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Free Probability Aspects of Random Matrices

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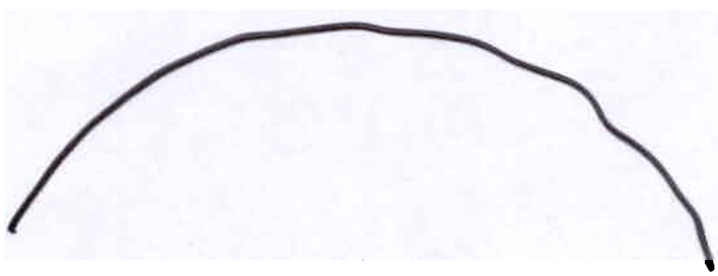
Large N limits of random Multimatrix Models



von Neumann Algebras of type II_1

$$T_{j,N} = T_{j,N}^{*}, \quad 1 \le j \le n$$

$$\varphi_N(P) = \frac{1}{N} E(\mathrm{Tr}\, P(T_{1,N}, \dots, T_{n,N}))$$

$$P \in \mathbb{C}\langle X_1, \dots, X_n \rangle = \mathbb{C}_{\langle n \rangle}$$

$$\varphi_N \longrightarrow \varphi_\infty$$

$\mathcal{H}$ Hilbert space

$\mathbb{C}_{\langle n \rangle}$, $\langle P, Q \rangle = \varphi_\infty(Q^* P)$

$X_j = X_j^*$, $\mathcal{H}$ left $\mathbb{C}_{\langle n \rangle}$ module

$\mathbb{C}_{\langle n \rangle} \hookrightarrow \mathcal{B}(\mathcal{H})$, $\varphi_\infty(P) = \langle P1, 1 \rangle$

(GNS-construction)

M = weak closure $\mathbb{C}\langle n \rangle$ in $\mathbb{B}(\mathcal{H})$

$\tau(\cdot) = \langle \cdot 1, 1 \rangle$

trace-state on M

$\tau(1) = 1$, $\tau(ab) = \tau(ba)$

$\tau(a^*a) \geq 0$

Large N limit:

$X_1, \dots, X_n$ in (M, τ)

cf $\quad n = 1$

$M = L^\infty(\mathbb{R}, \sigma)$

limit distr. of eigenv.

$\tau(f) = \int f \, d\sigma$, $X(t) = t$.

Typical (M, τ) $(n \geq 2)$

II_1 - factors $(Z(M) = \mathbb{C}I$

$\dim M = \infty)$

projectors $P = P^* = P^2 \in M$

∞-dim geometry of linear subspaces with dimensions $\tau(P) \in [0,1]$

G discrete group with ∞ conjugacy classes (i.c.c.)

$M = L(G) =$ left reg. repr. w-closed lin. span

$$\tau\left(\sum c_g \lambda(g)\right) = c_e$$

(M, τ) Noncommutative probability space

$X_1, \ldots, X_n$ Noncommutative random variables

Free Probability Th. =

= Noncommutative Prob. Th

+ Free Independence

$1 \in A_\iota \subset (M, \tau)$ subalgebras $\iota \in I$

freely independent

$\tau(a_{\iota_1} a_{\iota_2} \ldots a_{\iota_p}) = 0$ whenever

$\tau(a_\iota) = 0$, $\iota \in I$, $a_\iota \in A_\iota$, $\iota_j \neq \iota_{j+1}$, $1 \leq j \leq p$

Example 1°

$$G = G_1 * G_2 \quad \text{discrete groups}$$

$$L(G_1), L(G_2) \subset L(G)$$

freely indep.

Example 2° (V)

μ_N prob. meas. on $(M_N^{sa})^{q_1+\cdots+q_p}$

joint distr. $T_{1,N} \cdots T_{n,N}$

$n = q_1 + \cdots + q_p$. $\mathrm{diam}_\infty(\mathrm{supp}\,\mu_N) \le R$

$(U(N))^{p-1}$ acting by conjugation on $(M_N^{sa})^{q_1+\cdots+q_p}$ (trivial on last group of elts.) μ_N invariant.

