Random growth and determinantal processes

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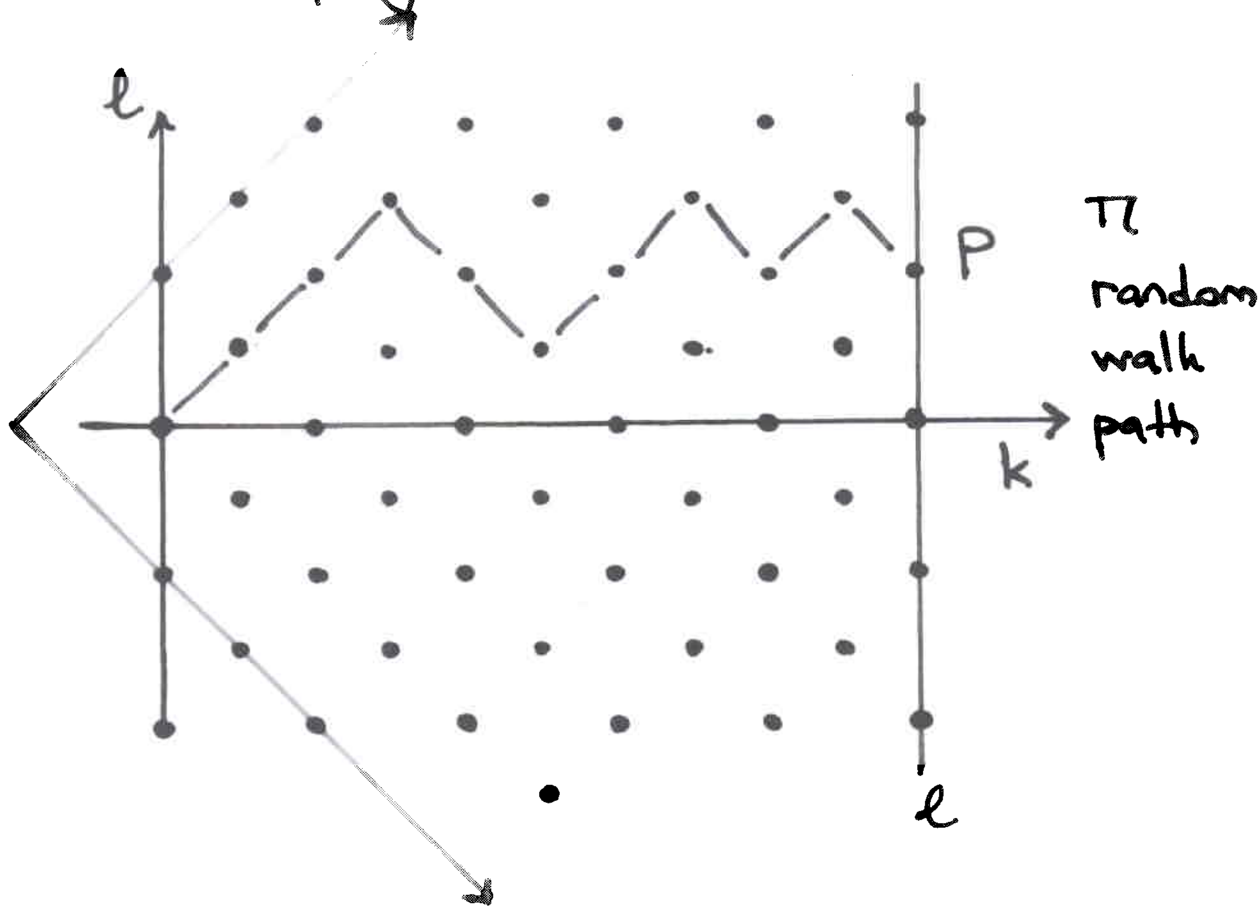
Directed polymer at zero temperature



$\omega(k,\ell)$ random energy at (k,ℓ)

(random environment)

$$F = \min_{\pi} \Big(\sum_{(i,j)\in\pi} \omega(i,j) \Big)$$

minimum over all paths from $(0,0)$ to ℓ (point to line) or from $(0,0)$ to P, some fixed point on ℓ (point to point)

Conjecture:

Fluctuations in F $\sim N^{1/3}$ (1)

