Virasoro and random matrices, permutations and walks.

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joint work with MARK ADLER

The main emphasis will be on the following two integrals

INTEGRAL (i)

$$I_n(t) := \int_{E^n} |\Delta_n(z)|^{2\beta} \prod_{k=1}^{n} \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k \right)$$

$E \subset \mathbb{R}$

e.g. $e^{-\sum_{i\geq 1} \frac{z^i}{i}} = 1-z$

INTEGRAL (ii)

$$I_n(t,s) = \int_{U(n)} \det(M)^{\gamma} e^{\sum_1^\infty \operatorname{Tr}(t_j M^j - s_j \bar{M}^j)} dM$$

$$= \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} z_k^{\gamma} e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k},$$

- relation to integrable systems
- relation to combinatorics & probability

These integrals appear in many problems: e.g.

1. Spectrum of random matrices

2. Integrals over groups, symmetric spaces, ...

$\approx$ { statistics of the length of longest increasing subsequences for { . random permutations

. random words

. random involutions

⋮

3. Testing independence of two normal populations.

(multivariate statistics)

4. non-intersecting random walks and Brownian motions.

I. Integrals over intervals $\subset \mathbb{R}$, Virasoro Constraints and associated integrable systems:

Measure: $\rho(z)dz := e^{-V(z)}dz$ on $E \subseteq \mathbb{R}$,

$$\begin{cases} -\dfrac{\rho'}{\rho} = \dfrac{g}{f} = \dfrac{\sum_{i\geq 0} b_i z^i}{\sum_{i\geq 0} a_i z^i} \\ \text{Disjoint union } \quad E = \bigcup_1^r \ [c_{2i-1}, c_{2i}] \subset \mathbb{R}. \end{cases}$$

(i) Virasoro constraints

The multiple integrals

$$I_n(t,c) := \int_{E^n} |\Delta_n(z)|^{2\beta} \prod_{k=1}^{n} \left(e^{\sum_1^\infty t_i z_k^i} \rho(z_k) dz_k \right)$$

satisfy the Virasoro constraints for all $N \geq -1$:

infinite subalgebra

$$\left(-\sum_1^{2r} c_i^{N+1} f(c_i) \frac{\partial}{\partial c_i} \right.$$

$$\left. + \sum_{i\geq 0} \left(a_i \ \mathbb{J}^{(2)}_{N+i}(t,n) - b_i \ \mathbb{J}^{(1)}_{N+i+1}(t,n) \right) \right) I_n(t,c) = 0,$$

linear diff. operator
in $t_1, t_2, \ldots$, at most
second order.

The generators $\mathbb{J}_k^{(2)}$ et $\mathbb{J}_k^{(1)}$ form an algebra

Virasoro: $$\left[\mathbb{J}_k^{(2)}, \mathbb{J}_\ell^{(2)}\right] = (k-\ell)\,\mathbb{J}_{k+\ell}^{(2)} + c\left(\frac{k^3-k}{12}\right)\delta_{k,-\ell}$$

Heisenberg: $$\left[\mathbb{J}_k^{(1)}, \mathbb{J}_\ell^{(1)}\right] = \frac{2k}{\beta}\delta_{k,-\ell},$$

with central charge:

$$c = 1 - 6\left((\beta)^{1/2} - (\beta)^{-1/2}\right)^2.$$

Note: duality $\beta \longleftrightarrow 1/\beta$

<u>Proof</u>: The substitution

$$z_k \mapsto z_k + \varepsilon f(z_k) z_k^{N+1}$$

leaves the integral unchanged!

Remark : Componentwise, we have

$\mathbb{J}_k^{(1)}(t,n) = J_k^{(1)} + nJ_k^{(0)}$ and $\mathbb{J}_k^{(0)} = nJ_k^{(0)} = n\delta_{0k}$

and

$$\begin{aligned} &\mathbb{J}_k^{(2)}(t,n) \\ &\quad = \beta\, J_k^{(2)} + \Big(2n\beta + (k+1)(1-\beta)\Big)\, J_k^{(1)} \\ &\qquad + n\Big((n-1)\beta + 1\Big) J_k^{(0)}, \end{aligned}$$

where

$$\begin{aligned} J_k^{(1)} &= \frac{\partial}{\partial t_k} + \frac{1}{2\beta}(-k)t_{-k} \\ J_k^{(2)} &= \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{1}{\beta} \sum_{-i+j=k} i t_i \frac{\partial}{\partial t_j} \\ &\quad + \frac{1}{4\beta^2} \sum_{-i-j=k} i t_i j t_j . \end{aligned}$$

(ii) Associated integrable system

$$I_n = \int_{\mathbb{R}^n} |\Delta_n(z)|^{2\beta} \prod_{k=1}^{n} e^{-V(z_k)+\sum_1^\infty t_i z_k^i} dz_k$$

- $\underline{\underline{\beta = 1}}$

$$\begin{aligned} I_n &= n!\,\det(m_n(t)) \\ &= \int_{\mathcal{H}_n} e^{tr(-V(M)+\sum_1^\infty t_k M^k)} dM \end{aligned}$$

↑ n×n Hermitian matrices.

m_∞ : **Hänkel matrix** of moments

$\langle y^i, z^j \rangle = \int_{\mathbb{R}} z^{i+j} e^{-V(z)+\sum_1^\infty t_k z^k} dz$

Shift matrix

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty \iff \frac{\partial \langle y^i, z^j\rangle}{\partial t_k} = \langle y^i, z^{j+k}\rangle$$

Borel decomposition:

$m_\infty = S^{-1} S^{\top -1}$

lower-triang. ↑ ↑ upper-triangular.

