Workshop MSRI 5/5-2003 ①

Perturbations of self-adjoint operators with periodic flow - branching levels.

joint work in progress with M. Hitrik

0. Introduction.

Melin-Sj (2001) found large stable classes of analytic non-selfadjoint pseudodiff. op. in dimension with spectrum given by a Bohr-S. condition. (absence of small denominators.)

Current project with Hitrik 2002-20XY:

Small non-s.a. perturbations of s.a. operators.

So far, mainly the case when the classical flow of the unperturbed part is periodic.

Let $M = \begin{cases} \mathbb{R}^2 \\ \text{or} \\ \text{compact 2-dim analytic mfld.} \end{cases}$

When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P^w(x, hD_x, \varepsilon; h)$$

$[0, \varepsilon_0[\ni \varepsilon \xrightarrow[C^\infty]{} P(x, \xi, \varepsilon; h) \in$

space of holomorphic function $f(x,\xi)$ in tubular neighborhood of $\mathbb{R}^4$ with $|f(x,\xi)| \le C\, m(\mathrm{Re}(x,\xi))$

$m \geq 1$

$$m(X) \le C_0 \langle X - Y \rangle^{N_0} m(Y)$$

$$P_\varepsilon(x,\xi,\varepsilon;h) \sim \sum_{j=0}^{\infty} P_{j,\varepsilon}(x,\xi)h^j, \quad h\to 0$$ 3

in this space

$$|P_{0,\varepsilon}(x,\xi)| \geq \frac{1}{C} m(x,\xi), \quad |(x,\xi)| \geq C$$

Write $P = P_{0,0}$, $P_\varepsilon = P_{0,\varepsilon}$

When M compact, assume P_ε is a h-differential operator ....

(Replace $m(x,\xi)$ by $\langle\xi\rangle^m$, $m>0$)

Assume $P_{\varepsilon=0}$ is self-adjoint.

Then near 0, $\sigma(P_\varepsilon)$ is discrete and contained in $\{|\operatorname{Im} z| \leq \mathcal{O}(\varepsilon)\}$.

Assume $p^{-1}(0)\cap T^*M$ is connected and $dp \neq 0$ there.

Let $H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial x}$. Assume

The H_p-flow is periodic on $p^{-1}(E) \cap T^*M$ with period $T(E) > 0$ analytic in E, $E \in \mathrm{neigh}(0, \mathbb{R})$.

Write $P_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2 m)$

Following the tradition in the self-adjoint case (A. Weinstein, Colin de Verdière, Guillemin, Boutet de Monvel, Helffer–Robert, Grigis, Dozias, Ivrii ...) introduce

$$\langle q \rangle = \frac{1}{T(E)} \int_{-T(E)/2}^{T(E)/2} q \circ \exp(tH_p)\, dt, \quad \text{on } p^{-1}(E).$$

Using phase space exponential weights $\approx$ conjugation by a Fourier int. op. with complex phase

Can assume

$$P_\varepsilon = P + i\varepsilon\langle q\rangle + \mathcal{O}(\varepsilon^2)$$

on (new) real phase space.

Assume $\langle q\rangle$ real or more generally

$\mathrm{Im}\langle q\rangle$ is an analytic function of p and $\mathrm{Re}\langle q\rangle$.

Let $F_0 \in \left[\min\limits_{p^{-1}(0)}\langle \mathrm{Re}\, q\rangle, \max\limits_{p^{-1}(0)}\langle \mathrm{Re}\, q\rangle\right]$.

Goal: determine all eigenvalues in a rectangle $\left]-\frac{1}{\mathcal{O}(1)}, \frac{1}{\mathcal{O}(1)}\right[+ i\varepsilon \left]F_0 - \frac{1}{\mathcal{O}(1)}, F_0 + \frac{1}{\mathcal{O}(1)}\right[$

for $h \ll \varepsilon \le \mathcal{O}(h^\delta)$ in general, and for $h^2 \ll \varepsilon \le \mathcal{O}(h^\delta)$ when the subprincipal symbol of $P_{\varepsilon=0}$ vanishes.

