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Random Matrices, Neutron Capture Levels, Quasicrystals and Zeta-function Zeros

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1. Ancient History

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Happy Birthday, Harold Widom! I am delighted and honored to be here to help celebrate Harold's seventieth birthday, and to celebrate the flowering of random matrix theory to which he has contributed so much. I have in my files a collection of wonderful handwritten letters that Harold wrote to me about ten years ago when he and Craig Tracy were embarked on their grand tour of Fredholm determinants and Painlevé transcendents. The letters are full of enthusiasm and full of equations. Here is a man who is in love with mathematics. As we are gathered here to do him honor, we can say about Harold what the inscription in St. Paul's Cathedral in London says about Christopher Wren, "Si monumentum quaeris, circumspice!", "If you are looking for his monument, look around you!"

I am talking to a group of experts on random matrix theory, but I am no longer an expert. I worked vigorously on random matrix theory for about ten years, roughly from 1962 to 1972, mostly in collaboration with Madan Lal Mehta, and then went on to other things. Any expert knowledge that I may have possessed is now thirty years out of date. I

have only a very superficial knowledge of the beautiful work that you people have been doing during the last thirty years. It would be absurd for me to try to talk about your recent work, and painful for you to listen. The only subject I can talk about with authority is ancient history, so I will talk about that. I will try to pick out aspects of ancient history that have not been published and are not well known. Some of the old problems that we used to worry about might be worth revisiting with the more modern tools that are now available.

Random matrix theory was invented by Eugene Wigner in 1951 as a branch of nuclear physics. In those days nuclear physics was a fashionable subject for leading physicists to work in. Now nuclear physics has been out of fashion for a long time, and the younger members of the audience have probably never imagined that nuclear physics could have any connection with interesting mathematics. In the 1960s when Mehta and I were working on random matrices, the connection with nuclear physics was always in our minds. The Wigner random matrix model was intended to be a model for the Hamiltonian of a heavy nucleus containing a large number of protons and neutrons with strong interactions. For such a nucleus, the standard models of nuclear physics gave a reasonably accurate picture of the ground state and the first few excited states. The low-lying states are simple enough to be described by a dynamical model with a few suitably chosen quantum numbers. But when you are looking at the highly excited states that are formed when slow neutrons are captured by heavy nuclei, there is no hope of describing them individually by a dynamical model. For example, when slow neutrons are captured in uranium 238, the common isotope of uranium, you see resonances at the energies of excited states of the unstable nucleus uranium 239. The binding energy of the neutron is about four million volts, and the spacings between the observed excited states are of the order of ten volts. This means that there are about four hundred thousand states between the ground state of uranium 239 and these excited states that are observed close to the neutron binding energy.

Wigner said, it makes no sense to try to write down a Hamiltonian or to solve the Schrödinger equation for these highly excited states, so let us go to the opposite extreme and assume that we know nothing at all about the Hamiltonian. Let us assume that the Hamiltonian for the uranium 239 nucleus is a matrix chosen at random from a suitable

ensemble. Wigner chose the ensemble to be the set of all real symmetric N by N matrices with each matrix element an independent Gaussian random variable. The assumption that a state of the nucleus is an eigenfunction of such a random matrix tells us nothing about the properties of the individual state. But, as Wigner discovered, the assumption that a set of states are eigenfunctions of the same random matrix tells us a great deal about the statistical properties of the eigenvalues. The eigenvalues of a random matrix are not randomly distributed but show strong correlations which can be calculated precisely. The distribution of spacings between nearest-neighbor eigenvalues was calculated analytically by Michel Gaudin in 1961, and was found to differ very little from the empirical formula conjectured earlier by Wigner. This was a beautiful piece of analysis. It was Gaudin who first understood the intimate connection between random matrices and Fredholm determinants.

When I began working on random matrices in 1962, the primary motivation was still, as it had been for Wigner, to use the random matrix model to interpret experiments in nuclear physics. The experiments in the 1960s were done at the Nevis cyclotron of Columbia University by James Rainwater. Rainwater had developed the art of time-of-flight neutron spectroscopy to a high degree of refinement. The trick was to use a rapidly rotating shutter to let out a short pulse of slow neutrons with a wide range of energies, then let them travel a long distance before measuring their absorption in uranium or some other absorbing material. The time-of-flight of each neutron was inversely proportional to the square root of its energy. The peaks of absorption as a function of time gave an accurate measurement of the neutron capture levels as a function of energy. Rainwater measured long series of capture levels in uranium, thorium and other heavy elements. The most interesting for testing the random matrix model turned out to be Erbium 166. The compound nucleus Erbium 167 happens to have the longest series of levels within the range that Rainwater could measure accurately. I worked with Rainwater and a graduate student called Henry Camarda analysing the experiments. Camarda is now a professor at Penn State. Rainwater, after sharing a Nobel Prize with Aage Bohr and Ben Mottelson for discovering that many heavy nuclei are non-spherical and have states characterized by rotational quantum numbers, died in 1986.

2. Cryptography and Error-Correcting Codes

When we analysed the Columbia experiments using random matrix theory, we were doing a kind of cryptography. The sequence of observed energy levels can be considered as a secret message. The nucleus is telling us something about itself, and the energy levels are the channel it is using for the communication. Our job as cryptographers is to extract all the information we can from the encrypted message. If the levels were randomly distributed and uncorrelated, the channel would be a hundred percent efficient and would carry the maximum amount of information without any redundancy. But the eigenvalues of a random matrix are strongly correlated, so that they carry less information than a random sequence. The energy level channel is redundant, and the redundancy causes a loss of information. If we could measure the loss of information, we might learn something about the nature of the channel.

To calculate the loss of information, we introduced another physical model which we called the Coulomb Gas model. The Coulomb Gas model is a set of equal point charges moving on a line, each pair of charges repelling each other with a force inversely proportional to the distance between them. It turns out that the statistical behavior of the charges in the Coulomb Gas model is mathematically identical to the statistical behavior of the eigenvalues of a random matrix. And the loss of information due to redundancy in the energy level channel is exactly equal to the negative entropy of the Coulomb Gas. The principle of equivalence of entropy and information, first discovered by Leo Szilard in 1928, says that information can be converted into negative entropy at a fixed rate. We avoid the discussion of the fixed rate by measuring entropy in information units, the unit of information being the bit.

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The one-dimensional Coulomb Gas belongs to the class of exactly integrable many-body systems, and so the entropy can be calculated exactly. The result which we calculated for the entropy was

$$(1/(2\log 2))(1-\gamma-\log\pi)N = -0.521N \text{ bits}, \tag{1}$$

where $\gamma = 0.577$ is Euler's constant, and N is the number of eigenvalues. This means that the loss of information due to redundancy is almost exactly one-half of a bit per energy level. The result was mathematically elegant but practically disappointing. One half of a bit is a miserably small amount of information. We wanted to verify whether the result (1) is actually true for the energy levels measured by Rainwater. We collected all the series of energy-levels measured in various elements up to that time. Given a finite sequence of levels, it should be possible to calculate the entropy and compare the result with the theoretical value (1). But the calculations turned out to be frustrating. Different level-series from different elements gave values for the entropy that were widely scattered. Most of the level-series were too short to be useful. The longest series were thorium 232 with 101 levels, uranium 238 with 96 and erbium 166 with 82. In spite of Rainwater's heroic efforts, the total number of levels measured was only around 2500. The wide scatter of the calculated entropies was probably due to the presence of missed or spurious levels. In a finite level-series, one or two missed or spurious levels could make a big difference to the entropy, producing a noise signal that drowned out the theoretical one-half-bit of entropy that we were trying to measure.

Having failed to verify the connection between random matrix theory and nuclear data, we decided to follow a different approach. The main trouble with the data was that it was probably corrupted with missing and spurious levels. The visibility of a level depended on the strength of the neutron absorption. Since the levels were of widely different strengths, there was always a chance that a very weak level would be missed. There was also a chance that a weak level arising from some state with higher angular momentum would be detected and included in the series. The random matrix model was only supposed to apply to a series of levels with the same angular momentum, and states of higher angular momentum were supposed to be too weak to be detected. But we had to expect that a small percentage of genuine levels would be missed and a small percentage of spurious levels would be included in the data.

Since, according to the theory, the level-series was a redundant channel of information, it should be possible to find an error-correcting code that would automatically correct the data. In principle, according to Shannon's theory of error-correcting codes, so long as the

information loss-rate due to errors is smaller than the redundancy of the channel, an error-correcting code should exist that converts corrupted data into uncorrupted data. Applied to the nuclear level data, the code should automatically eliminate spurious levels and restore missed levels. This was a grand dream, that by creative use of pure mathematics we could clean up the messy results of nuclear experiments. It would be a great triumph for mathematics if an error-correcting code could correct Nature's mistakes as well as our own.