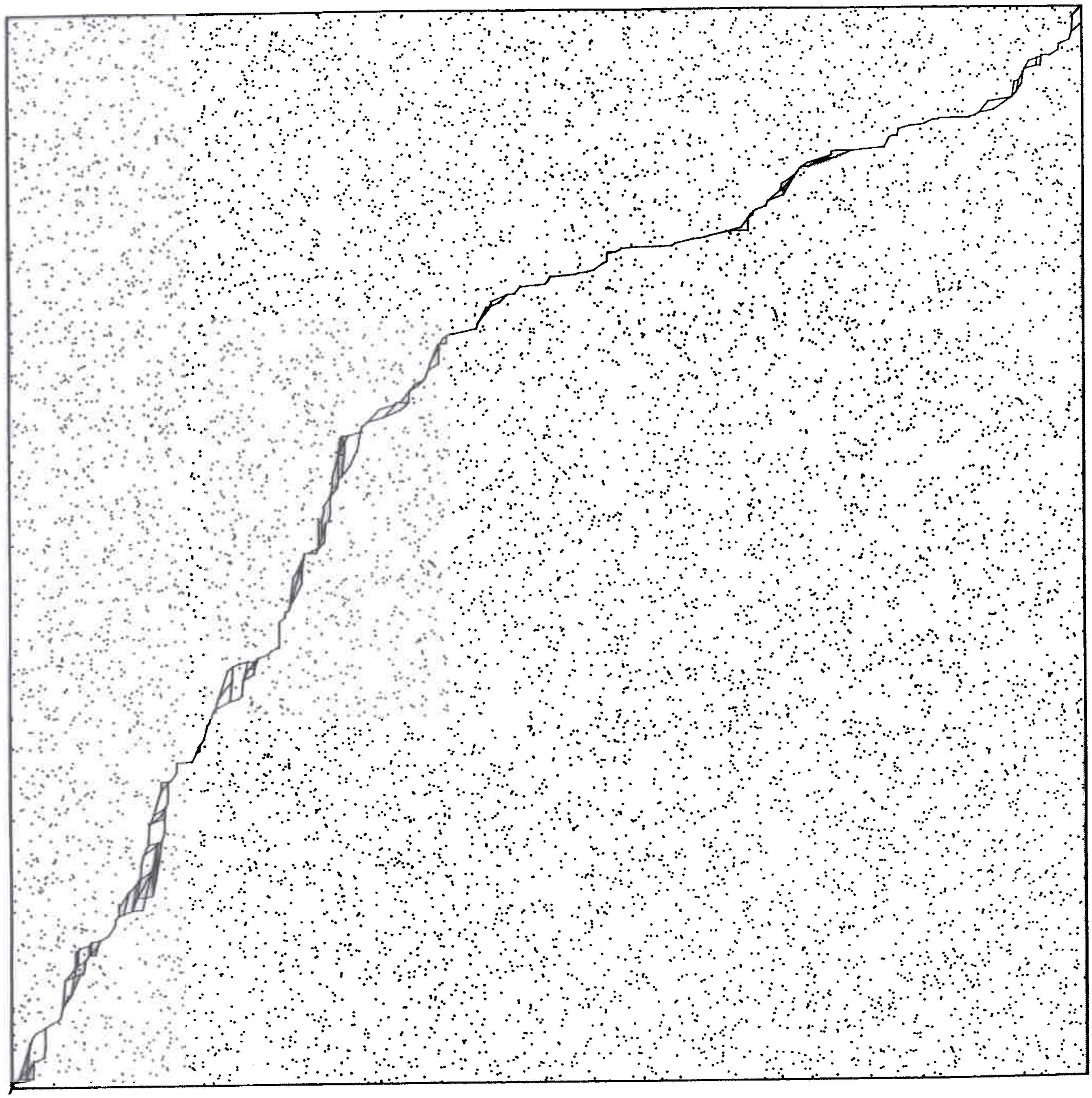
Fluctuations in the endpoint $\sim N^{2/3}$ (2)
(point to line) (superdiffusive)

ℓ has $k = 2N$

The results of Baik-Deift-J. on longest increasing subsequences in random permutations establishes (1) for a special random environment.
Also gives <u>law</u> of the fluctuations.

10 000 random points ; N = 10000

B



A

(Franz Merkl)

Up/right paths through the random points between A and B. Pick up as many as possible.

Mean $\sim 2\sqrt{N}$; SD $\sim (\sqrt{N})^{1/3}$

A directed last-passage percolation model

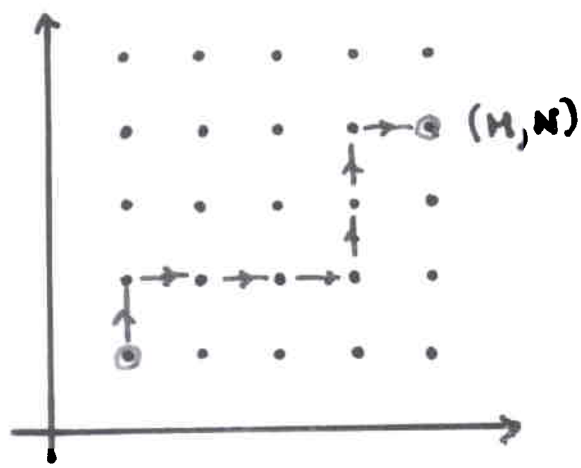
$w(i,j)$ independent geometric random variables