Let

$$\Lambda_{0,F_0} = \{\rho \in T^*M ;\ p(\rho)=0,\ \mathrm{Re}\langle q\rangle = F_0\}.$$

Assume

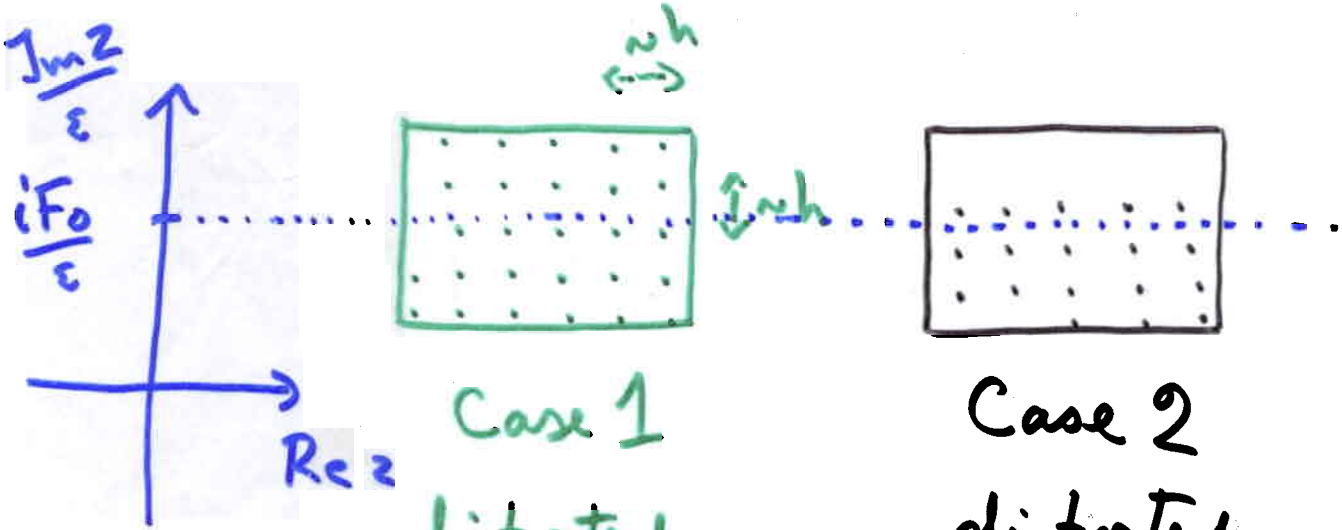
$T(0)$ is the minimal period for the H_p-flow on Λ_{0,F_0}.

Then $p^{-1}(0)/\exp \mathbb{R} H_p$ is a 2-dim. symplectic mfld near the image of Λ_{0,F_0}. $\mathrm{Re}\langle q\rangle$ is an analytic function on this mfld.

Case 1. F_0 is not a critical value of $\mathrm{Re}\langle q\rangle$. $\Rightarrow \Lambda_{0,F_0}$ is a Lagrangian torus.

Case 2. F_0 is a non-deg. maximum or minimum, attained at a single point.

Spectrum (Hitrik–S; 2003):



Case 1: distorted lattice

Case 2: distorted half-lattice

(In case 1 result still valid for ε up to $\varepsilon_0 > 0$ small but independent of h. S; 2002)

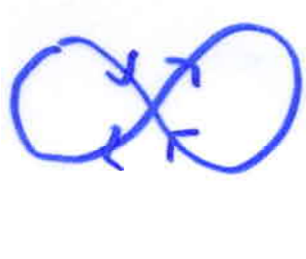
Remaining case generically:

Case 3: F_0 is a non-degenerate saddle point level, attained at only 1 critical point.

This talk

1. Reductions:

$\mathrm{Re}\langle q \rangle = F_0$:



Conjugation by microlocally defined elliptic Fourier integral operator (multivalued)

$\rightsquigarrow \tilde{P}_\varepsilon$ on $L^2_{(S_0,S_1,S_2)}(S^1_t \times \mathbb{R}_x)$

a space of microlocally defined Floquet periodic functions with Floquet parameters $e^{iS_0/h}$, $e^{iS_1/h}$, $e^{iS_2/h}$,

S_j = action differences.

Leading symbol of $\tilde{P}_\varepsilon$:

$$\tilde{P}_\varepsilon = \underset{\uparrow \text{real}}{g(\tau)} + i\varepsilon \langle q \rangle (\tau, x, \xi) + \mathcal{O}(\varepsilon^2)$$

real, $g' \neq 0$.