We worked hard to find an effective error-correcting code. We never found one that fulfilled our dreams. The best we could do was to find an algorithm that could identify with reasonable probability a single missed or spurious level in a long series of otherwise correct levels. The algorithm could correct at most one error at a time. The way the algorithm worked was simple. It was based on an optimized weight-function $f(x)$ defined by

$$f(x) = (1/2)\log[(1+\sqrt{(1-x^2)})/(1-\sqrt{(1-x^2)})], \quad |x| < 1, \tag{2}$$

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$$f(x) = 0, \quad |x| > 1. \tag{3}$$

For a picture of $f(x)$ see Figure 1. Given a series of levels E_j, and a particular level E_i to be tested, you choose an interval L and form the sum

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$$F_i = \sum_{j \neq i} f((E_j - E_i)/L). \tag{4}$$

This is a score to test the validity of level E_i. It is a sum over all the levels E_j other than E_i between $(E_i - L)$ and $(E_i + L)$. The interval L is chosen to be about ten times the average level-spacing D.

Given an observed set of levels E_i, you test them by calculating the scores F_i. If the levels are an error-free sequence of eigenvalues of a random matrix, all the F_i will have the same expected value

$$n - \Delta, \tag{5}$$

with

$$n = (\pi L/D), \quad \Delta = \log n + 3\log 2 + \gamma - 2 = \log n + 0.656, \tag{6}$$

and their values will be scattered around this value with a statistical standard deviation of $\sqrt{\log n}$. If the level E_i is spurious, the score F_i has expectation value n, and the scores of neighboring levels will also be raised above the expected value (5). If a level E_k is missing, the scores of neighboring levels will be lowered below the expected value (5). Figure 2 shows the scores calculated by Henry Camarda for the 96 measured levels of uranium 238. The results are, as you see, horrible. The deviations of the scores from the expected values are so large that they cannot be explained by a few missing or spurious levels. It looks as if there must be a heavy concentration of spurious levels in the range between level 30 and level 60. In this case the error-correcting algorithm has failed completely. Figure 3 shows the scores calculated by Camarda for the 82 levels of erbium 166, the only nucleus for which the algorithm works reasonably well. Here the scores cluster around the expected value with the expected statistical scatter, except for the deviation around level 60 which could be explained as the effect of one or two spurious levels. But even in this most successful case, the algorithm is not precise enough to identify the spurious levels and correct the errors.

If this attempt to construct an error-correcting code using random matrix theory had succeeded, we would have published the results long ago and I would not be talking about it today. I decided to talk about it because it failed. Because it failed, we abandoned it, and nobody has tried to do anything with it for thirty years. The reason it failed was that the nuclear level data was qualitatively and quantitatively insufficient. We had too few levels and too many errors, so that an error-correcting code had no chance. As a contribution to nuclear physics, our work was worthless. I decided to talk about it because I still think it is an interesting mathematical problem. Now, thirty years later, I am no longer interested in the nuclear level data. I am interested in the general question, whether error-correcting codes for random matrix eigenvalues are possible in principle. I would like to know how to construct an efficient code and how high an error-rate it could in principle correct. These problems are unsolved, and that is why I talk about them today and recommend them to your attention. I suspect that they might lead you to some interesting mathematics.

In the theory of random matrices there are three types of matrix ensemble which we call orthogonal, unitary and symplectic. The corresponding Coulomb gas models have three different temperatures. The orthogonal model has temperature 1, the unitary model has temperature 1/2, and the symplectic model has temperature 1/4. In nuclear physics, because the nuclear Hamiltonian has rotational symmetry, only the orthogonal ensemble is relevant. This is unfortunate, because the orthogonal ensemble has the weakest repulsion between neighboring levels and the lowest redundancy of the three types of ensemble. The main reason why our error-correction scheme failed was the low redundancy of the information channel in the orthogonal case, calculated to be only 0.521 bits per level. The higher the temperature, the weaker the repulsion between levels, the smaller the negatave entropy, and the lower the redundancy. The unitary ensemble has a stronger repulsion of levels and a substantially higher redundancy, calculated to be $(\gamma/\log 2)$ or 0.833 bits per level. The symplectic ensemble is even better with a redundancy of 1.223 bits per level, more than twice the redundancy of the orthogonal ensemble. If we are studying the possibility of error-correcting codes as a problem of pure mathematics, not as a problem of nuclear physics, we may as well study unitary or symplectic eigenvalue series, for which the redundancies are higher and the chances of successful error-correction are more favorable. We will encounter the unitary ensemble again in a little while when I discuss the zeta-function zeros.

3. Quasicrystals

I am presenting a challenge to young mathematicians to study quasi-crystals and use them to prove the Riemann Hypothesis. Those of you who are experts in number theory may consider the challenge frivolous. Those who are not experts may consider it uninteresting or unintelligible. Nevertheless I am putting it forward for your serious consideration. You should always be guided by the wisdom of the physicist Leo Szilard, who wrote his own Ten Commandments to stand beside those of Moses. Szilard's second commandment says, "Let your acts be directed toward a worthy goal, but do not ask if they will reach it; they are to be models and examples, not means to an end". The Riemann Hypothesis is a worthy goal, and it is not for us to ask whether we can reach it. I will give you some hints describing how it might be achieved. Here I will be giving voice to the mathematician that I was fifty years

ago before I became a physicist.

There were until recently two supreme unsolved problems in the world of pure mathematics, the proof of Fermat's Last Theorem and the proof of the Riemann Hypothesis. Four years ago my Princeton colleague Andrew Wiles polished off Fermat's Last Theorem, and only the Riemann Hypothesis remains. Wiles' proof of the Fermat Theorem was not just a technical stunt. It required the discovery and exploration of a new field of mathematical ideas, far wider and more consequential than the Fermat Theorem itself. It is likely that any proof of the Riemann Hypothesis will likewise lead to a deeper understanding of many diverse areas of mathematics and perhaps of physics too. The Riemann Hypothesis says that one specific function, the zeta-function that Riemann named, has all its complex zeros upon a certain line. Riemann's zeta-function, and other zeta-functions similar to it, appear ubiquitously in number theory, in the theory of dynamical systems, in geometry, in function theory and in physics. The zeta-function stands at a junction where paths lead in many directions. A proof of the hypothesis will illuminate all the connections. Like every serious student of pure mathematics, when I was young I had dreams of proving the Riemann Hypothesis. I had some vague ideas that I thought might lead to a proof, but never pursued them vigorously. In recent years, after the discovery of quasi-crystals, my ideas became a little less vague. I offer them here for the consideration of any young mathematician who has ambitions to win a Fields Medal.

The fact that the zeta-function zeros have something to do with random matrices was discovered thirty years ago by Hugh Montgomery [Montgomery, 1973]. Montgomery was studying the pair-correlation function of zeta-function zeros. He assumed the Riemann Hypothesis to be true, so that all the non-trivial zeros are on the critical line. The pair-correlation function $p(x)$ is the probability density for finding two zeros, not necessarily nearest neighbors, with imaginary parts differing by xD, where D is the average level-spacing. Montgomery conjectured on the basis of a non-rigorous calculation that $p(x)$ should be an elementary function,

$$p(x) = 1 - (\sin \pi x / \pi x)^2. \tag{7}$$

He was able to prove only a weaker statement, that the Fourier transform of Eq. (7) holds for all frequencies not greater than one. But nobody doubts that Eq. (7) is true. Andrew Odlyzko [Odlyzko, 1989], computed eight million zeros near to zero number 10^{20} and computed their pair-correlation numerically. The results are shown together with Montgomery's conjecture (7) in figure 4. The agreement is perfect. You can even see the third wiggle around $x = 3$ in the numerical data. This reminds me of the third wiggle in the spectrum of the microwave background radiation anisotropy, recently measured by heroic efforts of microwave radio-astronomers, which provided the conclusive piece of evidence that we live in a flat accelerating universe. If you can see three wiggles in the right places, then you believe the results. So we believe that Eq. (7) is the correct pair-correlation function for the zeta-function zeros. But now comes the beautiful surprise. Eq. (7) is also the exact pair-correlation function for the unitary ensemble of random matrices, defined as the ensemble of complex Hermitian N by N matrices with the real and imaginary parts of each element independent Gaussian random variables. As you can see in figure 4, the unitary ensemble has a strong repulsion between pairs of levels. The repulsion is twice as strong in the unitary ensemble as it is in the orthogonal ensemble that we used as a model for heavy nuclei. The wiggles demonstrate that the zeta-function zeros have a vestigial long-range order, a memory-trace of a regular crystalline structure which makes differences between zeros prefer to be integer multiples of the average spacing. This long-range order is a well-known feature of random matrix eigenvalues.