$$\mathbb{P}[w(i,j) = m] = (1-q)q^m, \quad m \geq 0,$$

$(i,j) \in \mathbb{Z}_+^2$

π

up/right path



point to point

$$G(M,N) = \max_{\pi} \sum_{(i,j)\in\pi} w(i,j)$$

$q = \alpha/N^2$

$G(N,N) \xrightarrow{\mathcal{D}} L(\alpha)$,

the length of the longest increasing subsequence in a random permutation from S_M, $M \in Po(\alpha)$.

(Ulam's problem)

Thm. (J. '99)

$$\mathbb{P}\left[\frac{G(N,N) - \omega N}{\sigma N^{1/3}} \leq \xi \right] \xrightarrow[N\to\infty]{} F_2(\xi)$$

(point to point)

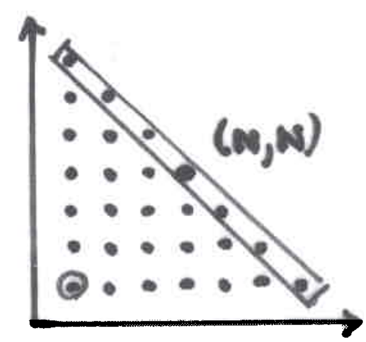
Tracy-Widom distribution (GUE, $\beta=2$)

Consider all the variables

$$G(N+K, N-K) \quad , \quad -N < K < N,$$

a process

(point to line)



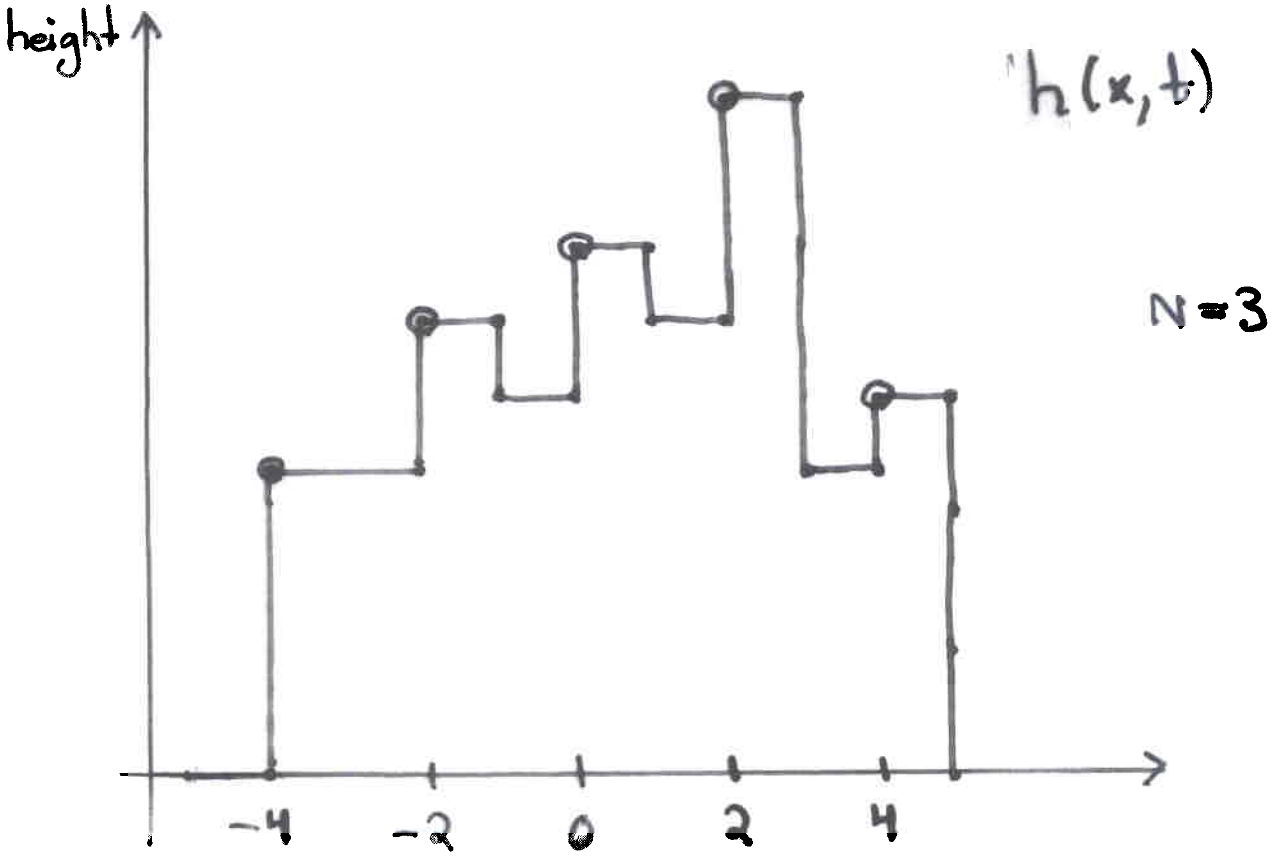
Thm. (Baik-Rains, '00)