If φ_∞ limit exists then

$\{X_1, \ldots, X_{q_1}\}, \{X_{q_1+1}, \ldots X_{q_1+q_2}\}, \ldots$

freely indep. in (M, τ)

<u>Cor.</u> Independent i.i.d. Gaussian Hermitian random matrices and constant hermitian (∞ norm bddness cond) asymptotically free.

In this case M for Gaussian matrices $\simeq L(F(n))$
free group

v. Neumann algebra applications of $L(F(n))$ asymptotically generated by n independent i.i.d. hermitian.

"Scaling" of $L(F(\infty))$
(v. Neumann fundamental group $= \mathbb{R}_{>0}$)
$L(F(\infty)) \simeq P\, L(F(\infty))\, P$
$P = P^2 \neq 0$, (V $\tau(P) \in \mathbb{Q}$, Rădulescu $\in \mathbb{R}$)

Open Problems:

Connes Problem: every separable II_1 factor asymptotically generated by some random matrices in above sense

For many models

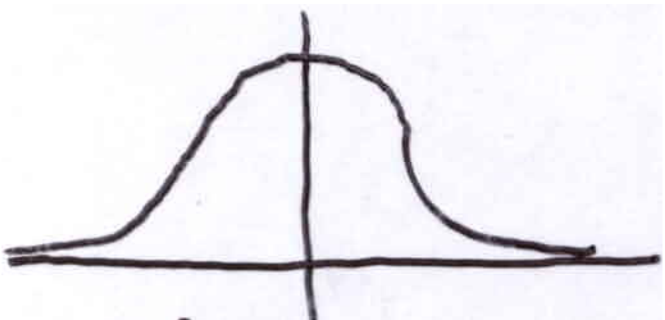
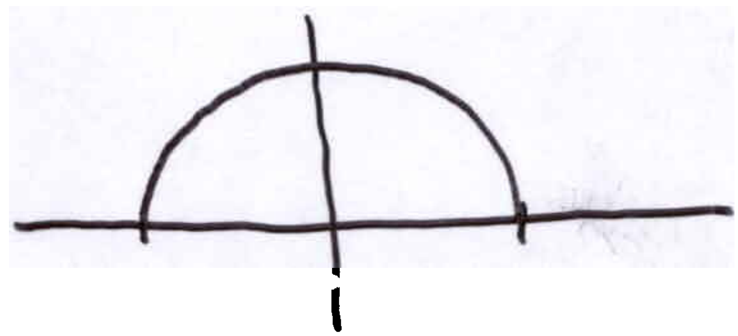
$\varphi_N \longrightarrow \varphi_\infty$ limit

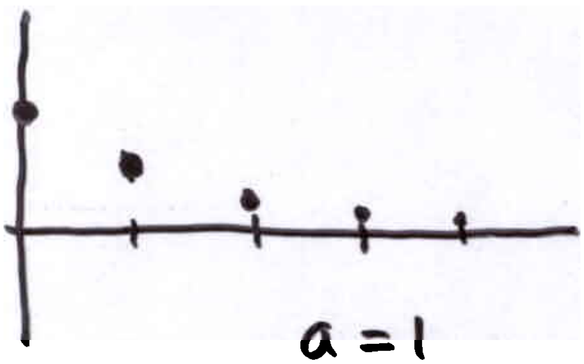
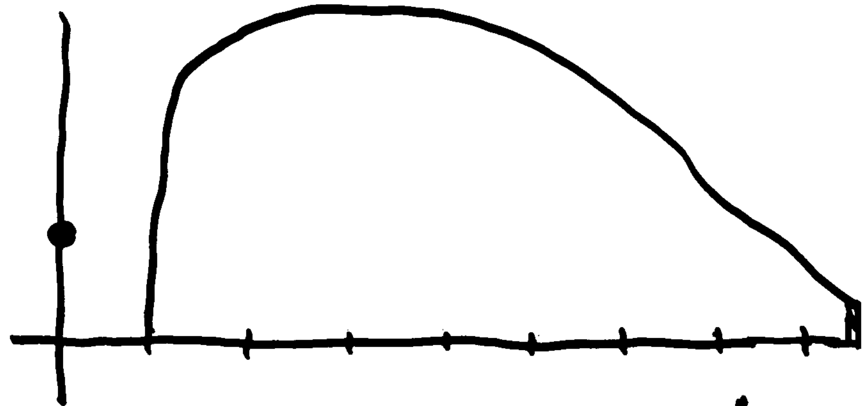
for instance if