So we have a striking but not yet understood connection between the zeta-function and random matrices. The zeta-function zeros behave as if they were the eigen-values of a random Hermitian matrix. This immediately suggests the question, whether we could pick out of the unitary ensemble the particular Hermitian matrix that has the zeta-function zeros as eigenvalues. If we could pick out the right matrix, we would know that all its eigenvalues are real, and we would know that all the zeta-function zeros are on the critical line. Peter Sarnak has extended Montgomery's work and found that not only the pair-correlation function but all the many-level correlation functions of the zeta-function zeros agree with those of the unitary random-matrix ensemble. The agreement of the many-level correlation-functions is still a conjecture, but nobody doubts that it is true. Sarnak has proved that the Fourier

transforms of the many-level correlation-functions are equal over a wide range of frequencies. Since Peter Sarnak will be talking here tomorrow and knows twenty times as much about this subject as I do, I will not say more about it today. There is an excellent and readable account of this story in a lecture by Andrew Granville [Granville, 2002], published in the MSRI news-letter "Emissary" in Spring 2202.

Now I jump from the Riemann Hypothesis to quasi-crystals. Quasi-crystals were one of the great unexpected discoveries of recent years. They were discovered in two ways, as real physical objects formed when alloys of aluminum and manganese are solidified rapidly from the molten state, and as abstract mathematical structures in Euclidean geometry. They were unexpected, both in physics and in mathematics. I am here discussing only their mathematical properties.

Quasi-crystals can exist in spaces of one, two or three dimensions. From the point of view of physics, the three-dimensional quasi-crystals are the most interesting, since they inhabit our three-dimensional world and can be studied experimentally. From the point of view of a mathematician, one-dimensional quasi-crystals are more interesting than two-dimensional quasi-crystals because they exist in far greater variety, and the two-dimensional are more interesting than the three-dimensional for the same reason. The official mathematical definition of a quasi-crystal is as follows, [Senechal, 1995]. First we define a Delone set and then we define a quasi-crystal. Roughly speaking, a Delone set is a discrete set of points that is nowhere too concentrated and nowhere too sparse. Viewgraph 9

Delone Set. A Delone Set is a set of points D in a Euclidean space of any number of dimensions, with positive numbers r and R such that (i) no sphere of radius r contains more than one point of D, and (ii) every sphere of radius R contains at least one point of D.

Quasicrystal. A quasicrystal is an aperiodic Delone set whose Fourier transform contains a countably infinite sum of delta-functions.

Speaking in a less formal way, we may say that a quasi-crystal is a distribution of discrete point masses whose Fourier transform is also a distribution of discrete point masses. Or, to put it more briefly, a quasi-crystal is a pure point distribution that has a pure point

spectrum. This definition includes as a special case the ordinary crystals which are exactly periodic point distributions with exactly periodic point spectra.

Excluding the ordinary crystals, quasi-crystals in three dimensions come in very limited variety, all of them being associated with the icosahedral rotation-group. The two-dimensional quasi-crystals are more numerous, roughly one distinct type associated with each regular polygon in a plane. The two-dimensional quasi-crystal with pentagonal symmetry is the famous Penrose tiling of the plane, discovered by Penrose before the general concept of quasi-crystal was invented. The Penrose tiling happens to belong to a special class of tilings called fractal. Here are the definitions of tiling and fractal.

Tiling. A tiling is a division of Euclidean space into a finite number k types of polyhedral cells.

Fractal Tiling. A tiling is fractal if a subset of its vertices forms an expanded tiling with tiles of the same k shapes increased in size by a fixed factor, and there exists a matrix A with integer elements such that every bigger tile of type i is formed by sticking together A_{ij} small tiles of type j. We call A the inflation-matrix of the tiling.

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The Penrose tiling has pentagonal symmetry. The tiles are rhombuses of two kinds, fat and skinny. All the angles are $(j\pi/5)$ with $j = 1, 2, 3$ or 4. The area of the fat tiles is $F = \sin(2\pi/5)$, the area of the skinny tiles is $S = \sin(\pi/5)$. The inflation rules for skinny and fat tiles are $S \to F, F \to (F + S)$. The inflation-matrix is $A = [11, 10]$, with eigenvalues $[F/S, -S/F]$. These eigenvalues are the two roots of $X^2 - X - 1 = 0$, numerically 1.618 and -0.618, the golden ratio and its negative inverse.

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Paul Steinhardt discovered a fractal tiling with heptagonal symmetry. The Steinhardt tiling has three kinds of rhombus, fat, medium, skinny. All their angles are multiples of $\pi/7$. The areas of the tiles are, fat $F = \sin(3\pi/7)$, medium $M = \sin(2\pi/7)$, skinny $S = \sin(\pi/7)$. The inflation rules for skinny, medium and fat are, $S \to F, M \to (F+M), F \to (F+M+S)$. The inflation-matrix is $A = [111, 110, 100]$, with the eigenvalues $[F/S, S/M, -M/F]$, roots of $X^3 - 2X^2 - X + 1 = 0$. The biggest eigenvalue is roughly 9/4, the others are roughly [4/8, -8/9] and are smaller than unity. Figure 5 shows a picture that Steinhardt drew of

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the heptagonal tiling, illustrating another inflation scheme that sticks together little tiles to make big ones. Unfortunately, Paul Steinhardt has stopped working on quasi-crystals and is now a famous cosmologist, studying inflation schemes in the history of the universe instead of inflation schemes in fractal tilings.

The connection between quasicrystals and fractal tilings was found by Enrico Bombieri and Jean Taylor. First we define some special algebraic numbers known as Pisot and Salem numbers.

Definition. A **Pisot number** is a positive real algebraic integer whose conjugates all lie inside the unit circle, [Bertin et al., 1992].

Definition. A **Salem number** is a positive real algebraic integer whose conjugates all lie inside or on the unit circle, at least one being on the circle, [Bertin et al., 1992].

Bombieri-Taylor Theorem. The vertices of a fractal tiling with inflation-matrix A are a quasicrystal if and only if the largest eigenvalue of A is a Pisot number, [Bombieri and Taylor, 1987].

Siegel Theorem. The smallest Pisot number is $X = 1.324..$, the largest root of $X^3 - X - 1 = 0$, [Siegel, 1944].

Conjecture. The smallest Salem number is $Y = 1.17628..$, the largest root of $Y^{10} + Y^9 - Y^7 - Y^6 - Y^5 - Y^4 - Y^3 + Y + 1 = 0$. This is known to be a Salem number, but it is not known to be the smallest. It is not even known whether or not there are Salem numbers arbitrarily close to 1.

The Bombieri-Taylor theorem shows that both the Penrose and Steinhardt tilings are two-dimensional quasicrystals. This construction for a tiling with the symmetry of a regular N-gon gives a quasicrystal for $N = 5, 7, 8, 9, 10, 12, 14, 15, 18, 30$ and for no other values of N.

4. One-dimensional Quasicrystals

Quasi-crystals in one dimension have a far richer structure than in two or three dimenstions, since in one dimension they are not tied to any rotational symmetries. So far as I know, no complete enumeration of one-dimensional quasi-crystals exists. It is known that a quasi-crystal exists corresponding to every Pisot number. The set of all Pisot numbers is infinite and has a remarkable topological structure. The set is discrete and nowhere dense, but possesses limit-points, limit-points of limit-points, and so on to all orders. The set of all one-dimensional quasi-crystals has a structure at least as rich as the set of all Pisot numbers and probably much richer. We do not know for sure, but it is likely that a huge universe of one-dimensional quasi-crystals not associated with Pisot numbers is waiting to be discovered.

The class of one-dimensional quasi-crystals associated with Pisot numbers is constructed as follows. Suppose there is a fractal tiling of a line, with k types of tile, each tile of type j being an interval of the line with length L_j. Suppose that the tiling is fractal with inflation-matrix A. Then the vector L of tile-lengths is an eigenvector of A with eigenvalue X, where X is an algebraic number of degree k. When the inflation-matrix A is applied to the tiles, the length of each type of tile is multiplied by the same factor X. The tiling is a quasicrystal if and only if X is a Pisot number. If X is a Pisot number, it is the largest eigenvalue of A, and all the other eigenvalues are inside the unit circle.

Here are some examples of one-dimensional quasi-crystals. First, the Penrose tiling with inflation-matrix $A = [11, 10]$. The sequence of repeated inflations is

$F \rightarrow FS \rightarrow FSF \rightarrow FSFFS \rightarrow FSFFSFSF...$

The number of tiles at the successive stages of inflation is the Fibonacci sequence

$[1, 2, 3, 5, 8, 13, 21, 34, 55, ..]$.

Second, the Steinhardt tiling with inflation-matrix $A = [111, 110, 100]$. The sequence of repeated inflation is

$M \rightarrow FM \rightarrow FMSFM \rightarrow FMSFMFFMSFM...$

The number of tiles at successive stages of inflation is the sequence

$[1, 2, 5, 11, 25, 56, 126, 283, ..]$.