Further averagings in t eliminate the t-dependence from the full symbol and we get:

a) General case:

$$\tilde{P}_\varepsilon(\tau,x,\xi;h) =$$

$$g(\tau)+\varepsilon\Big[i\langle q\rangle(\tau,x,\xi)+\mathcal{O}(\varepsilon)+\frac{h}{\varepsilon}p_1+h\frac{h}{\varepsilon}p_2+\dots\Big]$$

ε, $\frac{h}{\varepsilon}$ small parameters if $h \ll \varepsilon \le h^\delta$.

b) The subprincipal symbol of $P_{\varepsilon=0}$ is 0:

$\frac{h}{\varepsilon}p_1 = h\tilde{p}_1$: $\tilde{P}_\varepsilon(\tau,x,\xi;h) =$

$$g(\tau)+\varepsilon\Big[i\langle q\rangle(\tau,x,\xi)+\mathcal{O}(\varepsilon)+\frac{h^2}{\varepsilon}p_2+h\tilde{p}_1+h^2\tilde{p}_2+\dots\Big]$$

ε, $\frac{h^2}{\varepsilon}$ small parameters if $h^2 \ll \varepsilon \le h^\delta$.

These reductions are mod $\mathcal{O}(h^\infty)$ in the space of holomorphic symbols.

To compute the spectrum "it suffices" to expand in Fourier series in t and study $\tilde{P}_\varepsilon\left(h\left(k-\frac{k_0}{4}\right)-\frac{S_0}{2\pi},x,hD_x;h\right)$ $k\in\mathbb{Z}$

$k_0 \in \mathbb{Z}$ is fixed.

Likely that this gives the spectrum in the respective cases $h \ll \varepsilon \leq h^{\delta}$, $h^2 \ll \varepsilon \leq h^{\delta}$, but we recently discovered a technical difficulty when the last operator becomes too non-self-adjoint. We then need to work in modified spaces to avoid pseudo-spectral behaviour and these spaces should vary with k. (covers the case of the damped wave eq.)

However, in case b) for $\varepsilon \sim h$ and probably in both cases a, b for $\varepsilon \leq \mathcal{O}(h)$, we can work on ordinary L^2 and we seem to get complete spectral results. Remainder of the talk: 1D-case.

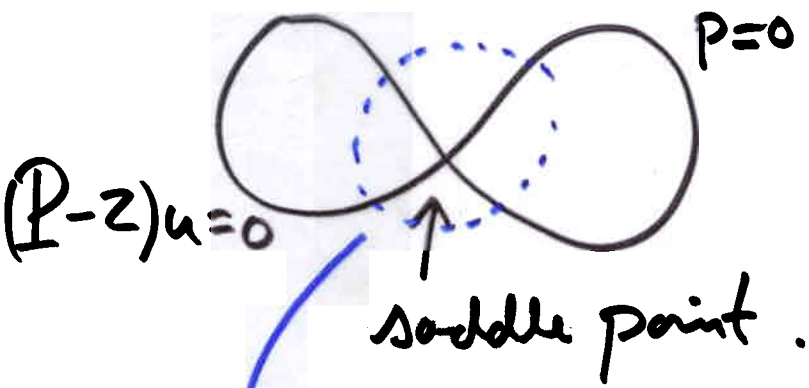
2. Spectrum of branching 1-dim pseudodiff. operators.

Let $P(x, hD_x; h)$ analytic pseudo defined near $(0,0) \in \mathbb{R}_x \times \mathbb{R}_\xi$. If p is real with a saddle point at $(0,0)$ and $p(0,0)$ then $\exists\, F(\tau; h) = f(\tau) + h f_1(\tau) + \cdots$ s.t.