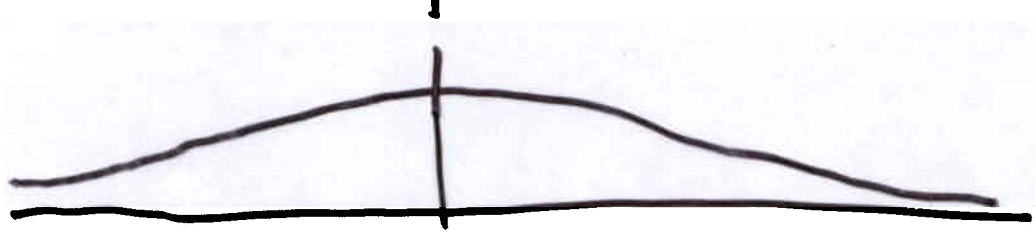
$$d\sigma_N = c_N e^{-N \operatorname{Tr} P(A_1, \ldots, A_m)} d\lambda(A_1, \ldots A_m)$$

[in this case, is
$M \simeq L(F(n))$??]

Basic Laws

Classical	Free
Gauss law	Semicircle law
	
$c\, e^{-x^2/2}\, dx$	$c \sqrt{4 - x^2}\, dx$ $-2 \le x \le 2$
Poisson	Free Poisson
$\sum_{n=0}^{\infty} e^{-a} \frac{a^n}{n!} \delta_n$ $a > 0$	$\begin{cases} (1-a)\delta_0 + \nu & \text{if } 0 \le a \le 1 \\ \nu & \text{if } a > 1 \end{cases}$ ν: $(2\pi t)^{-1} \sqrt{4a - (t-(1+a))^2}$ $(1-\sqrt{a})^2 \le t \le (1+\sqrt{a})^2$

Classical	Free
Poisson	Free Poisson
	 $a = 1/4$
Cauchy	Free Cauchy = Cauchy
$c \dfrac{1}{x^2+a^2}\,dx$	$c \dfrac{1}{x^2+a^2}\,dx$



Free Prob. Th.

Counterparts of Classical:

— addition and multiplication of independent variables

— additive and multiplication convolution operations

— Gaussian processes

— Stochastic Integration (Biane, Speicher)

— Infinitely divisible and stable laws (Bercovici + V)

— Combinatorics of cumulants (Speicher)

Combinatorial Approach
to Free Prob.
(Speicher):

all partitions of $\{1, \dots n\}$
replaced by non-crossing
(No: $a < b < c < d$)
Planar diagram.

Applications of asymptotic
random matrix model
to v. Neumann algebras:
V., Dykema, Rădulescu,
Shlyakhtenko
(to other operator algebras:
Haagerup)

In one-random-matr. th.
logarithmic energy:

$$\iint d\mu(s)\,d\mu(t)\log|s-t|$$

limit distr. of eigenv.

In Free Prob. this is

$\chi(X)$ free entropy

of X in (M,τ).

Analogue of Shannon
differential entropy.

$\chi(X_1,\dots,X_n)$

(2 approaches χ, χ^*
and unification problem)

$\chi(X_1, \ldots, X_n)$, $X_j = X_j^* \in M$ (12
$1 \le j \le n$

Matricial Microstates
($S = k \log W$ idea)

$\Gamma_R(X_1, \ldots, X_n : m, k, \varepsilon)$

$(A_1, \ldots, A_n) \in (M_k^{sa})^n$

$\|A_j\| < R$

$|k^{-1} Tr_k(A_{i_1} \cdots A_{i_p}) - \tau(X_{i_1} \cdots X_{i_p})| < \varepsilon$

$1 \le p \le m$

$\limsup_{k \to \infty} \left(k^{-2} \log vol \Gamma + \frac{n}{2} \log k \right)$