$$G_{pl}(N) = \max_{|K|<N} G(N+K, N-K)$$

$$\mathbb{P}\left[\frac{G_{pl}(N) - \omega N}{\sigma N^{1/3}} \leq \xi \right] \xrightarrow[N\to\infty]{} F_1(\xi)$$

TW-distr.

Random growth



$$G(N+K, N-K) = h(2K, 2N-1)$$

<u>Growth rule</u> based on

$$G(M,N) = \max\left(G(M-1,N), G(M,N-1)\right) + w(M,N)$$

Polynuclear growth model

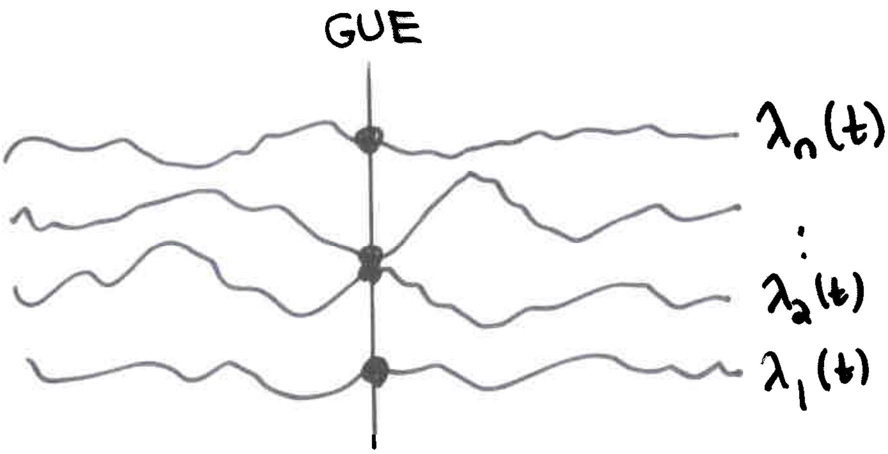
(Prähofer–Spohn in the permutation limit)

Introduce the <u>rescaled process</u>

$$H_N(cN^{-2/3}k) = \frac{1}{bN^{1/3}}\left(G(N+k, N-k) - aN\right);$$

extend to $t \to H_N(t)$, $t \in \mathbb{R}$, by linear interpolation.

Dyson's GUE Brownian motion



Airy process (Prähofer – Spohn)

$$A(t) = \lim_{n\to\infty} \sqrt{2}\, n^{1/6}\left(\lambda_n(n^{-1/3}t) - \sqrt{2n}\right)$$

Γ $\tilde{\lambda}_n(t) = \sqrt{2n}\, \lambda_n(t/n)$; exponents $1/3$, $2/3$.

Finite dimensional distributions

$$\mathbb{P}[\, A(\tau_1) \leq \xi_1, \dots, A(\tau_m) \leq \xi_m \,]$$

$$= \det\left(I + f_{\xi_1, \dots, \xi_m} K^{\text{ext. Airy}} \right)_{L^2(\{\tau_1, \dots, \tau_m\} \times \mathbb{R})}$$

where

$$f_{\xi_1, \dots, \xi_m}(\tau_j, x) = -\chi_{(\xi_j, \infty)}(x)$$

Note

$$\mathbb{P}[A(\tau) \leq \xi] = F_2(\xi) \qquad \text{TW-distribution}$$

Extension of the Tracy-Widom distribution.

<u>Thm.</u> (J. '02) (Functional limit theorem)

$$H_N(t) \Rightarrow A(t) - t^2$$

as $N \to \infty$ in $C(-T,T)$, any $T > 0$.

Together with the Baik-Rains result this implies:

<u>Thm.</u> (J. '02)

$$F_1(\xi) = \mathbb{P}\left[\sup_{t \in \mathbb{R}} (A(t) - t^2) \le \xi\right].$$

Let K_N be the leftmost point where $H_N(t)$ assumes its maximum, and K the same thing for $A(t) - t^2$.