Orthogonal polynomials:

$p_n(t;z) = (S(t)\chi(z))_n$ $\chi(z) := \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \end{pmatrix}$

$\langle p_n, p_m \rangle = \delta_{nm}$

$L = S\Lambda S^{-1} \left\{ \begin{array}{l} \textit{symmetric} \\ \textit{tridiagonal} \end{array} \right.$

$Lp = zp$ (*Three-step relations*)

Lax pair:

$$\frac{\partial L}{\partial t_i} = \left[-\frac{1}{2}\pi_{\mathrm{bo}}(L^i), L\right]$$

(Toda lattice)

for the decomposition

$$gl(\infty) = \text{Borel} + \text{skew}$$

KP-equation for $\tau_n := \det m_n$:

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right) \log \tau_n$$
$$+ 6\left(\frac{\partial^2}{\partial t_1^2} \log \tau_n\right)^2 = 0,$$

$$I_n = \int_{\mathbb{R}^n} |\Delta_n(z)|^{2\beta} \prod_{k=1}^{n} e^{-V(z_k)+\sum_1^\infty t_i z_k^i} dz_k$$

- $\beta = 1/2, 2$

$$\begin{aligned} I_n &= n! \,\mathrm{pf}\left(m_{\left\{ \begin{array}{c} n \\ 2n \end{array} \right\}}(t)\right) \\ &= \left\{ \begin{array}{l} \int_{\mathcal{S}_n} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_i X^i)} dX \\ \int_{\mathcal{T}_{2n}} e^{\mathrm{Tr}(-V(X)+\sum_1^\infty t_i X^i)} dX \end{array} \right. \end{aligned}$$

$\mathcal{S}_n = \{$ real symm. matrices $\}$

$\mathcal{T}_{2n} = \{$ Hermitian quaternionic $n \times n$ matr. $\}$

m_∞ : **skew-symmetric matrix:**

$$\langle y^i, z^j \rangle_t := \int\int_{\mathbb{R}^2} y^i z^j e^{\sum t_i (y^i + z^i)} 2D^\alpha \delta(y-z) \tilde{\rho}(y) \tilde{\rho}(z) dy\, dz \quad \left\{ \begin{array}{l} \alpha = -1 \\ \alpha = 1 \end{array} \right.$$

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty \Lambda^{\top k}$$

Skew-Borel decomposition:

$m_\infty = S^{-1} J S^{\top -1}$

$$J = \begin{pmatrix} \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} & & 0 \\ & \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} & \\ 0 & & \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \end{pmatrix}$$

Skew-orthogonal polynomials:

$p_n(t; z) = (S(t)\chi(z))_n$

$\langle p_n, p_m \rangle = J_{nm}$

$L = S \Lambda S^{-1}, \quad Lp = zp$

Lax pair:

$$\frac{\partial L}{\partial t_i} = [-\pi_{\mathfrak{k}}(L^i), L]$$

. **(Pfaff lattice)**

for the decomposition

$$gl(\infty) = \mathfrak{k} + sp(\infty)$$

Pfaff-KP-equation for $\tau_n :=$ pf (m_n):

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_2}\right)^2 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right)\log \tau_n$$
$$+6\left(\frac{\partial^2}{\partial t_1^2}\log \tau_n\right)^2 = 12\frac{\tau_{n-2}\tau_{n+2}}{\tau_n^2}.$$

II. Integrals over circle: Virasoro Contraints and associated integrable systems:

The multiple integrals

$$I_n(t,s) = \int_{U(n)} \det(M)^{\gamma} e^{\sum_1^\infty \operatorname{Tr}(t_j M^j - s_j \bar{M}^j)} dM$$

$$= \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} z_k^{\gamma} e^{\sum_1^\infty (t_j z_k^j - s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k},$$

(i) satisfy an algebra of Virasoro constraints:

$$\mathbb{V}_k I_n(t,s) = 0, \quad \text{for } k = -1, 0, 1,$$ (finite algebra)

with Virasoro operators:

$$\begin{aligned}
\mathbf{V}_{-1} &= \sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} \\
&\quad + n\left(t_1 + \frac{\partial}{\partial s_1}\right) - \gamma\frac{\partial}{\partial s_1} \\
\mathbf{V}_0 &= \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right) + \gamma n \\
\mathbf{V}_1 &= -\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i} \\
&\quad + n\left(s_1 + \frac{\partial}{\partial t_1}\right) + \gamma\frac{\partial}{\partial t_1}.
\end{aligned}$$

(ii) Associated integrable system

$$I_n = n! \det(m_n(t,s))$$

$$= \int_{U(n)} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)} dM$$

m_∞: **Toeplitz matrix:**

$\langle z^i, z^j \rangle = \int_{S^1} z^{i-j} e^{\sum_1^\infty (t_k z^k - s_k z^{-k})} dz$

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty, \qquad \frac{\partial m_\infty}{\partial s_k} = -m_\infty \Lambda^{\top k}$$

Borel decomposition:

$m_\infty = S_1^{-1}(t,s) S_2(t,s)$

lower upper-triangular

Bi-orthogonal polynomials:

$p_n^{(1)}(t,s;z) = (S_1(t,s)\chi(z))_n,$

$p_n^{(2)}(t,s;z) = (h\ S_2^{\top -1}(t,s)\chi(z))_n$

$\chi(z) := \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \end{pmatrix}$

$$\langle p_n^{(1)}, p_m^{(2)} \rangle = \delta_{nm} h_n$$

$$L_1 = \widehat{S}_1 \Lambda \widehat{S}_1^{-1}, \qquad L_2 = S_2 \Lambda^\top S_2^{-1}$$

$$L_1 := \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \\ & & & & \ddots \end{pmatrix}$$

and

$$L_2 := \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \\ & & & & \ddots \end{pmatrix}$$

with the variables

$$\begin{aligned} x_n(t,s) &:= (-1)^n \frac{\mathcal{I}_n^+(t,s)}{\mathcal{I}_n(t,s)} = p_n^{(1)}(t,s;0) \\ y_n(t,s) &:= (-1)^n \frac{\mathcal{I}_n^-(t,s)}{\mathcal{I}_n(t,s)} = p_n^{(2)}(t,s;0). \end{aligned}$$

Lax pair: (2-Toda lattice)

$$\frac{\partial L_i}{\partial t_n} = \left[\left(L_1^n\right)_+, L_i\right] \quad \text{and} \quad \frac{\partial L_i}{\partial s_n} = \left[\left(L_2^n\right)_-, \hat{L}_i\right]$$

preserve the shape of L_1 and L_2.

$$I_n^{\pm} = \int_{U(n)} (\det M)^{\pm 1} e^{\sum_1^\infty \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM$$

$I_n(t,s) = \int_{U(n)} e^{\sum_i \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)} (\det M)^{\gamma+\varepsilon} \, dM$

III. Unitary integral for special $(t_1, t_2, \ldots)$ and $(s_1, s_2, \ldots)$

$$\mathcal{L} = \begin{cases} it_i := \begin{cases} u_i - (\gamma_1' d_1^i + \gamma_2' d_2^i), & \text{for } 1 \le i \le N_1 \\ -(\gamma_1' d_1^i + \gamma_2' d_2^i) & \text{otherwise} \end{cases} \\ \\ is_i := \begin{cases} -u_{-i}, & \text{for } 1 \le i \le N_2 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

The following integral over the group $U(n)$:

$$I_n^{(\varepsilon)} := \int_{U(n)} e^{\mathrm{Tr}(P_1(M)+P_2(\bar{M}))} (\det M)^{\gamma+\varepsilon} \det(I - d_1 M)^{\gamma_1'} \det(I - d_2 M)^{\gamma_2'} \; dM$$

$$= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left(z_k^{\varepsilon} \rho(z_k) \frac{dz_k}{2\pi i z_k} \right)$$

(Toeplitz determinant)

for the weight ($N_1 = \deg P_1$ and $N_2 = \deg P_2$ differ by at most 1)

$$\rho(z) := e^{P_1(z)+P_2(z^{-1})} z^{\gamma} (1 - d_1 z)^{\gamma_1'} (1 - d_2 z)^{\gamma_2'}$$

The variables:

$$x_n = (-1)^n \frac{I_n^+}{I_n} \quad , \quad y_n = (-1)^n \frac{I_n^-}{I_n}$$

satisfy two inductive *rational* $N_1 + N_2 + 4$-step relations

$$\begin{aligned} x_n &= F_n(x_{n-1}, y_{n-1}, \cdots, x_{n-N_1-N_2-3}, y_{n-N_1-N_2-3}) \\ y_n &= G_n(x_{n-1}, y_{n-1}, \cdots, x_{n-N_1-N_2-3}, y_{n-N_1-N_2-3}) \end{aligned}$$

Then, the integrals I_n can be expressed as rational functions of x_i and y_i:

$$I_n = n! \; I_1^n \prod_1^{n-1} (1 - x_i y_i)^{n-i}.$$

satisfy:
"singularity confinement property".

Nature of the rational relations:

Set

$$v_n := 1 - x_n y_n.$$

Then

$$0 = \frac{x_n y_n}{v_n} \left\{ \begin{array}{l} \frac{1}{\tau_n^+}\left(a\mathbb{V}_{-1}^+ + b\mathbb{V}_0^+ + c\mathbb{V}_1^+\right) I_n^+ \\ + \frac{1}{I_n^-}\left(a\mathbb{V}_{-1}^- + b\mathbb{V}_0^- + c\mathbb{V}_1^-\right) I_n^- \\ - \frac{2}{I_n}\left(a\mathbb{V}_{-1} + b\mathbb{V}_0 + c\mathbb{V}_1\right) I_n \end{array} \right\}$$

$$= -\partial_n \left(\sum_{i\geq 1} \alpha_{i,n}\cancel{(t)} L_1^i - \sum_{i\geq 1} \beta_{i,n}\cancel{(t)} L_2^i \right)_{n,n} + (aL_2 - cL_1)_{nn}$$

Taking $\partial/\partial t_1$ of the relation above leads to a second difference relation:

$$\partial_n^2 \sum_{i\geq 1} \left(v_{n-1}\alpha_{i,n-1}\cancel{(t)} L_1^i - \beta_{i,n-1}\cancel{(t)} L_2^i \right)_{n,n-1} + \partial_n \left(c(L_1^2)_{nn} + b(L_1)_{nn} \right) = 0.$$

(Adler-PvM)

$$I_n = \int_{U(n)} e^{\sum \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)} dM$$

• **Example 1: Weight** $e^{\sqrt{t}(z+z^{-1})}$. Corresponding to locus:

$$\mathcal{L}_1 = \{\text{all } s_k = t_k = 0, \text{ except } t_1 = s_1 = \sqrt{t}\}$$

For the uniform probability $P(\pi_k) = 1/k!$ on the permutation group S_k and

$$L(\pi_k) = \left\{ \begin{array}{c} \text{length of the longest (strictly)} \\ \text{increasing subsequence of } \pi_k \end{array} \right\},$$

the generating function can be expressed as det(Toeplitz matrix):