$\sup_{R > 0} \inf_{\varepsilon > 0} \inf_{m \in \mathbb{N}}$

get $\chi(X_1, \ldots, X_n)$

Many nice properties similar to Shannon differential entropy

$$H(X_1 \cdots X_n) = -\int_{\mathbb{R}^n} p \log p \, d\lambda$$

indep $\longrightarrow$ free indep
Gauss law $\longrightarrow$ semicircle

$X_1 \cdots X_n$ free
$$\Rightarrow \chi(X_1, \cdots X_n) = \chi(X_1) + \cdots + \chi(X_n)$$

Gaussian bound

$$\chi(F_1(X_1 \cdots X_n) \cdots F_n(X_1 \cdots X_n))$$
$$= \log \underbrace{|\det| (\mathcal{J}(F))}_{\text{Kadison-Fuglede det}} + \chi(X_1 \cdots X_n)$$

Kadison-Fuglede det of Jacobian involving "free "difference quotient" deriv."

Applications to v. Neumann algebras
(V, Ge, Stefan, Shlyakhtenko, Dykema, Jung)

<u>Example</u> (Ge)
$L(F(n))$ is <u>prime</u>
i.e. <u>not isomorphic</u> $\underbrace{M_1 \otimes M_2}_{\infty\text{-dim v. Neuman}}$

[Key $\chi(S_1, \dots S_n) > -\infty$
large lim of Gaussian RM
while $\chi(\text{generator } M_1 \otimes M_2) = -\infty$]

Connections to large deviations for random matrices
(Ben Arous, Cabanal-Duvillard, Guionnet, Petz)

"Microstates - free" approach

$\chi^*(X_1, \dots, X_n)$

Infinitesimal approach: Free Fisher Information

Classically: distr. pdt

Score function: $\frac{p'}{p} = -\left(\frac{d}{dt}\right)^* 1$ in $L^2(\mathbb{R}, p\,dt)$

Fisher information

$$\left\| \frac{p'}{p} \right\|^2_{L^2(p\,dt)} = \int \frac{(p')^2}{p}\,dt =$$

$$= \left\| \left(\frac{d}{dt}\right)^* 1 \right\|^2$$

Also derivative of entropy along Brownian motion

$$= 2 \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\left(H(X + \varepsilon^{1/2} G) - H(X)\right)$$

Free analogue

$\frac{d}{dt} \rightsquigarrow \partial_X$

free difference quotient

$\partial_{X:B} : B\langle X\rangle \to B\langle X\rangle \otimes B\langle X\rangle$

$B \subset (M, \tau)$ v. Neumann subalg

$X = X^* \in M$

B and X algebraically free (i.e. no alg. relation)

$\partial_{X:B}\, b_0 X b_1 X \cdots b_n =$

$= \sum_{1 \le j \le n} b_0 X \cdots b_{j-1} \otimes b_j X \cdots b_n$

$J(X:B) = \partial^*_{X:B}\, 1 \otimes 1$

$\partial^*_{X:B}$ in $L^2(M, \tau)$

$\langle a, b \rangle = \tau(b^* a)$.

$B = \mathbb{C}1$

$$\partial_{X:\mathbb{C}} P \simeq \frac{P(X) - P(Y)}{X - Y}$$

In general

$$\Phi^*(X_1, \ldots, X_n) =$$

$$= \sum_{1 \leq k \leq n} \tau(\mathcal{J}_k^2)$$

$$\mathcal{J}_k = \mathcal{J}(X_k : W^*(X_1, \ldots, \hat{X_k}, \ldots X_n))$$

$$\chi^*(X_1, \ldots, X_n) =$$

$$= \frac{n}{2} \log 2\pi e + \int_0^\infty \left(\frac{n}{1+t} - \Phi^*(X_1 + t S_1, \ldots) \right) dt$$

$S_1, \dots, S_n$ large N limit of Gaussian i.i.d. n-tuple of random matrices and freely indep of $X_1, \dots X_n$.

Theory of χ and χ^* incomplete (i.e. would like complete list of free analogues of properties of differential entropy to be proved) in addition unification problem (show $\chi = \chi^*$).

