First order asymptotics for matrix integrals

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I shall discuss in this talk the first order asymptotics of matrix integrals of the form

$$\int e^{-N \operatorname{tr} P(A_1^N, \ldots, A_d^N)} \, dA_1^N \ldots dA_d^N$$

Lebesgue measure on $\begin{cases} S(N) (\beta=1) \\ H(N) (\beta=2) \end{cases}$

where P is a polynomial function of d non-commutative variables.

In the case where $P(A_1, \ldots A_d) = \sum P_i(A_i) + \sum \alpha_{ij} A_i A_j$ for some $\{\alpha_{ij}\}$, I shall also discuss the limit points for the spectral measures of $\{A_1^N \ldots A_d^N\}$ under the Gibbs measure

$$\mu_N(dA_1^N \ldots dA_d^N) = \frac{1}{Z_N(P)} e^{-N \operatorname{tr} P(A_1^N \ldots A_d^N)} dA_1^N \ldots dA_d^N$$

with $Z_N(P)$ the partition function

$$Z_N(P) = \int e^{-N \operatorname{tr} P(A_1^N \ldots A_d^N)} dA_1^N \ldots dA_d^N$$

My favorite tools are large deviations techniques + free probability.

Compare with - orthogonal polynomial methods?
Riemann-Hilbert problem?
- character expansion?
enumeration of maps?

I) Case $d=1$

To illustrate the kind of results we shall obtain, let us mention what they would be for the easy case $d=1$

$$\int e^{-N\,\mathrm{tr}\,P(A_1^N)}\,dA_1^N$$

$$=\int e^{-N\sum_{i=1}^{N}P(\lambda_i)}\prod_{i<j}|\lambda_i-\lambda_j|^{\beta}\,\frac{\prod_{i=1}^{N}d\lambda_i}{Z_N}$$

$\Rightarrow$ $$\lim_{N\to\infty}\frac{1}{N^2}\log\int e^{-N\,\mathrm{tr}\,P(A_1^N)}\,dA_1^N=-\inf_{\mu\in\mathcal{P}(\mathbb{R})}\left\{\mu(P)-\frac{\beta}{2}\Sigma(\mu)\right\}+c$$

with $$\Sigma(\mu)=\iint\log|x-y|\,d\mu(x)\,d\mu(y)$$

$\Rightarrow$ Because $\mu\to\mu(P)-\Sigma(\mu)$ is convexe, there exists a unique minimizer μ^* in the above infimum

$\Rightarrow$ It satisfies

$$\beta H\mu^*(x)=P'(x)\quad \mu^*\text{ as}$$

$\Rightarrow$ Under the corresponding Gibbs measure, the spectral measure of A_1^N converges weakly towards μ^*.

II AB interaction

In physics, many matrix models appear with a "AB" type interaction

1) Ising model on random graphs

$$Z^N_{Ising} = \int e^{N \operatorname{tr} AB - N \operatorname{tr} P(A) - N \operatorname{tr} Q(B)} dA\, dB$$

Lebesgue on $\uparrow \begin{cases} \mathcal{H}(N \\ S(N \end{cases}$

P, Q polynomials

$P(x) \geq cx^2 + d \qquad Q(x) \geq ax^2 + b \qquad ac > \frac{1}{2}$

2) Potts model on random graphs

$$Z^N_{Potts} = \int \prod_{i=2}^{q} e^{N \operatorname{tr}(A_1 A_i) - N \operatorname{tr} P_i(A_i) - N \operatorname{tr} P_1(A_1} dA_1 \dots dA$$

3) Chain model

$$Z^N_{chain} = \int \prod_{i=2}^{q} e^{N \operatorname{tr}(A_{i-1} A_i) - N \operatorname{tr} P_i(A_i) - N \operatorname{tr} P_1(A} dA_1 \dots dA_q$$