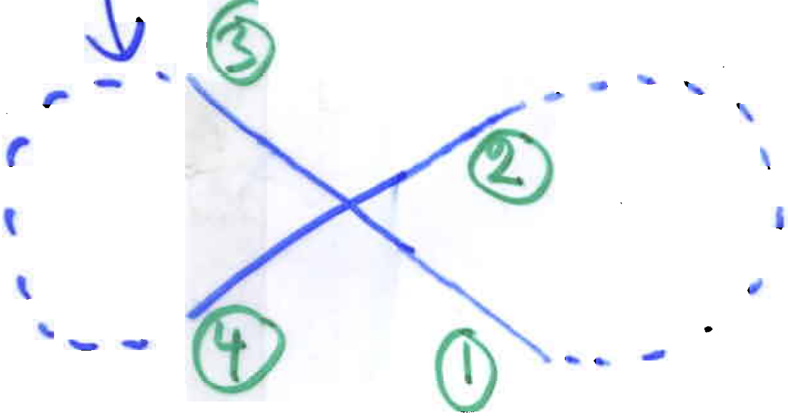
$$U^{-1} F(P; h) U = -P_0$$

$P_0 = \frac{1}{2}\left((hD_x)^2 - x^2\right)$. Still true after making p slightly complex (U is then a Fourier int. op. with complex phase) (Helffer-Sj, Horn, C. März, Parisse-Colin de Verdière, Fujiie-Ramond, Buslaev-Grigis ....)

Let p be real defined near



and add perturbations making P non-self-adjoint.



$(P_0 + \mu) v = 0$

cf $(-h^2 \partial_x^2 + V + \mu) = 0$

V: 

Near j: $v = v_j \times$ normalized WKB solution

Transition matrix:

$$\begin{pmatrix} v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} b_{2,3} & b_{2,4} \\ b_{1,3} & b_{1,4} \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$$

$\det B = 1$ $\quad b_{j,k}$ can be computed explicitly

Let $S_{3,4}^{(\mu)}$ = exterior action from 4 to 3

$S_{1,2}$ = — " — 2 to 1

<u>Proposition</u> μ is an eigenvalue of $P \iff G(\mu; h) = 0$

where $G(\mu; h) = a_1 + a_2 + a_3 + a_4$

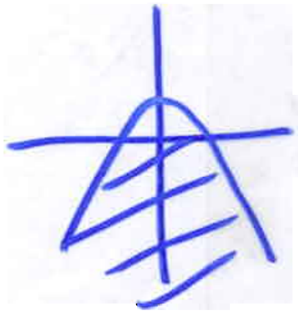
$$= e^{-\frac{\pi\mu}{2h}}\left(e^{\frac{i}{h}(S_{1,2}+S_{3,4})} b_{2,3} + e^{\frac{i}{h}S_{1,2}+\pi\frac{\mu}{h}} + e^{\frac{i}{h}S_{3,4}+\pi\frac{\mu}{h}} + b_{1,4}\right)$$

Cf: Fujiié-Ramond

Davies, Redparth, Shkalikov

If $\frac{|\mu|}{h} \gg 1$, $\left|\arg\mu - \frac{\pi}{2}\right| < \pi - \frac{1}{C}$:

Case 1.



$$a_4 = a_{4+} + a_{4-}$$

$$a_{4\pm} = e^{-\Theta_{2,3}(h)+\Theta_-\left(\frac{h}{\mu}\right)+\frac{i}{h}\left(-\mu\ln\frac{\mu}{i}-\frac{1-\mu^2}{2}\right)} \times e^{\pm\frac{\pi\mu}{h}}$$

$$= e^{\frac{i}{h}\varphi_{4\pm}}$$

$$a_2 = e^{\frac{i}{h}S_{1,2}+\frac{\pi\mu}{2h}} = e^{\frac{i}{h}\varphi_2}$$

$$a_3 = e^{\frac{i}{h}S_{3,4}+\frac{\pi\mu}{2h}} = e^{\frac{i}{h}\varphi_3}$$

$$a_1 = e^{\frac{i}{h}\left(S_{1,2}+S_{3,4}+\mu\ln\frac{\mu}{i}+\frac{1-\mu^2}{2}\right)-\Theta_-\left(\frac{h}{\mu}\right)+\Theta_{2,3}(h)}$$

No zero if one a_j, $j = 1, 2, 3, 4^{\pm}$ dominates over the others.

Look for the curves $\Gamma_{j,k}$ where $|a_j| = |a_k|$ and especially where $|a_j|, |a_k|$ dominate over the other a_ν.