Third, the Perrin minimum-growth quasi-crystal with inflation-matrix

$A = [011, 100, 010]$. The sequence of inflations is

$S \to M, M \to F, F \to (M + S)$.

The number of tiles at successive stages of inflation is the Perrin sequence,

$[0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, ..]$.

This is the sequence of integers P_n satisfying the recurrence relation $P_{n+3} = P_{n+1} + P_n$. As n tends to infinity, the nth root of P_n tends to the minimum Pisot number $X = 1.324...$ Much more than this is true. For all $n > 9$, P_n is the nearest integer to X^n.

The Perrin sequence is as rich in interesting properties as the Fibonacci sequence. It is easy to prove that n divides P_n when n is prime. You can check this for

$n = 2, 3, 5, 7, 11, 13, 17, 19$

by looking at the numbers. Several people conjectured that the converse is true, so that the divisibility of P_n by n would be a necessary and sufficient primality test for n. If you look at the numbers up to $n = 10^5$, you do not see any composite n dividing P_n. If no such composite n existed, this would be the quickest primality test for a general integer n. But that would be too good to be true. No primality test for a general integer n is anywhere near that fast. It was a big achievement when some mathematicians at the Indian Institute of Technology at Kanpur recently found a primality test that requires only d^{12} computer operations for a general integer with d digits. To test the divisibility of P_n by n requires only a constant times d multiplications. If this were a sufficient test, we would have a primality test requiring only $(d^2 \log d)$ computer operations. In fact the Perrin test is not sufficient, but the first composite n for which it fails is $n = 271441 = 521^2$, [Adams and Shanks, 1982].

5. Riemann Hypothesis

Now we come to the punch-line of the argument connecting one-dimensional quasi-crystals with the Riemann Hypothesis. Let Z be the set of imaginary parts of the complex zeroes of the zeta-function. If the Riemann Hypothesis is true, the Fourier transform of Z is a sum of delta-functions with coeficients $1/k$ at all points $\pm \log p^k$, where p is a prime and k is a positive integer. If the Riemann Hypothesis is not true, then the Fourier transform of Z is a messy expression involving the zeros that are off the critical line, and it is definitely not a sum of delta-functions. Andrew Odlyzko, [Odlyzko, 1990], [figures 5 and 6] has published

a beautiful computer calculation of the Fourier transform of the zeta-function zeros. The calculation shows precisely the expected structure of the Fourier transform, with a sharp discontinuity at every logarithm of a prime-power integer and nowhere else. Z does not qualify as a quasicrystal according to the official definition, because it is not a Delone set. Nearest-neighbor distances between points of Z do not have a positive lower bound. But Z has the essential property of a quasicrystal, a discrete Fourier transform. I take the liberty of generalizing the definition of a quasi-crystal so that Z will qualify.

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Definition. A **generalized quasicrystal** is a set of points which is (i) aperiodic, (ii) locally finite, (iii) has points in every sphere of some radius R, and (iv) has a discrete Fourier transform. According to this definition, Z is a generalized quasicrystal if and only if the Riemann Hypothesis is true.

My proposal is the following. Let us pretend that we do not know that the Riemann hypothesis is true. Let us tackle the problem from the other end. Let us try to obtain a complete enumeration and classification of all one-dimensional quasi-crystals. That is to say, we enumerate and classify all point distributions that have a discrete point spectrum. We shall then find the well-known quasi-crystals associated with Pisot numbers, and also a whole slew of other quasi-crystals, known and unknown. Among the slew of other quasi-crystals we should be able to identify one corresponding to the Riemann zeta-function and one corresponding to each of the other zeta-functions that resemble the Riemann zeta-function. Suppose that we can prove rigorously that one of the quasi-crystals in our enumeration has properties that identify it with the zeros of the Riemann zeta-function. Then we have proved the Riemann hypothesis and can wait for the telephone call announcing the award of the Fields Medal.

I propose as an exercise for students of mathematics in the audience the following simple task. Classify all one-dimensional generalized quasicrystals and make a list of them. After you have done this, look at the list and see whether Z is there.

These are of course idle dreams. The problem of classifying one-dimensional quasi-crystals is horrendously difficult, probably at least as difficult as the problems that Andrew

Wiles took seven years to fight his way through. But still, the history of mathematics is a history of horrendously difficult problems being solved by young people too ignorant to know that they were impossible. The classification of quasi-crystals is a worthy goal, and might even turn out to be achievable. Problems of that degree of difficulty will not be solved by old people like me. Let the young people now have a try.

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(1)

Random Matrices, Neutron Capture Levels, Quasicrystals and Zeta-function Zeros

Freeman J. Dyson, Institute for Advanced Study, Princeton, New Jersey

Talk given at Mathematical Sciences Research Institute
Berkeley, California, September 23, 2002

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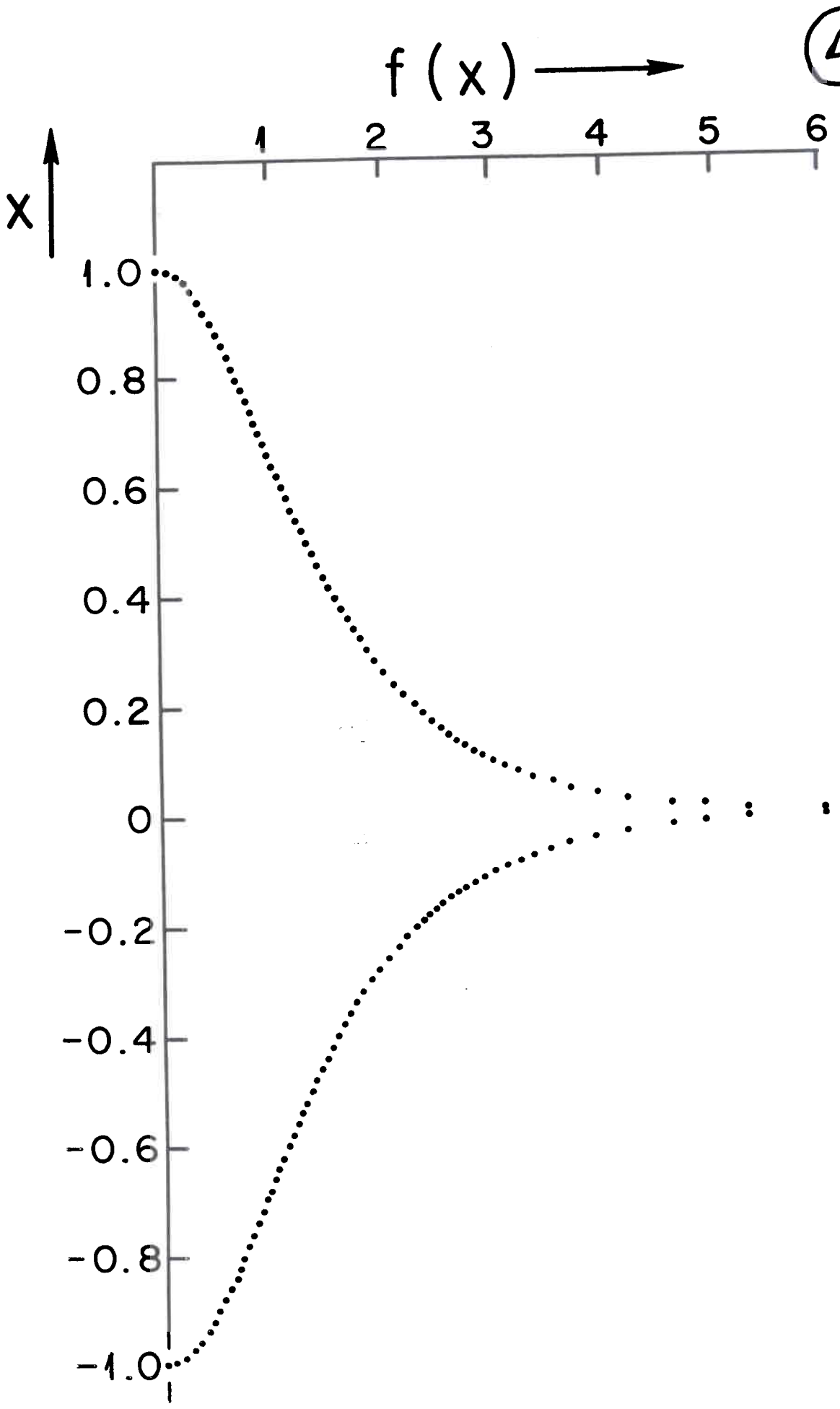
Entropy of classical one-dimensional Coulomb gas, equal to minus the redundancy of information in a series of N eigenvalues of a random real symmetric matrix,

$$(1/(2\log 2))(1-\gamma-\log\pi)N = -0.521N \text{ bits.} \tag{1}$$

Optimized weight-function $f(x)$ for discriminating spurious or missing levels,

$$f(x) = (1/2)\log[(1+\sqrt{(1-x^2)})/(1-\sqrt{(1-x^2)})], \quad |x|<1, \tag{2}$$