4) Induced QCD model

$$Z^N_{QCD} = \int \prod_{i=1}^{q} \int e^{N \sum_{\mu=1}^{D} \operatorname{tr}(U_\mu A_i U_\mu^* A_{i+\mu})} \prod_{1 \leq i \leq \mu} dm^N_\beta(U_i)$$

$$\prod_{i=1}^{q} e^{-N \operatorname{tr} P_i(A_i)} dA_1 \dots dA_q$$

m^N_β = Haar measure on $\begin{cases} U(N) & \beta = 2 \\ O(N) & \beta = 1 \end{cases}$

In all these models, the interaction is given in terms of a spherical integral

$$I_N^{(\beta)}(A_N, B_N) = \int e^{N \operatorname{tr}(UA_NU^* B_N)} dm_\beta^N(U)$$

With O. Zeitouni [JFA 2002] we proved:

<u>Thm</u>: Let for $A \in \begin{cases} \mathcal{H}(N) \\ \mathcal{S}(N) \end{cases}$ $\hat{\mu}_A^N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_A^i}$

with $(\lambda_A^1, \ldots \lambda_A^N)$ be the eigenvalues of A

Assume $\exists\, \mathcal{K}$ compact $\subset \mathbb{R}$ st $\operatorname{supp} \hat{\mu}_{A_N}^N \subset \mathcal{K}\ \forall N$

$\exists\, M < \infty$ st $\hat{\mu}_{B_N}^N(x^2) \leq M \quad \forall N$

$\hat{\mu}_{A_N}^N \xrightarrow[N\to\infty]{w} \mu_A \qquad \hat{\mu}_{B_N}^N \xrightarrow[N\to\infty]{w} \mu_B$

Then

$$I^{(\beta)}(\mu_A, \mu_B) = \lim_{N\to\infty} \frac{1}{N^2} \log I_N^{(\beta)}(A_N, B_N)$$

$$= -J_\beta(\mu_A, \mu_B) + I_\beta(\mu_B) + \frac{1}{2}\int x^2 d\mu_A(x)$$

with

$$I_\beta(\mu) = \frac{1}{2}\int x^2 d\mu(x) - \frac{\beta}{2}\int \log|x-y| d\mu(x) d\mu(y)$$
$$- \inf_\nu \left\{ \frac{1}{2}\int x^2 d\nu - \frac{\beta}{2}\int \log|x-y| d\nu(x) d\nu(y) \right.$$

$$J_\beta(\mu_A, \mu_B) = \frac{\beta}{4} \inf \left\{ \int_0^1 \int (\partial_x k_t)^2 \mu_t(dx)\, dt \,; \right.$$

$\mu_t(dx) = \rho_t(x) dx \quad t \in (0,1) \qquad \mu_0 = \mu_A \quad \mu_1 = \mu_B$

We left open the question of the minimizing path in J_β but mentionned heuristics from Matytsin [Nuclear physics 94] who argued that it should be achieved at $\{\mu_t^*, t\in[0,1]\}$ such that

$$\mu_t^*(dx) = \rho_t^*(x)\,dx \quad , \quad t\in[0,1]$$

$$\mu_0^* = \mu_A \qquad \mu_1^* = \mu_B$$

and if we set

$$\Pi_t^*(x) = k_t^*(x) + \int \log|x-y|\, d\mu_t^*(y)$$

then

$$\begin{cases} \partial_t \rho_t^* + \partial_x(\rho_t^*\, \partial_x \Pi_t^*) = 0 \\ \partial_t \Pi_t^* = -\frac{1}{2}(\partial_x \Pi_t^*)^2 + \frac{\pi^2}{2}\rho_t^*(x)^2 \end{cases}$$

In other words, $(\rho^*, u^*) = (\rho^*, \partial_x \Pi^*)$ satisfies the Euler equation for isentropic flow with negative (!) pressure $p(\rho) = -\frac{\pi^3}{3}\rho^3$

$$\begin{cases} \partial_t \rho_t^* + \partial_x(\rho_t^* u_t^*) = 0 \\ \partial_t(\rho_t^* u_t^*) + \partial_x\left(\frac{1}{2}\rho_t^*(u_t^*)^2 + p(\rho_t^*)\right) = 0 \end{cases}$$