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} P(L(\pi_k) \le n) = \int_{U(n)} e^{\sqrt{t}\,\mathrm{Tr}(M+\bar{M})} dM = I_n \Big|_{\mathcal{L}_1}$$

(Gessel/Rains)

$$= \exp \int_0^t \log\left(\frac{t}{u}\right) v_n(u)\, du$$

(Tracy-Widom)

(Adler - P v M)

Also

$$x_n = y_n = (-1)^n \frac{\int_{U(n)} (\det M) e^{\sqrt{t}\,\mathrm{Tr}(M+\bar{M})}}{\int_{U(n)} e^{\sqrt{t}\,\mathrm{Tr}(M+\bar{M})} dM} = p_n^{(1)}(0)$$

$$I_n \Big|_{\mathcal{L}_1} = n!\, I_1^n \prod_1^{n-1} (1 - x_i^2)^{n-i}$$

satisfies the 3-step McMillan equation

$$n x_n + \sqrt{t}(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0.$$

(Borodin)

(discrete P II)

and

$$v_n := 1 - x_n^2 = -\left.\frac{\partial^2 \log I_n}{\partial s_1 \partial t_1}\right|_{\mathcal{L}} = 1 - \left(p_n^{(1)}(0)\right)^2.$$

satisfies

$$\begin{cases} v_n'' - \dfrac{v_n'^2}{2}\left(\dfrac{1}{v_n - 1} + \dfrac{1}{v_n}\right) + \dfrac{v_n'}{t} - \dfrac{n^2}{2t^2}\dfrac{(v_n - 1)}{v_n} + \dfrac{2}{t} v_n(v_n - 1) = 0 \\ \qquad \textbf{(Painlevé V)} \\ \text{with } \; v_n(t) = 1 - \dfrac{t^n}{(n!)^2} + O(t^{n+1}), \text{ near } t = 0. \end{cases}$$

We have here two phenomena :

- rational maps related to discrete P II.
- integral representation in terms of cont. P V

This is obtained by eliminating the following functions

$$\frac{\partial}{\partial t_1} \log \frac{\tau_n}{\tau_{n-1}}\bigg|_{\mathcal{L}}, \quad \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}}\bigg|_{\mathcal{L}}, \quad \frac{\partial^2 \log \tau_n}{\partial s_2 \partial t_1}\bigg|_{\mathcal{L}},$$

from the 2-Toda equations and combinations of Virasoro's

$$\frac{\mathbb{V}_0 \tau_n}{\tau_{n_L}}\bigg|_{\mathcal{L}} = \frac{\mathbb{V}_0 \tau_n}{\tau_n} - \frac{\mathbb{V}_0 \tau_{n-1}}{\tau_{n-1}}\bigg|_{\mathcal{L}} = \frac{\partial}{\partial t_1} \frac{\mathbb{V}_{-1} \tau_n}{\tau_n}\bigg|_{\mathcal{L}} = 0$$

yielding an equation in

$$v_n = -\frac{\partial^2 \log \tau_n}{\partial s_1 \partial t_1}\bigg|_{\mathcal{L}}$$

- **Example 2: Weight** $(1+z)^{\alpha} e^{-sz^{-1}}$. Locus:

$\mathcal{L} = \{\text{all } it_i = -(-1)^i \alpha, \text{ all } s_i = 0, \text{ except } s_1 = s\}$.

Consider uniform probability $P(\pi_k) = 1/\alpha^k$ on

$$S_{k,\alpha} = \left\{ \begin{array}{l} \text{words } \pi_k \text{ of length } k \text{ from} \\ \text{an alphabet of } \alpha \text{ letters} \end{array} \right\}.$$

Then we have

$$\sum_{k=0}^{\infty} \frac{(\alpha s)^k}{k!} P\left\{ \begin{array}{l} \pi_k \in S_{k,\alpha} \\ L(\pi_k) \le n \end{array} \right\} = \int_{U(n)} \det(I+M)^{\alpha} e^{-s\, tr \bar{M}} dM$$

$$= \exp\left(sn + (n+\alpha) \int_0^{s} \frac{h_n(u)}{u} du \right)$$

(Tracy-Widom)
Adler-PvM

Thus spelled out, the variables

$$\left.\begin{array}{l} x_n \\ y_n \end{array}\right\} = (-1)^n \frac{\int_{U(n)} (\det M)^{\pm 1} \det(I+M)^{\alpha} e^{-s\, tr \bar{M}} dM}{\int_{U(n)} \det(I+M)^{\alpha} e^{-s\, tr \bar{M}} dM}$$

$$I_n = n!\, I_1^n \prod_1^{n-1} (1 - x_i y_i)^{n-i}$$

satisfy a 3-step and a 4-step relation (Adler-PvM)

$$-(n+\alpha+1)\underline{\underline{x_{n+1}}}y_n - s x_n \underline{\underline{y_{n+1}}} + (n+\alpha-1)x_n y_{n-1} + s x_{n-1} y_n =$$

$$-v_n\big((n+\alpha+1)\underline{\underline{x_{n+1}}}y_{n-1} - s\big) + v_{n-1}\big((n+\alpha-2)x_n y_{n-2} - s\big)$$
$$+ x_n y_{n-1}(x_n y_{n-1} - 1) = v_1(s-(2+\alpha)x_2) + x_1(x_1 - 1)$$