$$f(x) = 0, \quad |x|>1. \tag{3}$$



Slide 1

Score for testing validity of level E_i,

$$F_i = \sum_{j \neq i} f((E_j - E_i)/L). \tag{4}$$

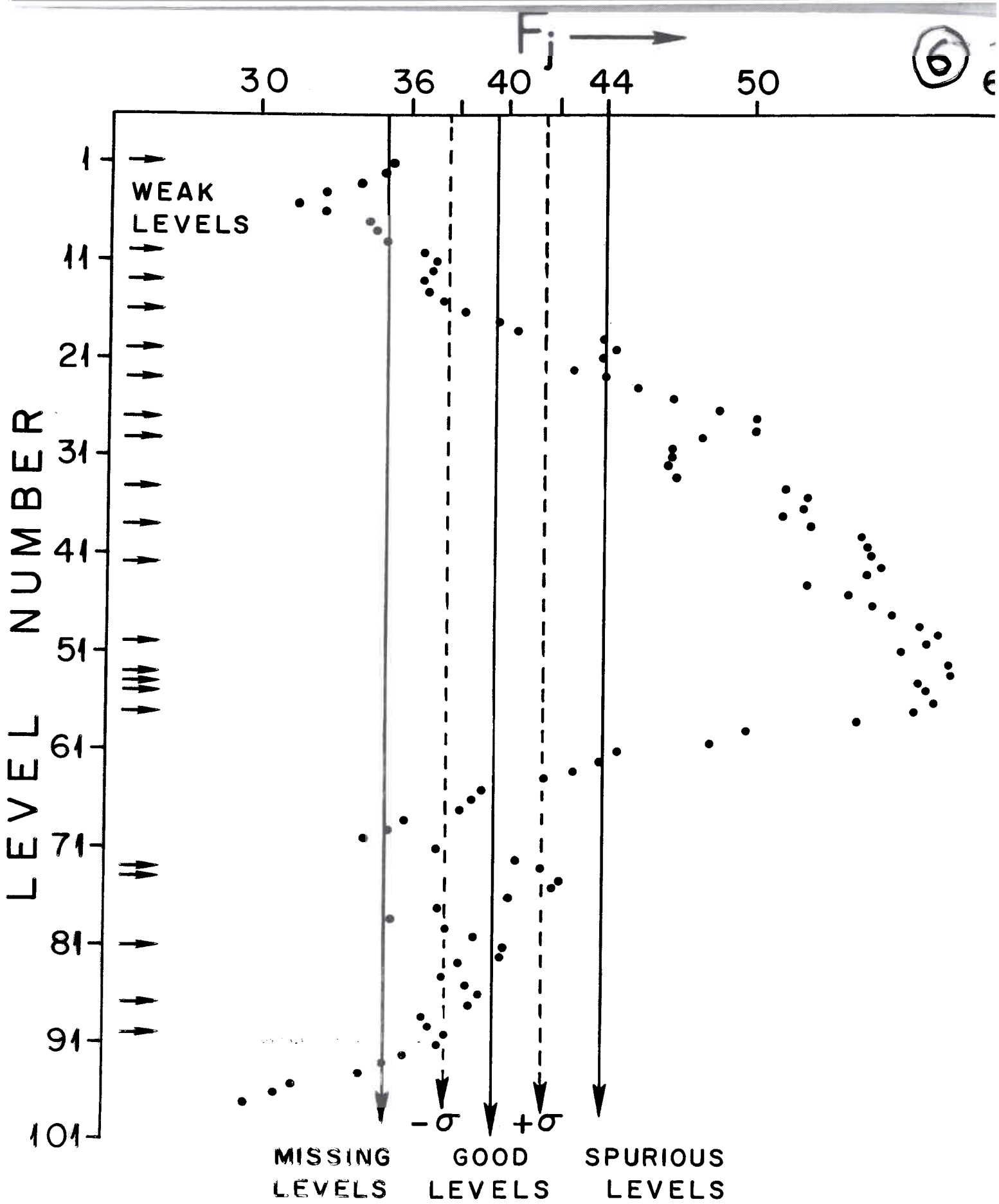
Expectation value of F_i is n for spurious level,

$$n - \Delta, \tag{5}$$

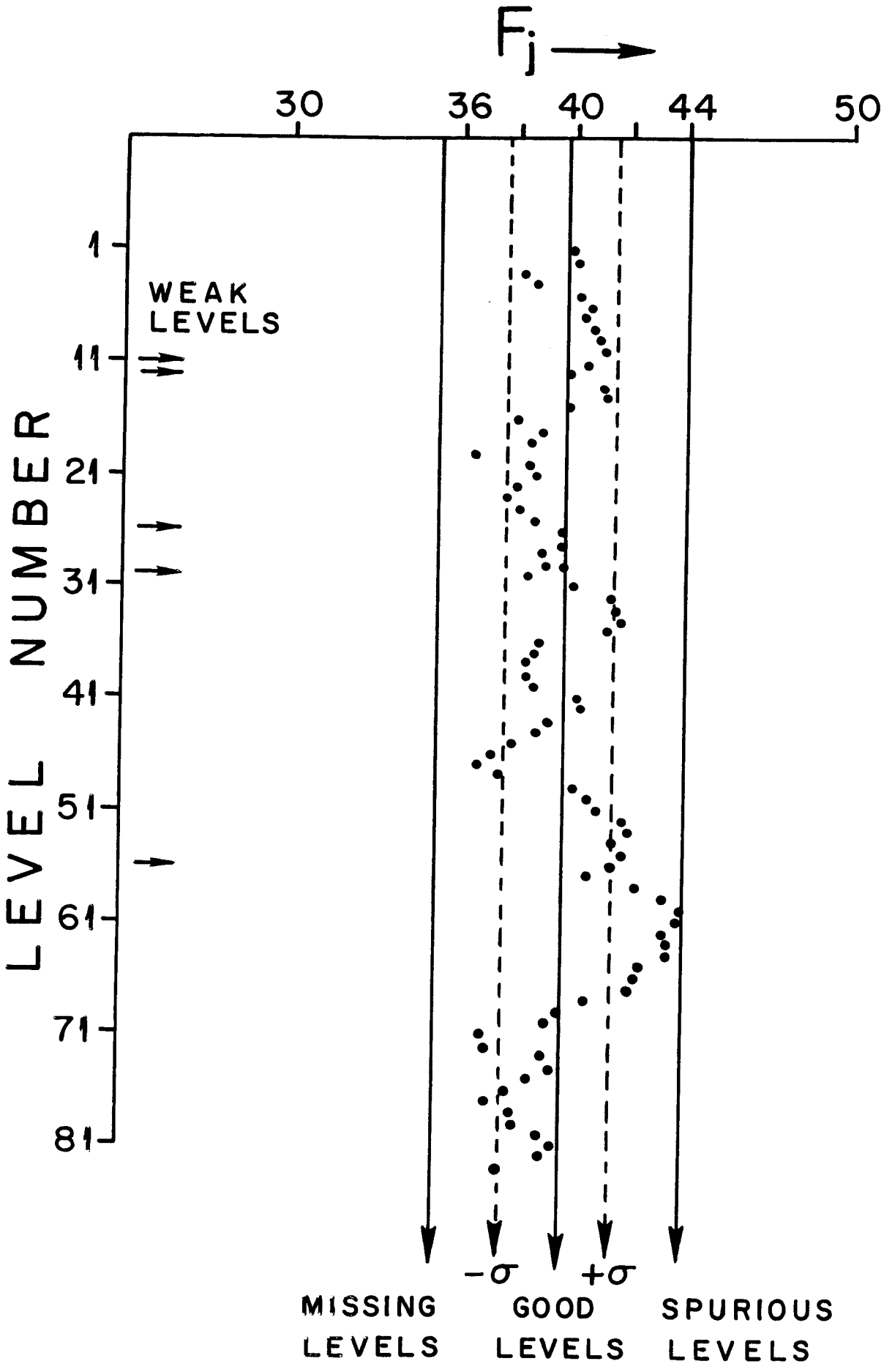
for valid level, with

$$n = (\pi L/D), \quad \Delta = \log n + 3\log 2 + \gamma - 2 = \log n + 0.656. \tag{6}$$

Statistical standard deviation of F_i is $\sqrt{\log n}$.



URANIUM 238 M = 14



ERBIUM 166 M = 14

Redundancy of information in unitary eigenvalue series,

$$(\gamma/\log 2)N = 0.833N \text{ bits.} \tag{7}$$

Redundancy in symplectic eigenvalue series,

$$(2\gamma + \log 2 - 1)/\log 2)N = 1.223N \text{ bits.} \tag{8}$$

Montgomery conjecture for pair-correlation function of zeta-function zeros, where x is the difference between two zeros divided by the mean spacing,

$$p(x) = 1 - (\sin \pi x/\pi x)^2, \tag{9}$$

identical with pair-correlation function of eigenvalues of random complex Hermitian matrix.