Thm [G] with great help of | O. Zeitouni, D. Serre, C. Villani, Y. Brenier

Assume μ_A, μ_B compactly supported. Then

a) The infimum is indeed achieved at the solutions of the Euler equation for isentropic flow.

b) For any given (μ_A, μ_B), there exists one and only one such a solution

c)
$$I_\beta(\mu_A,\mu_B) = -\frac{\beta}{4} \inf_{(\mu_\cdot, u_\cdot)\in C(\mu_A,\mu_B)} \Big\{ \underbrace{\int_0^1 \int u_t^2 \, d\mu_t \, dt}_{\int_0^1 \int (\partial_t \mu_t)^2 d\mu_t dt} + \frac{\pi^2}{3}\int_0^1\int \rho_t^3 \, dx \, dt \Big\}$$
$$- \frac{\beta}{4}\left(\Sigma(\mu_A) + \Sigma(\mu_B)\right) + \frac{1}{2}\int x^2 (d\mu_A + d\mu_B)(x)$$
$$- \inf_{\nu \in P(\mathbb{R})} \left(\frac{1}{2}\int x^2 d\nu(x) - \frac{\beta}{2}\Sigma(\nu)\right)$$

$$(\mu_\cdot, u_\cdot) \in C(\mu_A,\mu_B) = \Big\{ (u_\cdot, \mu_\cdot) : \mu_t(dx) = \rho_t(x)\,dx \quad t \in (0,1)$$
$$\mu_0 = \mu_A \quad \mu_1 = \mu_B$$
$$\partial_t \rho_t + \partial_x(\rho_t u_t) = 0 \Big\}$$

The above infimimum is taken at the solution of the isentropic Euler equation

Rmk: c) $\Rightarrow$ b) since with $m = \rho u$
$$\int_0^1 \int u_t^2 \, d\mu_t \, dt + \frac{\pi^2}{3}\int_0^1\int \rho_t^3 \, dx \, dt$$
$$= \int_0^1 \int \frac{m_t^2}{\rho_t} \, dx \, dt + \frac{\pi^2}{3}\int_0^1 \int \rho_t^3 \, dx \, dt \quad \text{is a}$$

Discussion: Following Matytsin, we would like to consider

$$f(x,t) = u_t^*(x) + i\pi \rho_t^*(x)$$

Then, the Euler equation for isentropic flow boils down to the Burgers equation

$$\partial_t f(x,t) + \frac{1}{2} \partial_x f(x,t)^2 = 0$$

If we assume that $\mu_A(dx) = \rho_A(x)dx$, $\mu_B(dx) = \rho_B(x)dx$ with (ρ_A, ρ_B) which can be extended analytically to $\mathbb{C}$, it is ~~plausible~~ likely that f also extends smoothly to $\mathbb{C}$. Then, using characteristic methods, one finds

$$f(f(z,0)t + z, t) = f(z,0)$$

So, setting

$$G_+(z) = z + f(z,0)$$

$$G_-(z) = z - f(z,1)$$

we obtain

$$\begin{cases} f(G_+(z)t + z(1-t), t) = G_+(z) \\ G_+ \circ G_-(z) = G_- \circ G_+(z) = z \end{cases}$$

with bc $\quad \mathrm{Im}\, G_+(x) = \rho_A(x) \quad \mathrm{Im}\, G_-(x) = \rho_B(x)$

$\Rightarrow$ by uniqueness, if such a solution exists, it is <u>The</u> solution.

Rmk:

* If one considers

$$\mu_t(dx) = \rho_t(x)\,dx = \mu \boxplus \sigma_t = \text{law of } (X + \sqrt{t}\, S)$$

$\sigma_t = \frac{1}{\pi t}\sqrt{4t - x^2}\,dx$, X dist μ, S semicirc, X free with S

and set
$$f(x,t) = H\rho_t + i\pi \rho_t$$

then

$$\begin{cases} \partial_t f(x,t) + \frac{1}{2}\partial_x f^2(x,t) = 0 \\ f(x,0) = H\mu_0 + i\pi \rho_0 \end{cases}$$

$\Rightarrow$ The minimizing path follows the same Burgers equation but needs a field $u^* = \frac{\partial_x \Pi^*}{\partial x}$ to match the b.c.