Complete theory of χ or χ^* would have important consequences in von Neumann algebra theory.

Unification would solve Connes problem in particular. Also important connections to large deviations for random matrices (can be viewed as refinement of unification — χ^* rate function).

Recent large deviation important progress: $\chi \leq \chi^*$ (Biane – Capitaine – Guionnet) Stochastic analysis techniques.

Progress on free entropy also necessary for dealing with <u>variational problem</u>

$$\chi(X_1, \ldots, X_n) - \tau(P(X_1, \ldots X_n))$$

which is related to multimatrix model with distribution density

$$c_N \, e^{-N \operatorname{Tr} P(A_1, \ldots, A_n)}.$$

Free Probability Angle on "Largest Eigenvalues" of Random Matrices

Large-N-limit for n i.i.d. Gaussian Hermitian matrices are $S_1, \dots, S_n$ in (M, τ) freely indep. and with τ(spectral meas.) =

i.e. "semicircular". So

$$\lim_{N\to\infty} E\left(\frac{1}{N}\operatorname{Tr}\left(P(T_{1,N}, \dots, T_{n,N})\right)\right) = \tau\left(P(S_1, \dots, S_n)\right)$$

Recent result of Haagerup-Thorbjornsen

$$\lim_{N\to\infty} \| P(T_{1N}, \dots T_{nN}) \| \underset{\text{almost surely}}{=} \| P(S_1, \dots, S_n) \|$$

(Pf. is combination of free prob. techniques with concentration of measure).

For $n = 1$ this contains largest eigenvalue results.

Connects with "Topological free entropy" (V)

A = C^*-algebra with unit (norm-closed selfadjoint alg. of bdd. operators in $\mathcal{B}(\mathcal{H})$ — "noncommutative $C(X)$" i.e. topological sp.)

[v. Neumann alg. weakly closed — "noncommutative $L^\infty(X, d\mu)$" i.e. measure space]

$a_1, \ldots, a_n$ selfadj. generators of A.

$$\Gamma_{top}(a_1, \ldots, a_n; N, \varepsilon, P_1, \ldots, P_r)$$

$$A_1, \ldots, A_n \in (\mathcal{M}_N^{sa})^n$$

$$\big| \| P_j(A_1, \ldots, A_n) \| - \| P_j(a_1, \ldots, a_n) \| \big| < \varepsilon$$

$1 \le j \le r$, $P_j \in \mathbb{C}\langle X_1, \ldots X_n \rangle$.

$$\limsup_{N\to\infty}\left(N^{-2}\log \mathrm{vol}\,\Gamma_{top} + \frac{n}{2}\log N\right)$$

$$\inf_{\varepsilon>0}\ \inf_{r\in\mathbb{N}}\ \inf_{P_1\ldots P_r\in\mathbb{C}_{\langle n\rangle}}$$

gives $\chi_{top}(a_1,\ldots,a_n)$

Competing quantity:

<u>free capacity</u>

$$K(a_1,\ldots,a_n) = \sup_{\tau\in TS(A)} \chi(a_1\ldots a_n;\tau)$$

$(\chi(a_1\ldots a_n;\tau) = \chi(a_1\ldots a_n$ / in $(A,\tau))$.

<u>Fact</u>: $\chi_{top}(a_1,\ldots,a_n) \le$
$\le K(a_1,\ldots a_n)$

<u>Problem</u> $\chi_{top} = K$?

[Maybe always ???]

$n = 1$ $\sigma(a)$ spectrum of a

$$e^{\kappa(a) - \theta} = \mathrm{Cap}(\sigma(a))$$

$\mathrm{cap}(\cdot)$ logarithmic capacity.

$\chi_{top} = K$ for $S_1, \dots, S_n$ (free semicircular n-tuple

$\chi_{top} = K$ for universal n-tuple $C_1, \dots, C_n$ of self-adjoint contractions

Maximizing trace-states
- for $S_1 \dots S_n$ trace-state which makes them free semicircular
- for $C_1 \dots C_n$ freely indep with arcsine distribution.

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