8a

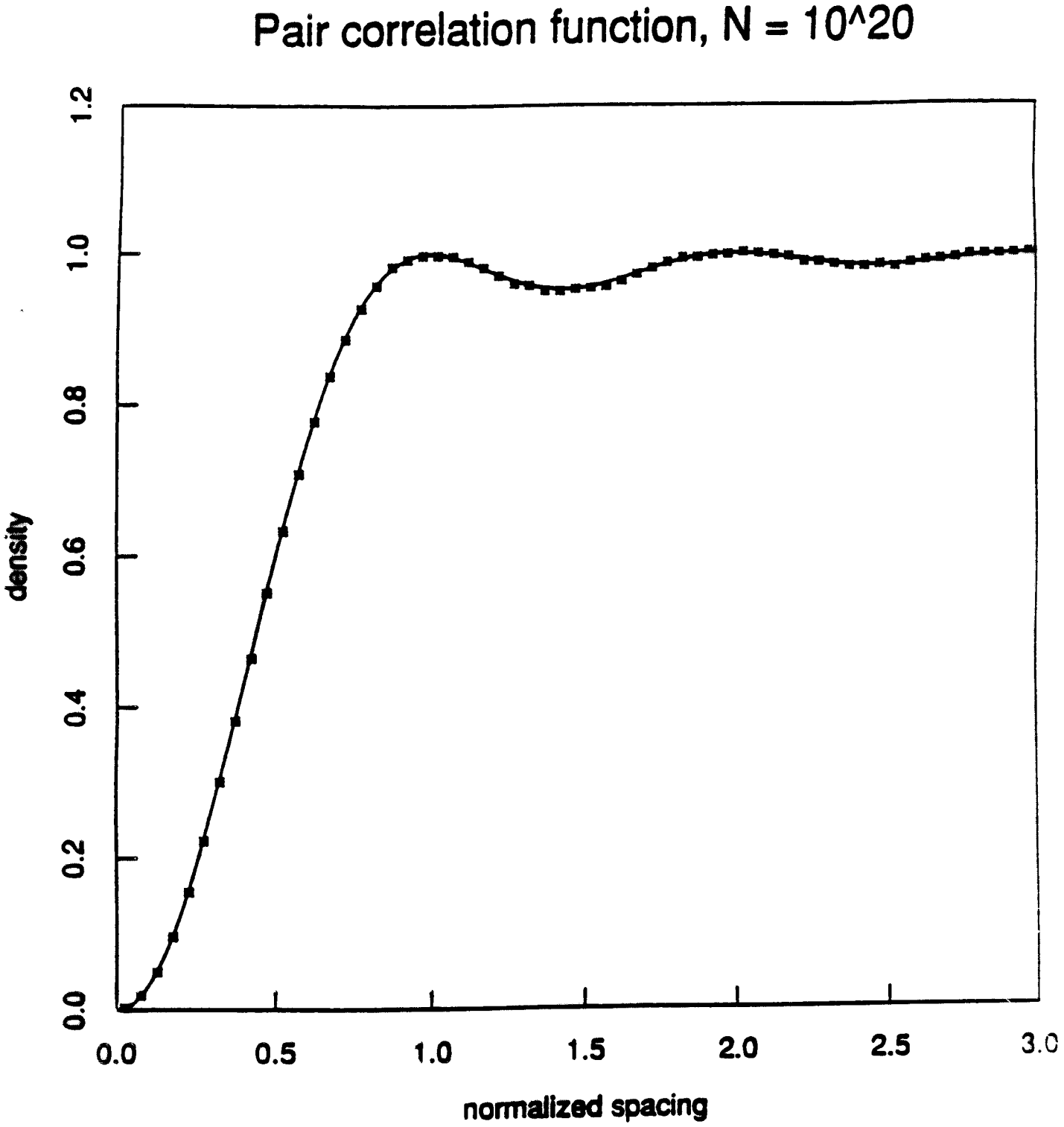


Figure 2.3.1. **Pair correlation of zeros of the zeta function. Solid line:** GUE **prediction. Scatterplot: empirical data based on** 8×10^6 **zeros near zero number** 10^{20}.

1. DEFINITIONS

Delone Set. A Delone Set is a set of points D in R_n with positive numbers r and R such that (i) no sphere of radius r contains more than one point of D, and (ii) every sphere of radius R contains at least one point of D.

Quasicrystal. A quasicrystal is an aperiodic Delone set whose Fourier transform contains a countably infinite sum of delta-functions.

Tiling. A tiling is a division of R_n into a finite number k types of polyhedral cells.

Fractal Tiling. A tiling is fractal if a subset of its vertices forms a tiling with bigger tiles of the same k types, and there exists a matrix A with integer elements such that every bigger tile of type i is formed by sticking together A_{ij} small tiles of type j. We call A the inflation-matrix of the tiling.

2. EXAMPLES

Penrose tiling with pentagonal symmetry. Two kinds of rhombus, fat and skinny. Area of tiles, $F = \sin(2\pi/5), S = \sin(\pi/5)$. Inflation rules, $S \to F, F \to (F + S)$.

Matrix $A = [11, 10]$, eigenvalues $[F/S, -S/F]$, roots of $X^2 - X - 1 = 0$.

Steinhardt tiling with heptagonal symmetry. Three kinds of rhombus, fat, medium, skinny.

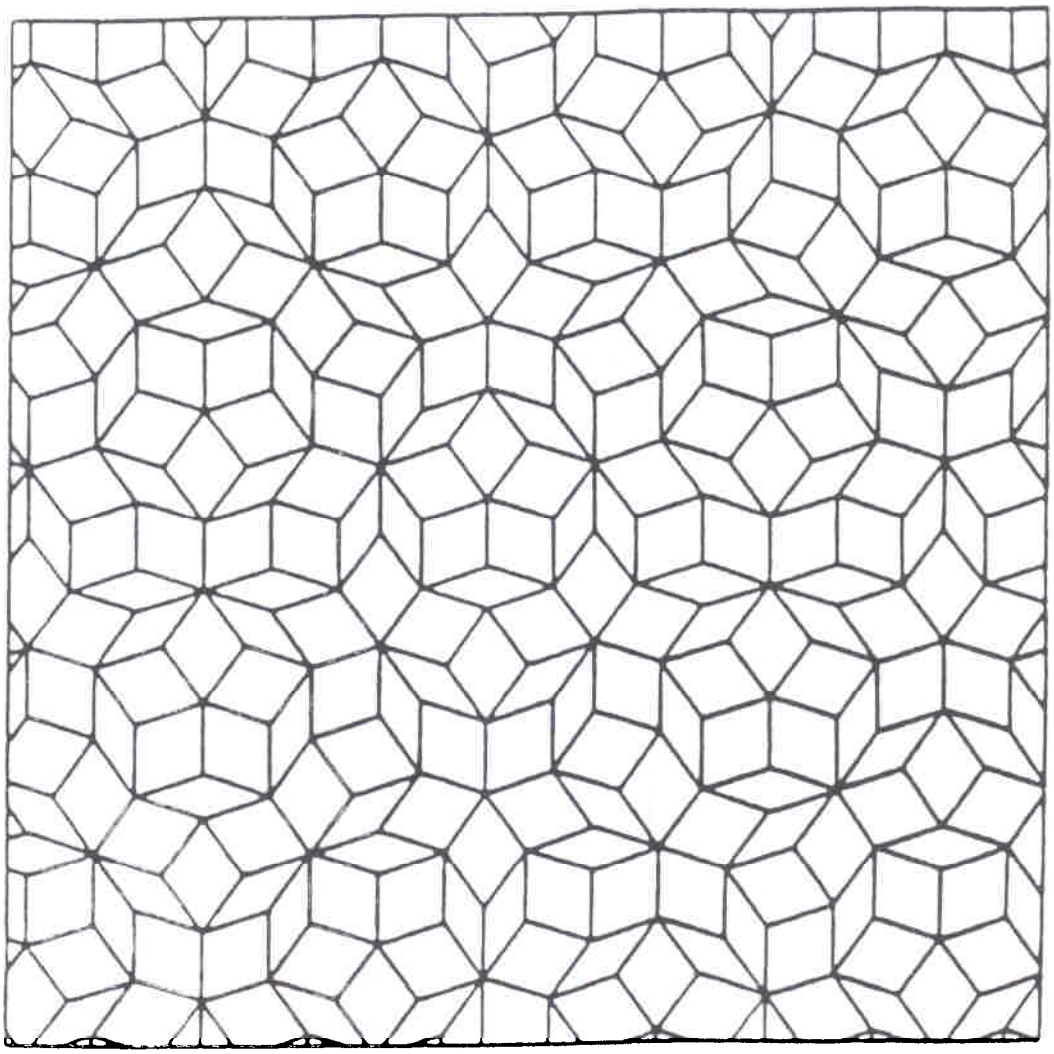
Areas of tiles, $F = \sin(3\pi/7), M = \sin(2\pi/7), S = \sin(\pi/7)$.