* In our case, we can also see that

$$\rho^*_t(x)\,dx = \text{law of } (tB + (1-t)A + \sqrt{t(1-t)}\, S)$$

B dist μ_B, A dist μ_A, S semicirc.

with S free with (A,B).

$\Rightarrow$ The isentropic Euler equation prescribes part of the joint law of (A,B) (namely law of $\{tB + (1-t)A,\ t \in (0,1)\}$) under the limiting Gibbs

Applications: by saddle point (Laplace) methods, we obtain

1) Ising model

$$F_{Ising} = \lim_{N\to\infty} \frac{1}{N^2} \log Z^N_{Ising}$$

$$= -\inf_{\mu,\nu\in\mathcal{P}(\mathbb{R})} \left\{ \mu(P) + \nu(Q) - I^{(\beta)}(\mu,\nu) - \beta\Sigma(\mu) - \beta\Sigma(\nu) \right\} + c$$

$$= -\inf_{\substack{\mu,\nu\in\mathcal{P}(\mathbb{R}) \\ \mu_t,\nu_t\in C(\mu,\nu)}} \Big\{ \mu\left(P - \tfrac{1}{2}x^2\right) + \nu\left(Q - \tfrac{1}{2}x^2\right) + \frac{\beta}{4}\left\{ \int_0^1 \int u_t^2\, d\mu_t(x)\, dx + \frac{\pi^2}{2} \iint \rho_t^3(x)\, dx\, dt \right\} - \frac{\beta}{2}\left(\Sigma(\mu) + \Sigma(\nu) \right) \Big\} + 2c$$

→ By convexity, there exists a unique minimizer $(\mu^*, \nu^*, \mu_t^*, u_t^*)$

→ Assume (μ^*, ν^*) compactly supported (clear according to physics litterature), Then $(\mu^*, \nu^*, \mu_t^*, u_t^*)$ should satisfy

$$P'(x) - x - \frac{\beta}{2} u_0^*(x) - \frac{\beta}{2} H\mu_0^*(x) = 0 \quad \mu_0^* \text{ as}$$

$$Q'(x) - x + \frac{\beta}{2} u_1^*(x) - \frac{\beta}{2} H\mu_1^*(x) = 0 \quad \mu_1^* \text{ as}$$

(μ_t^*, u_t^*) follows the Euler equation for Isentropic flow with $\mu_0^* = \mu^*$, $\mu_1^* = \nu^*$

→ Under $\mu^N_{Ising}(dA^N, dB^N)$,

$$\hat{\mu}^N_{A_N} \to \mu^* \qquad \hat{\mu}^N_{B_N} \to \nu^*$$

2) ~~Potts~~ model

$$F_{Potts} = -\inf \Big\{ \mu_1 (P - \tfrac{q}{2} x^2) + \sum_{i=2}^{q} \mu_i (P - \tfrac{x^2}{2})$$
$$+ \frac{\beta}{4} \sum_{i=1}^{q} \Big(\int_0^1 \int (u_t^i)^2 d\mu_t^i \, dt + \int_0^1 \frac{\pi^2}{3} \int (\rho_t^i)^2 dx \, dt$$
$$- \frac{\beta}{4} \sum_{i=2}^{q} \Sigma(\mu_i) + \frac{\beta}{4} (q-3) \Sigma(\mu_1) \Big\} + qc$$

→ Where inf taken on $(\rho^i, u^i) \in C(\mu_1, \mu_i)$, $\mu_i \in P(\mathbb{R})$

→ Convexity in μ_1 is lost and therefore uniqueness of the minimizers becomes unclear (for $q \geq 3$)

→ the critical points can be described as for Ising model.

3) Chain model, Induced QCD:

similar results are obtained convexity is lost in general.

Discussion:

Matytsin and all studied phase transition for various models with AB intera using these equations.