<u>Thm</u> (J. '02) If $A(t) - t^2$ has a unique global maximum almost surely, then

$$K_N \xrightarrow{\mathcal{D}} K \quad \text{as } N \to \infty$$

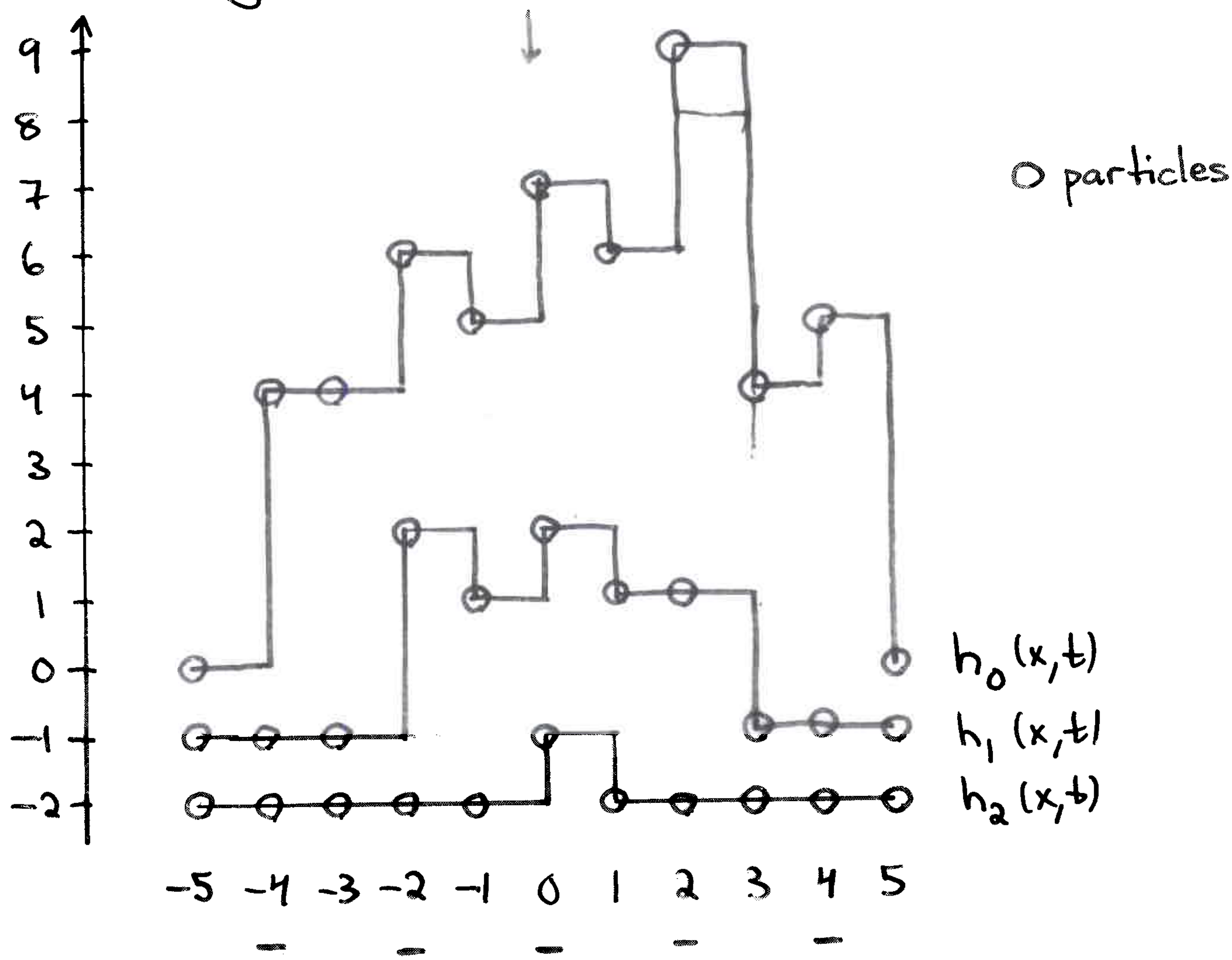
Law of K = Law of transversal fluctuations

$N = 3$

$M = 2 \cdot 3 - 1 = 5$

$$\begin{array}{ccccc} 0 \\ 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \to 3 \\ 1 \to 2 \to 1 & & 0 & 1 \end{array}$$