Inflation rules, $S \to F, M \to (F + M), F \to (F + M + S)$.

Matrix $A = [111, 110, 100]$, eigenvalues $[F/S, S/M, -M/F]$, roots of $X^3 - 2X^2 - X + 1 = 0$.

Biggest eigenvalue roughly 9/4, others [4/8, -8/9] smaller than unity. Tiling is a quasicrystal by Bombieri-Taylor theorem.

This construction for a tiling with the symmetry of a regular N-gon gives a quasicrystal for $N = 5, 7, 8, 9, 10, 12, 14, 15, 18, 30$ and for no other values of N.



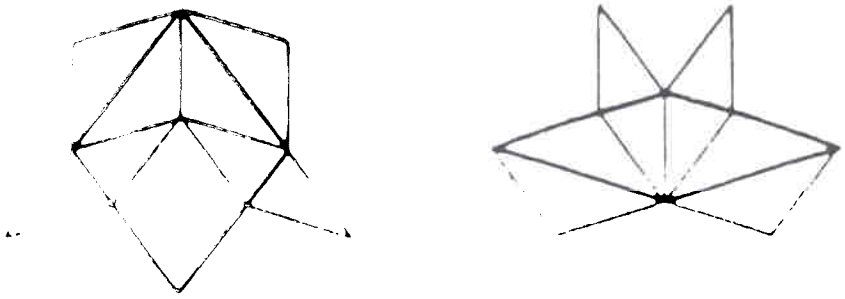
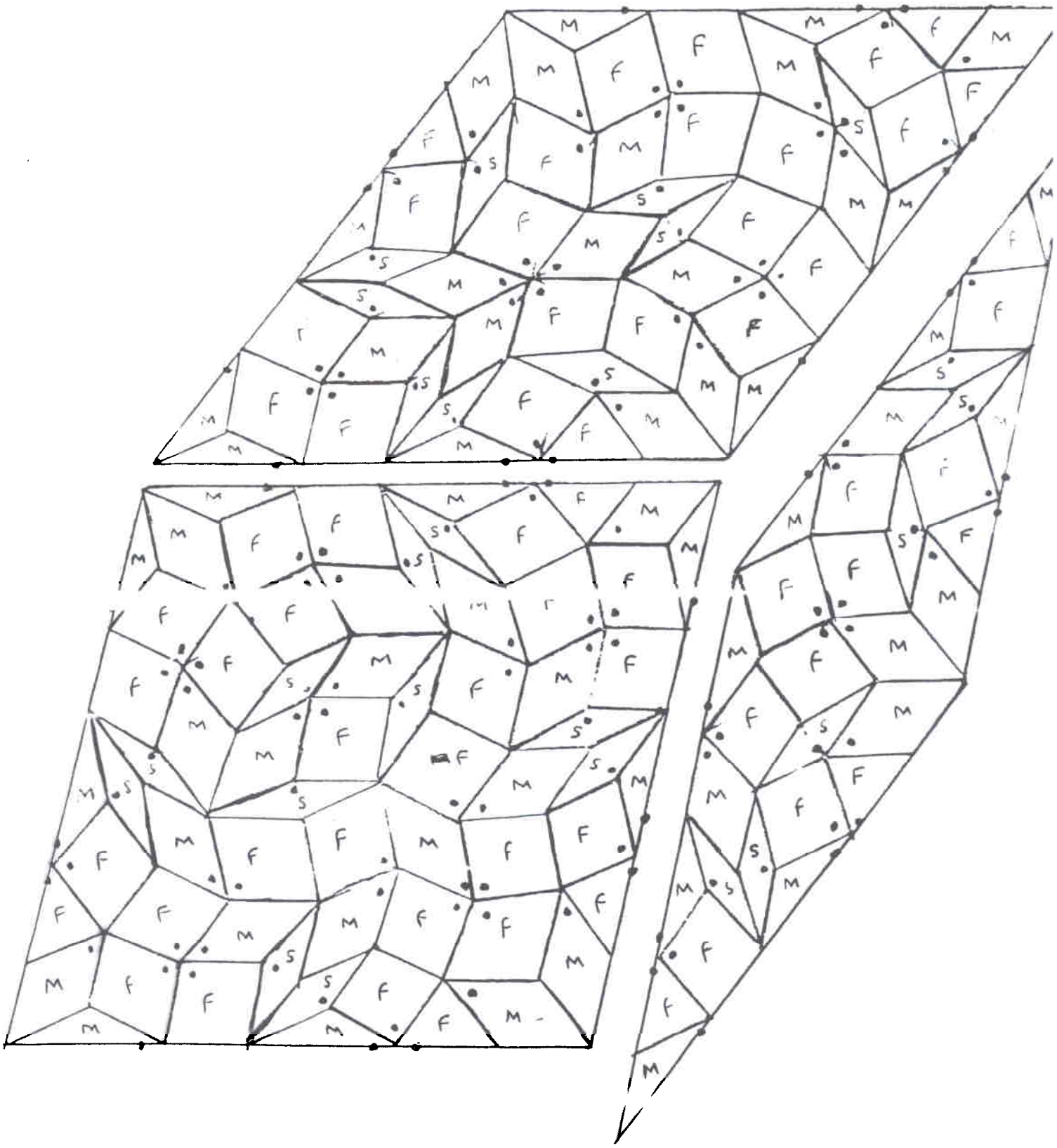


Figure 1. Portion of a Penrose tiling with fat and skinny rhombus cells. The deflation rules for the two cells are shown at the bottom.

STEINHARDT HEPTAGONAL QUASICRYSTAL

INFLATION SCHEME



3. Bombieri-Taylor Theorem

Definition. A **Pisot number** is a positive real algebraic integer whose conjugates all lie inside the unit circle.

Definition. A **Salem number** is a positive real algebraic integer whose conjugates all lie inside or on the unit circle, at least one being on the circle.

Bombieri-Taylor Theorem. A fractal tiling with inflation-matrix A is a quasicrystal if and only if the largest eigenvalue of A is a Pisot number.

Siegel Theorem. The smallest Pisot number is $X = 1.324..$, the largest root of $X^3 - X - 1 = 0$.

Conjecture. The smallest Salem number is $Y = 1.17628..$, the largest root of $Y^{10} + Y^9 - Y^7 - Y^6 - Y^5 - Y^4 - Y^3 + Y + 1 = 0$. It is not even known whether or not there are Salem numbers arbitrarily close to 1.

4. One-dimensional Quasicrystals

Take a fractal tiling with inflation-matrix A. Each tile is an interval on a line. Length of tile of type j is L_j. Vector L is eigenvector of A with largest eigenvalue X.

When inflation-matrix A is applied to the tiles, the length of each type of tile is multiplied by the same factor X. Tiling is a quasicrystal if and only if X is a Pisot number.

Examples

Penrose 5-fold. $A = [11, 10]$. Inflation sequence is

$F \to FS \to FSF \to FSFFS \to FSFFSFSF...$

Number of tiles is Fibonacci sequence [1,2,3,5,8,13,21,34,55,..].

Steinhardt 7-fold. $A = [111, 110, 100]$. Inflation sequence is

$M \to FM \to FMSFM \to FMSFMFFMSFM...$

Number of tiles is sequence [1,2,5,11,25,56,126,283,..].

5. Failed Primality Test

Perrin minimum-growth quasicrystal. $A = [011, 100, 010]$. Inflation rules $S \to M, M \to F, F \to (M + S)$.

Number of tiles is Perrin sequence

[0,2,3,2,5,5,7,10,12,17,22,29,39,51,68,90,119,158,209,..]

Let n'th term in Perrin sequence be P_n. As n tends to infinity, the nth root of P_n tends to the minimum Pisot number $X = 1.324...$ Much more than this is true. For all $n > 9$, P_n is the nearest integer to X^n.

Easy to prove that n divides P_n when n is prime, as you can see for $n = 2, 3, 5, 7, 11, 13, 17, 19$.

Several people conjectured that the converse is true, so [n divides P_n] would be a necessary and sufficient primality test for n. If that were true, it would be the quickest primality test for general n. But it ain't true. The first n for which it fails is $n = 271441 = 521^2$.

6. Riemann Hypothesis

Let Z be the set of imaginary parts of the complex zeroes of the zeta-function. If RH is true, the Fourier transform of Z is a sum of delta-functions with coeficients $1/k$ at all points $\pm \log p^k$, where p is a prime and k is a positive integer. Odlyzko verified this numerically for various long segments of Z.