The function h_n satisfies **Painlevé V**

$$\left\{\begin{array}{l} h''' - \dfrac{h''^2}{2}\left(\dfrac{1}{h'+1}+\dfrac{1}{h'}\right) + \dfrac{h''}{u} + \dfrac{2(n+\alpha)}{u}h'(h'+1) \\ -\dfrac{1}{2u^2h'(h'+1)}\Big((u-n)h' - h - n\Big)\Big((2h+u+n)h' + h + n\Big) = \\ \\ \text{with} \quad h_n(u) = u\dfrac{\alpha-n}{\alpha+n} - \dfrac{u^{n+1}}{(n+1)!}\dbinom{\alpha+n-1}{n} + O(u^{n+2}). \end{array}\right.$$

Setting

$$h_n = \frac{\partial}{\partial t_1} \log \tau_n \Big|_{\mathcal{L}},$$

one eliminates the following functions

$$\frac{\partial}{\partial t_1} \log \frac{\tau_n}{\tau_{n-1}} \Big|_{\mathcal{L}}, \quad \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} \Big|_{\mathcal{L}}, \quad \frac{\partial^2 \log \tau_n}{\partial s_2 \partial t_1} \Big|_{\mathcal{L}},$$

from the Toda equations and combinations of Virasoro's

$$\frac{(\mathbb{V}_0 + \mathbb{V}_1)\tau_n}{\tau_n} - \frac{(\mathbb{V}_0 + \mathbb{V}_1)\tau_{n-1}}{\tau_{n-1}} \Big|_{\mathcal{L}} = \frac{\partial}{\partial t_1} \frac{(\mathbb{V}_0 + \mathbb{V}_{-1})\tau_n}{\tau_n} \Big|_{\mathcal{L}} = 0$$

IV. Examples of integrals over intervals

The integrals

$$I_n = \int_{E^n} |\Delta_n(z)|^{2\beta} \prod_{k=1}^{n} (1-z_k)^a (1+z_k)^b e^{\sum_1^\infty t_i z_k^i} dz_k$$

- **Example 1** Setting $\beta = 1$ and $\mathcal{L} = \{\text{all } t_i = 0, \text{ except } t_1 = 2x\}$, one finds

$$\int_{E^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} (1-z_k)^{\pm 1/2} (1+z_k)^{\mp 1/2} e^{\sum_1^\infty t_i z_k^i}$$
$$= e^{\mp x} \int_{O(2n+1)_\pm} e^{x \operatorname{Tr} M} dM$$

The generating function (Rains)

$$2 \sum_{k \geq 0} \frac{x^{2k}}{(2k)!} \# \left\{ \begin{array}{l} \pi \in S_{2k},\ \pi^2 = 1, \\ \pi(y) \neq y,\ L(\pi) \leq n+1 \end{array} \right\}$$
$$= \int_{O(n+1)_-} e^{x \operatorname{Tr} M} dM + \int_{O(n+1)_+} e^{x \operatorname{Tr} M}$$
$$= \exp\left(\int_0^x \frac{f_n^-(u)}{u} du \right) + \exp\left(\int_0^x \frac{f_n^+(u)}{u} du \right),$$

where $f = f_n^{\pm}$, satisfies Painlevé V equation :

$$\begin{cases} f''' + \dfrac{1}{u}f'' + \dfrac{6}{u}f'^2 - \dfrac{4}{u^2}ff' - \dfrac{16u^2+n^2}{u^2}f' + \dfrac{16}{u}f + \dfrac{2(n^2-1)}{u} = \\ f_n^{\pm}(u) = u^2 \pm \dfrac{u^{n+1}}{n!} + O(u^{n+2}), \text{ near } u = 0. \end{cases}$$

(Adler-PvM)

- **Example 2** $\mathbb{F} = \begin{cases} \mathbb{C} & \leftrightarrow \beta = 1 \\ \mathbb{R} & \leftrightarrow \beta = 1/2 \\ \mathbb{H} & \leftrightarrow \beta = 2 \end{cases}$

$$\int_{[0,1]^p} |\Delta_p(z)|^{2\beta} \prod_1^p e^{xz_i} z_i^{\beta(q-p+1)-1} (1-z_i)^{\beta(n-q-p+1)-1} dz_i$$

$$= \int_{\underbrace{Gr(p,\mathbb{F}^n)}_{\simeq G/K}} e^{x \operatorname{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-\beta(q-p)} d\mu(Z) \quad (*)$$

$(n-p)\times p$

(i) Density of sample canonical correlation coeff. between two Gaussian populations

$$(X_1, \ldots, X_p) \text{ et } (Y_1, \ldots, Y_q), \qquad (p \leq q)$$

are $p+q$ normal random variables, centered at zero, whose covariance matrice is given by

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^\top & \Sigma_{22} \end{pmatrix}.$$

Non-singular linear transformations on the X's and Y's lead to the canonical form (Hotelling)

$$\Sigma_{canonical} = \begin{pmatrix} I & P & 0 \\ P & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad P = \operatorname{diag}(\rho_1, \ldots, \rho_p),$$

where $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_p$,

$$\begin{aligned} \rho_i &= \textit{canonical correlation coefficients} \\ &= \textit{roots of } \det(\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^\top - \rho^2\Sigma_{11}) = 0. \end{aligned}$$