Can neglect a_{4-} when $\operatorname{Re}\mu \geq 0$

a_{4+} –"– $\operatorname{Re}\mu \leq 0$

Can neglect $\Gamma_{2,3}$ since

$$a_2 a_3 = a_1 a_{4+}$$

$\operatorname{Re}\mu \geq 0$:

$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \begin{cases} \operatorname{Im} S_{3,4}(\mu) + \mathbb{X}(\mu) & : \Gamma_{3,4^+} = \Gamma_{1,2} \\ \operatorname{Im} S_{1,2}(\mu) + \mathbb{X}(\mu) & : \Gamma_{2,4^+} = \Gamma_{1,3} \\ \frac{1}{2}(\operatorname{Im} S_{1,2} + \operatorname{Im} S_{3,4}) + \mathbb{X}(\mu) & : \Gamma_{1,4+} \end{cases}$$

$\operatorname{Re}\mu \leq 0$:

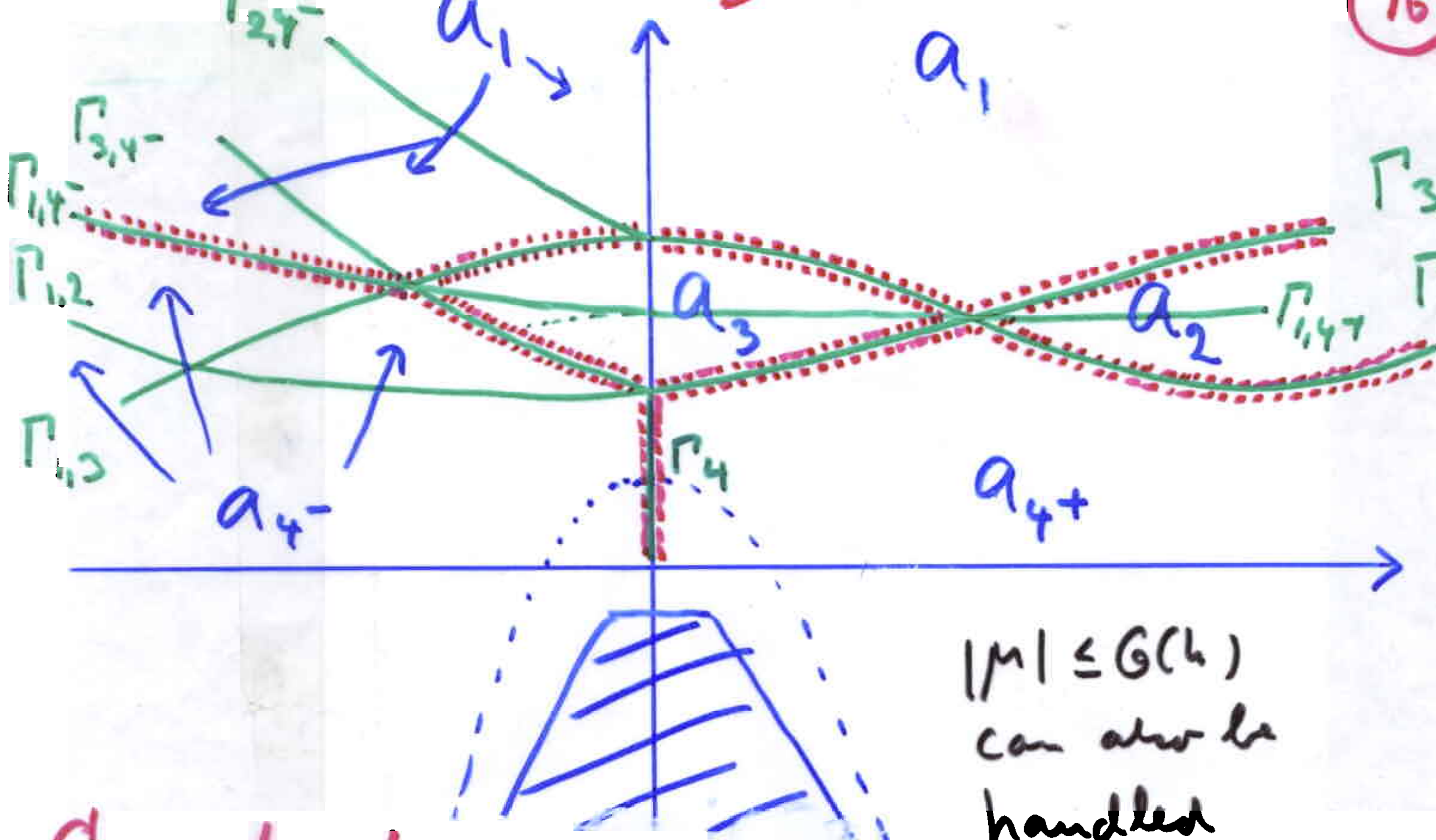
$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \begin{cases} \operatorname{Im} S_{3,4} + \mathbb{X}(\mu) - 2\pi\operatorname{Re}\mu & : \Gamma_{3,4^-} \\ \operatorname{Im} S_{1,2} + \mathbb{X}(\mu) - 2\pi\operatorname{Re}\mu & : \Gamma_{2,4^-} \\ \operatorname{Im} S_{1,2} + \mathbb{X}(\mu) & : \Gamma_{1,3} \\ \operatorname{Im} S_{3,4} + \mathbb{X}(\mu) & : \Gamma_{1,2} \\ \frac{1}{2}(\operatorname{Im} S_{1,2} + \operatorname{Im} S_{3,4}) + \mathbb{X} - \pi\operatorname{Re}\mu & : \Gamma_{1,4^-} \end{cases}$$