Z is not a quasicrystal because it is not a Delone set. Nearest-neighbor distances between points of Z do not have a positive lower bound. But Z has the essential property of a quasicrystal, a discrete Fourier transform.

Definition. A **generalized quasicrystal** is a set of points which is (i) aperiodic, (ii) locally finite, (iii) has points in every sphere of some radius R, and (iv) has a discrete Fourier transform. Then Z is a generalized quasicrystal if and only if RH is true.

Exercise for students. Classify all one-dimensional generalized quasicrystals. After you have done this, look at the list and see whether Z is there. If Z is there, you have proved RH.

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ODLYZKO

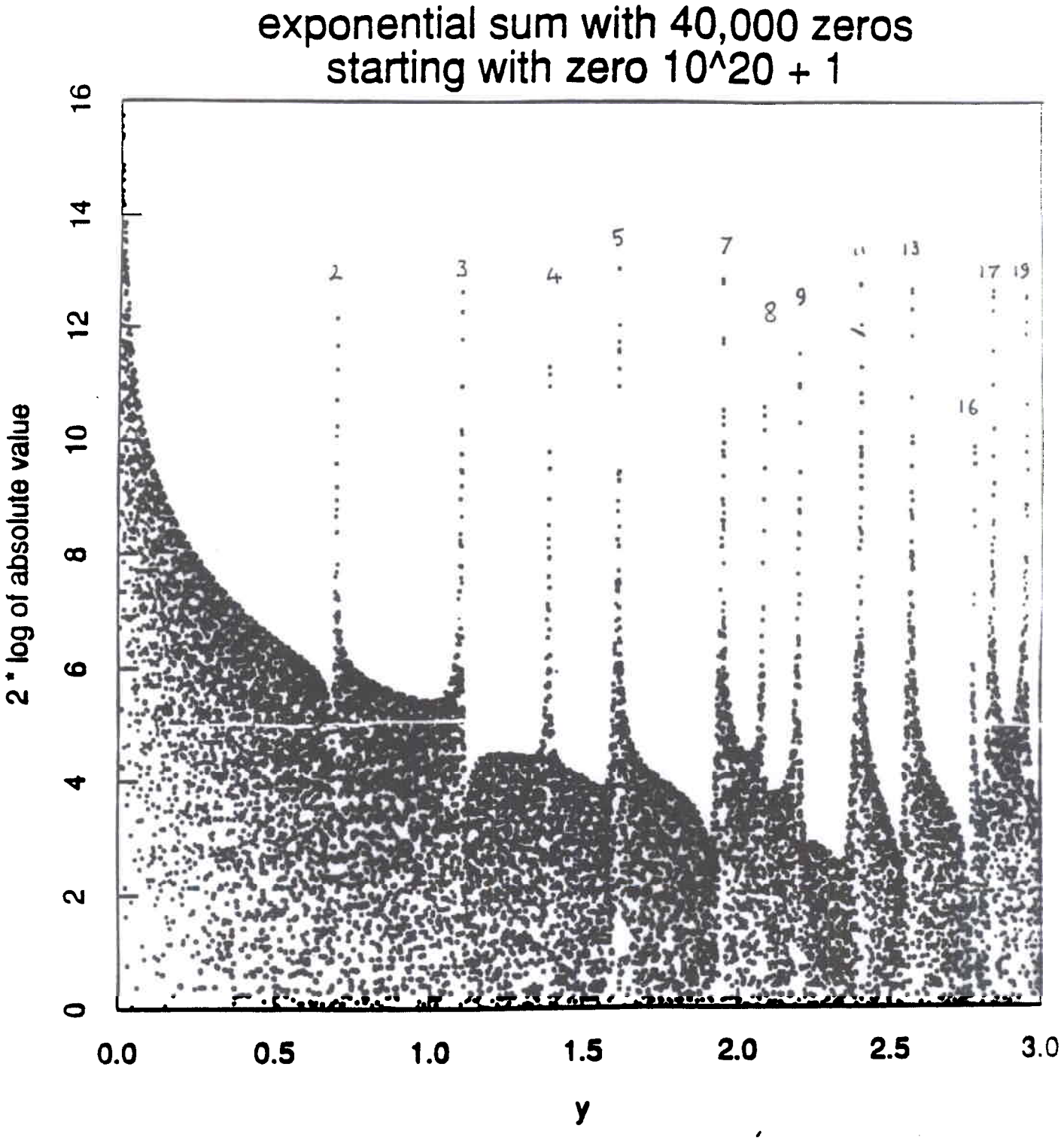

Figure 2.6.1. Graph of $2\log|\sum \exp(i\gamma_n y)|$, where n runs over $10^{20}+1 \le n \le 10^{20}+40{,}000$, and values < 0 and > 16 are deleted.

ODLYZKO

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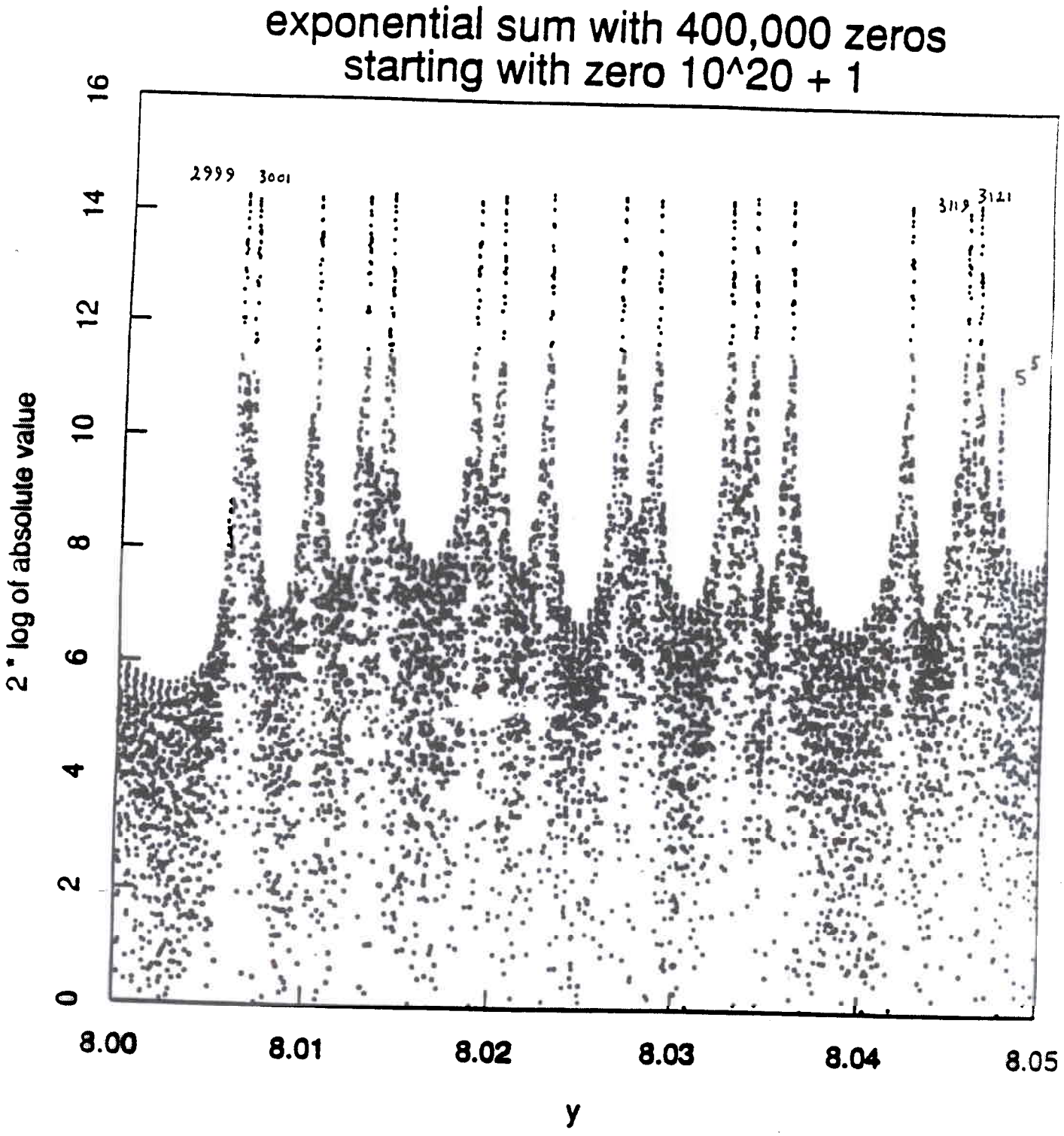

Figure 2.6.3. Graph of $2\log|\sum\exp(i\gamma_n y)|$, where n now runs over $10^{20}+1 \le n \le 10^{20}+400{,}000$, and values < 0 are deleted.