H.$$

Burq proved (5) or (6) in the case of Exp 2. with χ_s - compactly supported.