There is a 1–1 – correspondence which maps this into a family of non-intersecting paths:



$$G(N+K, N-K) = h_0(2K, 2N-1)$$

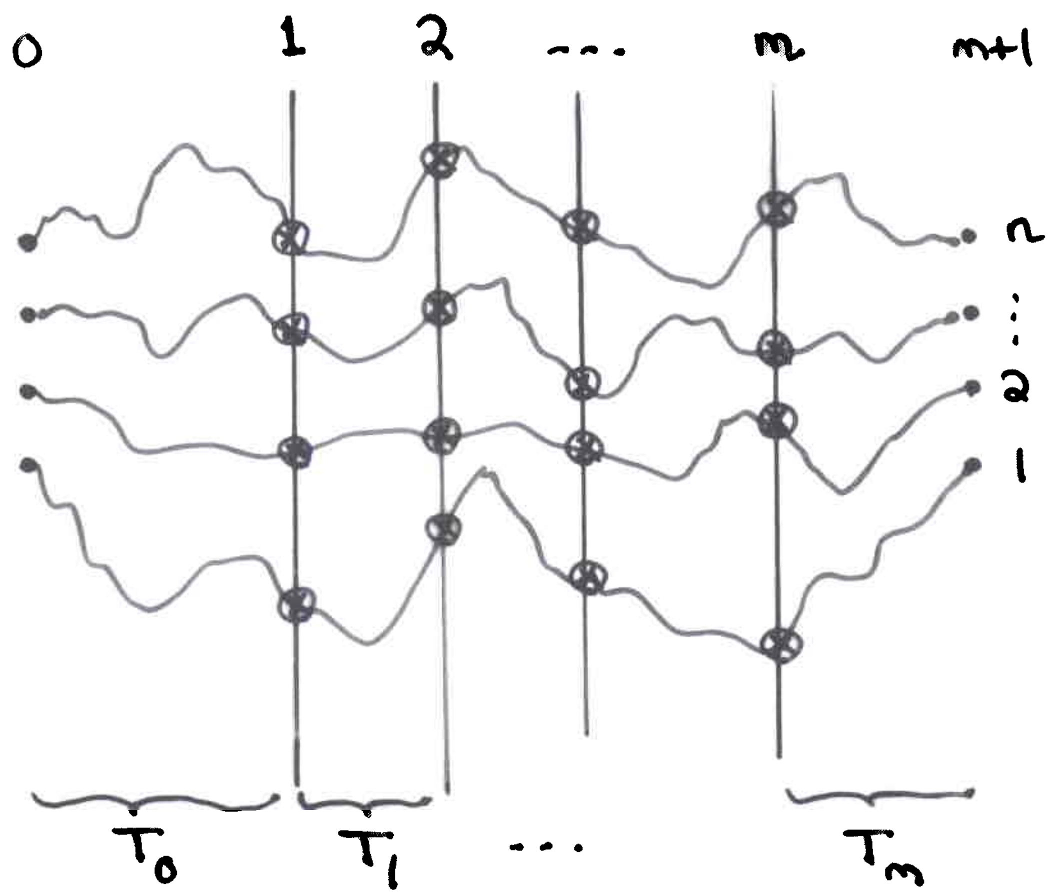
The measure on the particle configuration is given by a product of determinants by the Karlin-McGregor thm. / Lindström-Gessel-Viennot method.

Fits into the Schur process introduced by Okounkov-Reshetikhin.

Determinantal process, i.e. the correlation functions are determinants with a certain kernel as in $\beta=2$ RMT.

There is an explicit double integral formula for the kernel.

<u>Eynard-Mehta analysis of coupled matrices</u>



Measure

$$\frac{1}{Z}\prod_{j=0}^{m}\det\Big(\underbrace{\frac{1}{\sqrt{2\pi T_j}}\, e^{-(x_s^{j+1}-x_r^{j})^2/2T_j}}_{\varphi_{j,j+1}(x_r^j,\, x_s^{j+1})}\Big)_{1\le r,s\le n}$$

(Karlin–McGregor thm.)

Determinantal correlation functions with kernel

$$K(j,x;k,y) = \tilde{K}(j,x;k,y) - \underbrace{\varphi_{j,k}(x,y)}_{=0 \text{ if } j \geq k}$$

$$\tilde{K}(j,x;k,y) = \sum_{r,s=1}^{n} \varphi_{r,m+1}(x, x_r^{m+1})(A^{-1})_{rs}\,\varphi_{0,s}(x_s^0, y)$$

where

$$A = \left(\varphi_{0,m+1}(x_r^0, x_s^{m+1})\right)_{r,s=1,\dots,n}$$

(Proved using ideas from Eynard-Mehta)