The $n \geq p+q$ independent samples of $(X_1, \ldots, X_p)$ and $(Y_1, \ldots, Y_q)$:

$$(x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1q})^\top, \quad \ldots, (x_{n1}, \ldots, x_{np}, y_{n1}, \ldots, y_{nq})^\top$$

form a matrix

$$\begin{pmatrix} x \\ y \end{pmatrix} = (p+q) \times n \text{ matrix.}$$

The maximum likelihood estimators r_i of ρ_i satisfy the equation

$$\det(S_{12}S_{22}^{-1}S_{12}^\top - r^2 S_{11}) = 0,$$

where

$$S = \begin{pmatrix} xx^\top & xy^\top \\ yx^\top & yy^\top \end{pmatrix} = \left\{ \begin{array}{l} \textit{sample} \text{ covariance} \\ \text{matrix} \end{array} \right. .$$

The $z_i := r_i^2 = \cos^2\theta_i$ $(\frac{1}{2}\pi > \theta_1 > \ldots > \theta_p > 0)$ have the density (James and Constantine)

$$c_{p,q,n}|\Delta(z)|^{2\beta} \prod_{i=1}^{p} (1-z_i)^{\beta(n-p-q+1)-1} z_i^{\beta(q-p+1)-1} dz_i$$

(*) is the Laplace transform of this density.

Combinatorial
Interpretation of the integral (*) (for $p=q$)

(ii) Consider the set of words S_k^p:

$$\sum_{k \geq p(n-p)} \frac{(px)^k}{k!} P^{kp} \left(\pi \in S_k^p \;\middle|\; \begin{array}{c} d_1(\pi) = p \text{ and} \\ i_{p-1}(\pi) \leq k-(n-p) \end{array} \right)$$

k

← see next slide for definition

$$= \quad \tilde{c} x^{p(n-p)} \int_{Gr(p,\mathbb{C}^n)} e^{x \operatorname{Tr}(I+Z^\dagger Z)^{-1}} d\mu(Z)$$

$$= \quad c x^{p(n-p)} \exp \int_0^x \frac{u(y) - p(n-p) + py}{y} dy$$

where $u(x) =$ unique solution to:

$$\begin{cases} x^2 u''' + x u'' + 6 x u'^2 - 4 u u' + 4 Q u' - 2 Q' u = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \textbf{(Painlevé V)} \\ \text{with initial condition} \\ u(x) = p(n-p) - \dfrac{p(n-p)}{n} x + \ldots + a_{n+1} x^{n+1} + O(x^{n+2}) \end{cases}$$

where

$$4Q \quad = \quad -x^2 + 2nx - (n-2p)^2$$

Subsequence σ of the word π is k-**increasing:**

$$\sigma = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_k,$$

$$\sigma_i = \text{increasing subsequences of } \pi.$$

By the RSK correspondence: (Greene)

$$d_1(\pi) = \left\{ \begin{array}{l} \text{maximal length of} \\ \text{a strictly decreasing} \\ \text{subsequence of } \pi \end{array} \right\} = \lambda_1^{\top}$$

$$i_k(\pi) = \left\{ \begin{array}{l} \text{maximal length of} \\ \text{the \textit{disjoint union}} \\ \text{of } k\text{-increasing} \\ \text{subsequences of } \pi \end{array} \right\} = \lambda_1 + \ldots + \lambda_k$$

<u>*Example:*</u>

$$\pi \ = \ (4, \underline{2}, \underline{3}, 6, \underline{5}, 1, \underline{7}) \longrightarrow (2, 3, 5, 7) \qquad i_1(\pi) =$$

$$\pi \ = \ (\underline{\underline{4}}, \underline{2}, \underline{3}, \underline{\underline{6}}, \underline{5}, 1, \underline{7}) \longrightarrow (2, 3, 5, 7) \cup (4, 6)$$

$$i_2(\pi) = 4 + 2 = 6$$

Consider a word such that:

$$d_1(\pi) = p$$

$$\Updownarrow$$

$$\pi = (\dots, p, \dots, p-1, \dots, p-2, \dots, 2, \dots, 1, \dots) \in S_k^p$$

(word of length ℓ)

Then

$$\begin{aligned} i_p(\pi) &= \ell = \text{ trivial} \\ i_{p-1}(\pi) &= \text{non-trivial} \leq k-(n-p) \end{aligned}$$

V. RANDOM WALKS: In the integral

$$\int_{U(n)} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)} dM,$$

the shifts $t_1 \longmapsto t_1 + z, \quad s_1 \longmapsto s_1 - z$ lead to

$$\begin{aligned} \tau_n(t,s) &= \int_{U(n)} e^{z \,\mathrm{Tr}(M+\bar{M})} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)} dM \\ &= \sum_{\substack{\lambda,\ \mu,\ \text{such that} \\ \lambda_1^\top,\ \mu_1^\top \le n}} a_{\lambda\mu}(z) \mathbf{s}_\lambda(t) \mathbf{s}_\mu(-s) \end{aligned}$$

(Fourier expansion in Schur polynomials)

with

$$\begin{aligned} \tau_n(0,0) &= \int_{U(n)} e^{z \ \mathrm{Tr}(M+\bar{M})} dM \\ &= \det\left(\int_{S^1} u^{k-\ell} e^{z(u+u^{-1})} \frac{du}{2\pi i u}\right)_{1\le i,j\le n} \\ &= \det\left(J_{k-\ell}(z)\right)_{1\le k,\ell\le n} \end{aligned}$$