Exponential concentration of the zeros to

$\Gamma_{2,4^+} \cup \Gamma_{3,4^+}$ for $\operatorname{Re}\mu \gg h$:

$$G = a_4^+\left[\left(1+\frac{a_2}{a_{4^+}}\right)\left(1+\frac{a_3}{a_{4^+}}\right) + e^{-2\pi\mu/h}\right].$$

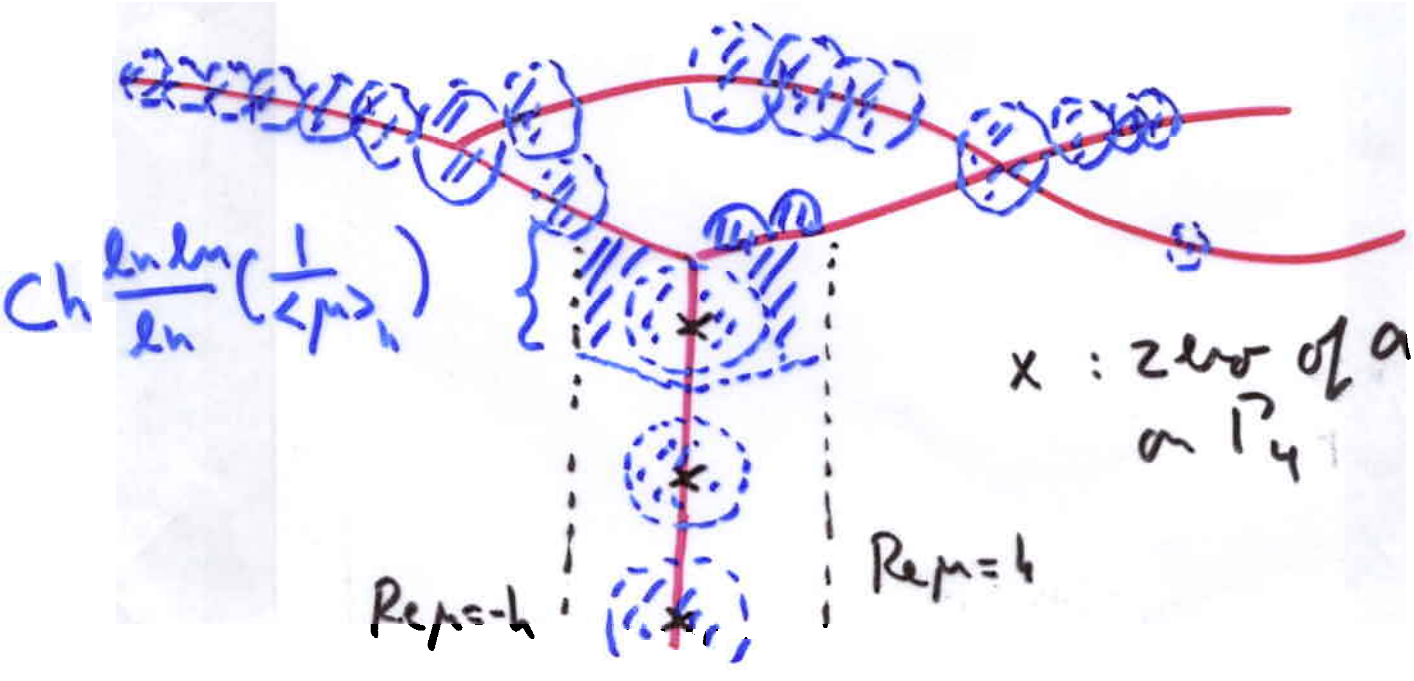
Skeleton



$|\mu| \le G(h)$ can also be handled

Concentration of zeros to the skeleton:

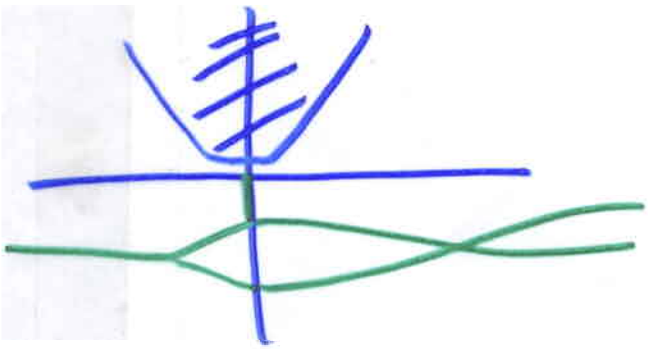
$$\langle\mu\rangle_h = (h^2+|\mu|^2)^{\frac{1}{2}} \qquad \bigcirc = D\left(\mu; \frac{Ch}{\ln\frac{1}{\langle\mu\rangle_h}}\right)$$



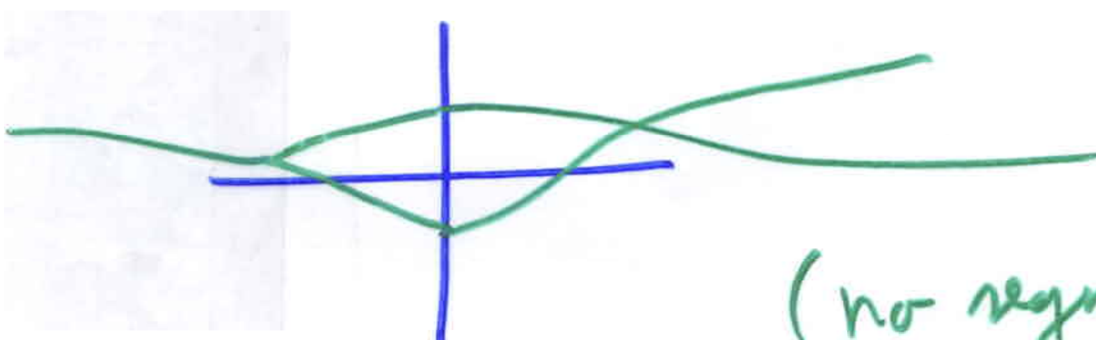
x : zero of a_4 on Γ_4

Other possibilities:

Case 2: $\left|\arg\mu + \frac{\pi}{2}\right| < \pi - \frac{1}{C}$:



When the skeleton reaches both of $i\mathbb{R}_\pm$:

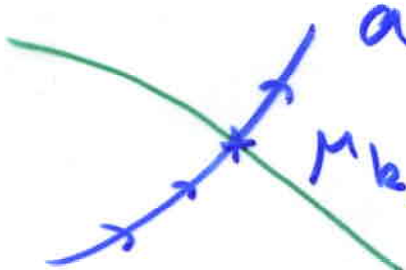


(no segment in $i\mathbb{R}$)

Discussion with Fujiié-Ramond:

Can determine the dominant term a_j away from the skeleton by Stokes line navigation n° of zeros inside a closed curve γ crossing the skeleton transversally at finitely many points $\mu_{k,e} = \mu_{k+1,s}$, $k = 0, 1, \dots, N-1$

$(N-1+1 = 0)$



$a_{\nu(k+1)} = e^{i\varphi_{\nu(k+1)}/h}$

$\mu_{k,e} = \mu_{k+1,s}$

$a_{\nu(k)} = e^{i\varphi_{\nu(k)}/h}$

$$\doteq \frac{1}{2\pi h} \sum_{k=0}^{N-1} \left(\varphi_{\nu(k)}(\mu_{k,e}) - \varphi_{\nu(k+1)}(\mu_{k+1,s}) \right) + \mathcal{O}(1) + \mathcal{O}\left(\max \ln \ln \frac{1}{\langle \mu_{k,e} \rangle_h} \right)$$

maximum over all crossing points in the exceptional region.

Inspiration: E.B. Davies : 1D-systems.