Scaling limit

$$T_j = t_j T \quad , \quad x_r^j = \sqrt{s_j T}\, \lambda_r^j$$

let $T \to \infty$,

$$\frac{1}{Z'} \prod_{r=1}^{n} \prod_{j=1}^{m} e^{-V_j(\lambda_r^j)} \Delta_n(\lambda^1) \Delta_n(\lambda^m)$$

$$\cdot \prod_{j=1}^{m-1} \det\left(e^{c_j \lambda_r^j \lambda_s^{j+1}}\right)_{1 \le r, s \le n} , \qquad (*)$$

where

$$V_j(\xi) = s_j \left(\frac{1}{t_{j-1}} + \frac{1}{t_j}\right) \xi^2$$

$$c_j = \frac{1}{t_j} \sqrt{s_j s_{j+1}}$$

$(*)$ corresponds to the coupled matrix model:

$$\frac{1}{\mathcal{Z}} e^{-\sum_{j=1}^{m} \mathrm{tr}\, V_j(A_j) + \sum_{j=1}^{m-1} c_j \,\mathrm{tr}(A_j A_{j+1})}$$

In this case the correlation kernel can be written as a double integral formula.

$$s_j = 2(t_0 + \dots + t_{j-1}) \quad , \quad t_m \to \infty$$

$$s_j = e^{2(\tau_1 + \dots + \tau_{j-1})} \quad , \quad s_1 = 1$$

The matrix $A_j = H(\tau_j)$, where $\tau \to H(\tau)$ is Dyson's GUE Brownian motion (independent matrix elements evolve according to Ornstein-Uhlenbeck processes).

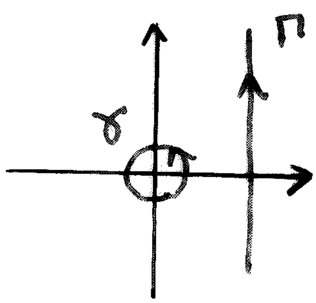
For Dyson's BM we get the correlation kernel (Forrester–Nagao–Honner)

$$K(\tau_1,\lambda;\tau_2,\mu)=\tilde{K}(\tau_1,\lambda;\tau_2,\mu)-\psi_{\tau_1,\tau_2}(\lambda,\mu)$$

where

$$\tilde{K}(\tau_1,\lambda;\tau_2,\mu)=\frac{\sqrt{2}\,e^{(\tau_1-\tau_2)/2}}{(2\pi i)^2}\,e^{n(\tau_1-\tau_2)}$$

$$\times\int_\gamma dz\int_\Gamma dw\,\frac{w^n}{z^n}\,\frac{1}{we^{\tau_1-\tau_2}-z}\,e^{-z^2/2+\sqrt{2}\lambda z+w^2/2-\sqrt{2}\mu w}$$



(Can be expanded in terms of Hermite polynomials.)

$$\psi_{\tau_1,\tau_2}(\lambda,\mu)=\frac{\sqrt{2}\,e^{(\tau_1-\tau_2)/2}}{\sqrt{2\pi(1-e^{2(\tau_1-\tau_2)})}}\,e^{-\frac{1}{1-e^{2(\tau_1-\tau_2)}}(e^{\tau_1-\tau_2}\lambda-\mu)^2}$$

if $\tau_1\le\tau_2$; $=0$ otherwise.

The Airy process is a natural limit process

Extension of the Tracy-Widom distribution

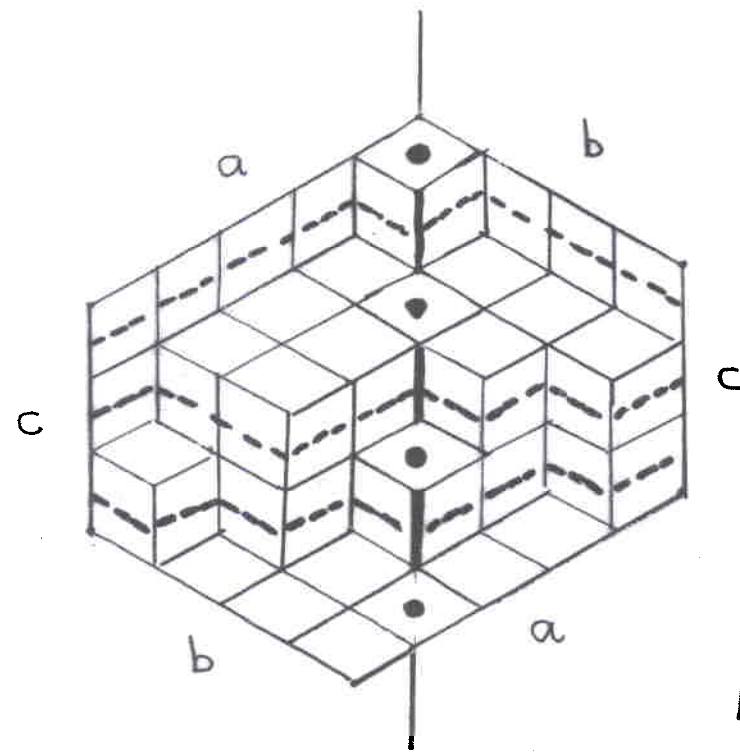
Are there differential equations for the joint distribution function?

$$F(\tau, \xi_1, \xi_2) = \mathbb{P}[\, A(0) \leq \xi_1 ,\ A(\tau) \leq \xi_2 \,]$$

$F(0, \xi, \xi)$ related to $P\underline{II}$.