$$e^{z(u+u^{-1})} = \sum_{n=-\infty}^{\infty} J_n(z) u^n$$

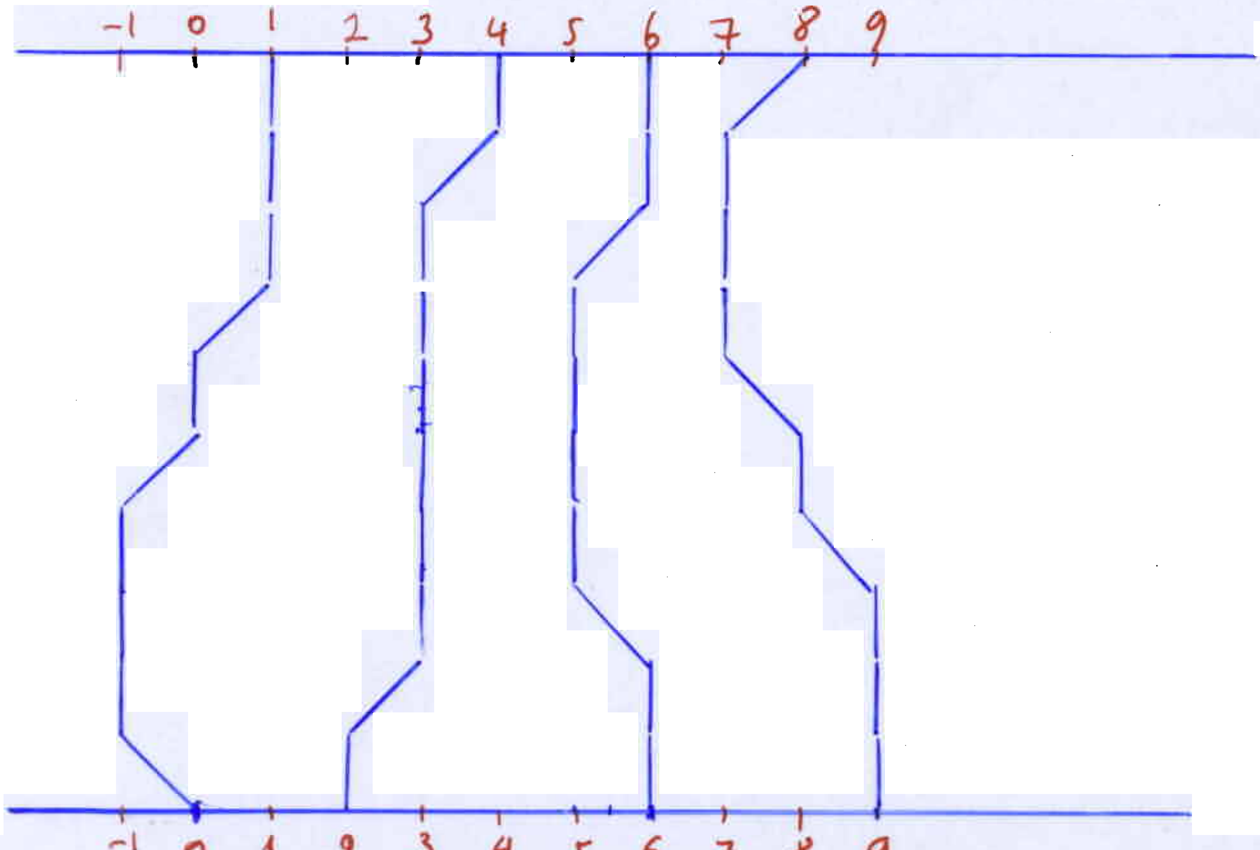
Then

$$\begin{aligned}
a_{\lambda\mu}(z) &= \int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) e^{z\,\mathrm{Tr}(M+\bar{M})} dM \\
&= \det\left(J_{(\lambda_i - i)-(\mu_j - j)}(z)\right)_{1\le i,j\le n} \quad \text{(Cauchy-Binet)} \\
&= \det\left(J_{y_i - x_j}(z)\right)_{1\le i,j\le n} \\
&= \det\left(e^{z(u+u^{-1})}\Big|_{u^{y_i - x_j}} \right)_{1\le i,j\le n} \\
&= \sum_{w\in W} (-1)^{\sigma(w)} \prod_1^n e^{z(u_i + u_i^{-1})}\Big|_{u_i^{y_i - x_{w(i)}}} \\
&= \sum_{k=0}^{\infty} \# \left\{ \begin{array}{l} \text{walks of } k \text{ steps} \\ \text{from } x \mapsto y \text{ in } \mathbb{Z}^n \\ \text{within } \{u_1 > \ldots > u_n\} \end{array} \right\} \frac{z^k}{k!} \\
&= \sum_{k\ge 0} \# \left\{ \begin{array}{l} n \text{ non-intersecting walks} \\ \text{in } \mathbb{Z} \text{ going in } k \text{ steps} \\ \text{from } x_1 < x_2 < \ldots < x_n \\ \text{to } y_1 < y_2 < \ldots < y_n \end{array} \right\} \frac{z^k}{k!} \\
&= \sum_{k\ge 0} b_{x,y}^{(k)} \frac{z^k}{k!},
\end{aligned}$$

where

$$\begin{aligned}
x &:= (0+\lambda_n, 1+\lambda_{n-1}, \ldots, n-1+\lambda_1) \\
y &:= (0+\mu_n, 1+\mu_{n-1}, \ldots, n-1+\mu_1)
\end{aligned}$$

This is the kind of random walk, where only one walker walks at each step.