Connection with integrable systems?

- Random rhombus tilings of a hexagon



k:th line from the left, $0 \le k \le a+b$

$a \geqslant b$

$\alpha = |a-k|$

$\beta = |b-k|$

$N = n + c$

random walk paths

Lock step model

- marks the position of the "vertical rhombi" that the line L intersects.

Let their positions, counting from the bottom and starting with zero, be $h_1, \dots, h_n$. The probability of having vertical rhombi at these positions is

$$\frac{1}{Z} \Delta_n(h)^2 \prod_{j=1}^{n} \frac{(N+\alpha-h_j)!\,(\beta+h_j)!}{h_j!\,(N-h_j)!}$$

Hahn ensemble

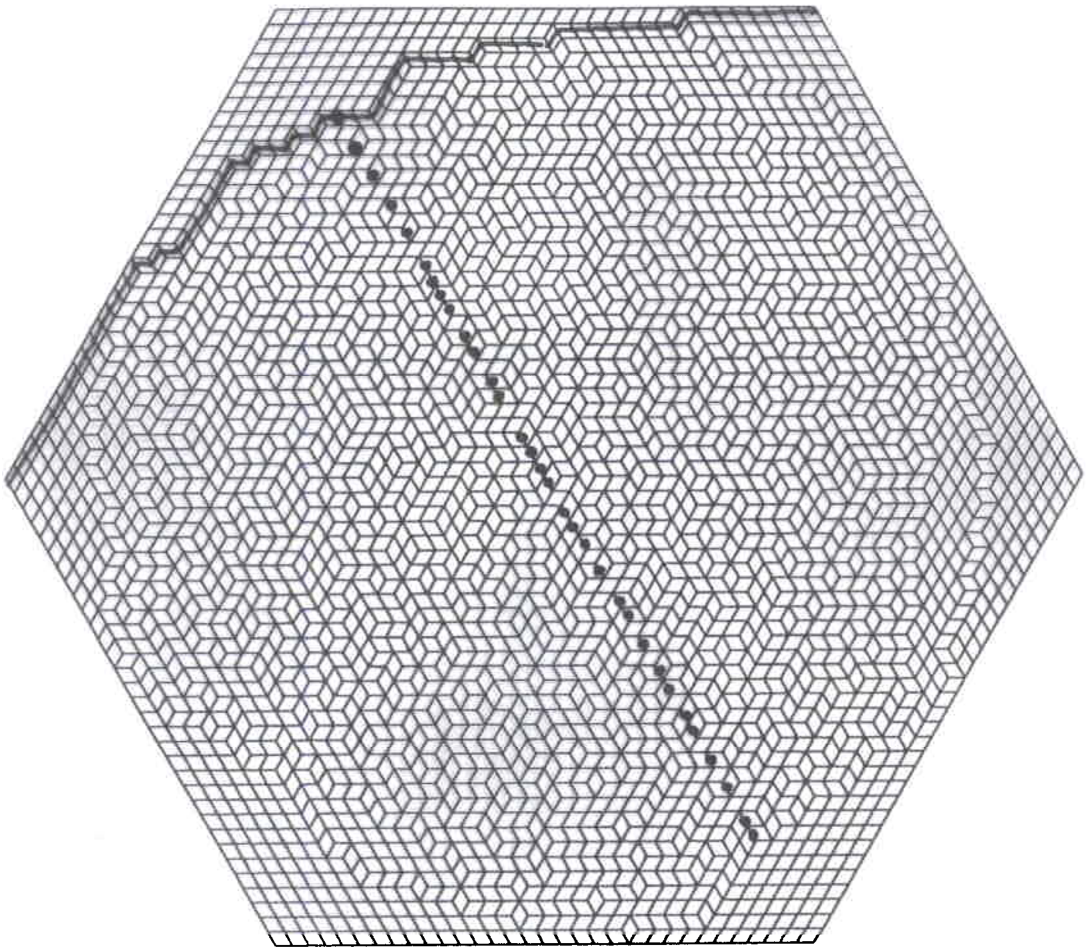


FIGURE 2. A random lozenge tiling of a 32, 32, 32 hexagon.