$$I_n = \int_{U(n)} e^{z\,\mathrm{Tr}(M+\bar{M})} e^{\sum_1^\infty \mathrm{Tr}(t_i M^i - s_i \bar{M}^i)} dM$$

$$= \sum_{\lambda,\mu} a_{\lambda\mu}(z)\, s_\lambda(t)\, s_\mu(-s)$$

$$\text{Virasoro} = \mathbb{V}_k = \mathbb{V}_k^{\text{diff}} + \mathbb{V}_k^{\text{mult}} \qquad k = -1, 0, 1$$

$$\begin{aligned}
0 &= \mathbb{V}_k I_n(t,s) \\
&= \mathbb{V}_k \Big(\sum_{\lambda,\mu} a_{\lambda\mu} \mathbf{s}_\lambda(t) \mathbf{s}_\mu(-s)\Big) \\
&= \sum a_{\lambda\mu} (\mathbb{V}_k^{\text{diff}} \mathbf{s}_\lambda(t)) \mathbf{s}_\mu(-s) \\
&\quad + \sum a_{\lambda\mu} \mathbf{s}_\lambda(t) (\mathbb{V}_k^{\text{diff}} \mathbf{s}_\mu(-s)) \\
&\quad + \sum a_{\lambda\mu} \mathbb{V}_k^{\text{mult}} \mathbf{s}_\lambda(t) \mathbf{s}_\mu(-s) \\
&= \sum_{\lambda,\mu} c_{\lambda\mu}(a) \mathbf{s}_\lambda(t) \mathbf{s}_\mu(-s)
\end{aligned}$$

Hence $c_{\lambda\mu}(a) = 0$.

Spelled out, this gives identities for $b_{x,y}^{(k)}$, on the next slide:

CLAIM 1

- $$\frac{\sum\limits_{1\leq i\leq n}(y_i - x_i)}{k} b_{xy}^{(k)} = \sum_{1\leq i\leq n}\left(b_{x+e_i,y}^{(k-1)} - b_{x,y+e_i}^{(k-1)}\right)$$

- $$\frac{1}{k}\sum_{1\leq i\leq n}\left((y_i+1)b_{x,y+e_i}^{(k)} - x_i b_{x-e_i,y}^{(k)}\right)$$
$$= n b_{x,y}^{(k-1)} - \sum_{1\leq i\leq n} b_{x,y+2e_i}^{(k-1)}$$

FACT 2

$$p(k,x,y) := \mathbb{P}\left(\begin{array}{l} n \text{ random walks in } \mathbb{Z} \\ \text{starting from } x = (x_1 < \dots < x_n) \\ \text{reach } y = (y_1 < \dots < y_n) \\ \text{in } k \text{ steps} \end{array} \middle| \begin{array}{c} \text{without} \\ \text{intersecting} \end{array} \right)$$

satisfies the "discrete Dyson diffusion"

$$\left(\left(\frac{\sum_1^n x_i}{\sqrt{k}} + \sqrt{k}\, \mathcal{L}_x \right) - \left(\frac{\sum_1^n y_i}{\sqrt{k}} + \sqrt{k}\, \mathcal{L}'_y \right) \right) p(k,x,y) = 0$$

where

$$\mathcal{L}_x := \Lambda^{-1} \sum_1^n \left(1 + \frac{\partial^+_{x_i} \Delta(x)}{\Delta(x)} \right) \partial^+_{x_i}$$

$$\mathcal{L}'_y = \Lambda^{-1} \sum_1^n \partial^+_{y_i} \left(1 - \frac{\partial^-_{y_i} \Delta(y)}{\Delta(y)} \right)$$

$\Delta(x)$ is the Vandermonde of $x = (x_1, \dots, x_n)$

$\partial^+_{x_i} f(x,y) := f(x+e_i, y) - f(x,y)$

$\partial^-_{y_i} f(x,y) := f(x,y) - f(x-e_i, y)$

Setting

$$x = \sqrt{\frac{2k}{nt}}\,\tilde{x} \quad , \quad y = \sqrt{\frac{2k}{nt}}\,\tilde{y}$$

and after proper rescaling, the equation above can be expanded in $\frac{1}{\sqrt{k}}$:

$$0 = f(\tilde{x},\tilde{y},t) + \left(\left(\mathcal{D}_{\tilde{x}} - \frac{\partial}{\partial t}\right) - \left(\mathcal{D}^{T}_{\tilde{y}} - \frac{\partial}{\partial t}\right)\right) p(t,\hat{x},\hat{y})\,\frac{c(t)}{\sqrt{k}} + O\left(\frac{1}{k}\right)$$

where $\mathcal{D}_{\tilde{x}}$ is the infinitesimal generator of the Dyson diffusion on next slide :

$$p(t,x,y)\,\widehat{dy} = P\left(\begin{array}{l|l} n \text{ Brownian motions} & \\ \text{in } \mathbb{R} \text{ starting from } x & \text{without} \\ \text{reach } y+dy \ \text{ at time } t & \text{intersecting} \end{array}\right)$$

$$= \frac{\Delta(y)}{\Delta(x)} \det \left(\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x_i-y_j)^2}{2t}} \right)_{1\le i,j\le n} dy_1 \ldots dy_n$$

satisfies Dyson's diffusion,

Laplacian

$$\frac{\partial}{\partial t} p(t,x,y) = \left(\frac{1}{2}\mathfrak{L}_y + \sum_i \frac{\partial}{\partial y_i} E(y_i) \right) p(t,x,y). = \mathcal{D}_y^{T} p$$

$$\frac{\partial}{\partial t} p(t,x,y) = \left(\frac{1}{2}\mathfrak{L}_x - \sum_i E(x_i) \frac{\partial}{\partial x_i} \right) p(t,x,y). = \mathcal{D}_x\, p$$

with

$$E(y_j) = \frac{\partial W}{\partial y_i} = -\frac{\partial}{\partial y_i} \sum_{1\le i<j\le n} \ln(y_i - y_j) = -\sum_{j\ne i} \frac{1}{y_i - y}$$

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