They always assume uniqueness of the minimizers, characteristic method's solut hold.

The minimizers are compactly supported

Phase transition seems to be due to the splitting of the support of the solutio

For instance, consider Ising model with $P(x) = Q(x) = \frac{x^2}{2} + g\,x^4$

→ subject to a new study.

Rmk: Power of large deviations techniques: Matytsin and all only consider $\beta = 2$ case!

III General case: link with non-commutative entropies

Micro-states entropy χ is related with asymptotics of general matrix integrals

free probability context:

τ: tracial state on a von Neuman algebra

$$\in \mathcal{M}_d = \{\tau \in \mathbb{C}\langle X_1,\dots,X_d\rangle^*,\ \tau(PQ) = \tau(QP)$$
$$\tau(QQ^*) \geq 0 \quad \tau(I)=1 \quad \forall P, Q \in \mathbb{C}\langle X_1 \dots X_d\rangle$$

Microstates: neighborhood of τ

$$\Gamma(\varepsilon, N, k, \tau) = \{A_1^N \dots A_d^N \in \mathcal{H}(N)^d :$$
$$|\mathrm{tr}_N\, P(A_1^N, \dots, A_d^N) - \tau(P)| < \varepsilon$$
$$\forall P \text{ monome of degree} \leq k\}$$

$$\chi(\tau) = \overline{\lim_{\substack{\varepsilon \downarrow 0 \\ k \uparrow \infty}}}\ \overline{\lim_{N\uparrow\infty}}\ \frac{1}{N^2} \log \mathbb{P}^{\otimes d}(\Gamma(\varepsilon, N, k, \tau))$$

$$\mathbb{P}(dA) = e^{-N\mathrm{tr}A^2} \frac{dA}{Z_N}$$

$$\Rightarrow \overline{\lim_{N\to\infty}}\ \frac{1}{N^2} \log \int e^{-N\mathrm{tr}P(A_1^N, \dots, A_d^N)} dA_1^N \dots dA_d^N$$

$$= \sup_{\tau \in \mathcal{M}_d} \{\chi(\tau) - \tau(P - \tfrac{1}{2}\sum_{i=1}^d A_i^2)\} + dc$$

In fact, for any smooth function,

$$\lim \frac{1}{N^2} \log \int e^{N^2 f(tr_N P_1 \dots tr_N P_m)} dA_1^N \dots dA_d^N$$

$$= \sup_{\tau \in \mathcal{M}_d} \left\{ \chi(\tau) + f(\tau(P_1), \dots \tau(P_m)) - \frac{1}{2} \sum \tau(A_i^2) \right\} + dc$$

Reciprocally, if

$$\exists \quad \Lambda(f, P_1 \dots P_m) = \lim_{N \to \infty} \frac{1}{N^2} \log \int e^{N^2 f(tr_N P_1 \dots tr_N P_m)} dA_1^N \dots dA_d^N$$

then [Bryc's lemma]

$$\chi(\tau) = \lim_{\substack{\varepsilon \downarrow 0 \\ k \uparrow \infty}} \lim_{N \uparrow \infty} \frac{1}{N^2} \log \mathbb{P}^{\otimes d}\left(\Gamma(R, \varepsilon, N, k) \right)$$

$$= - \sup_{f, P_i} \left\{ -\Lambda(f, P_1 \dots P_m) + f(\tau(P_1) \dots \tau(P_m)) \right\}$$

Rmk: Classical Sanov's thm: it is enough to take $f(\tau(P_1) \dots \tau(P_m)) = \tau(P_1)$

<u>Conjecture</u> (personnal): It is enough to take $f(\tau(P_1) \dots \tau(P_m)) = \tau(P_1)\tau(P_2)$

Let χ^* be the generalization of

$$S(\mu) = \int \underbrace{\frac{d\beta}{d\gamma}}_{\gamma \sim N(0,1)} \ln \frac{d\mu}{d\gamma}\, d\gamma = \frac{1}{2}\int_0^1 F(\mu^t)\, dt$$

Fisher information; μ^t: law of Brownian bridge between δ_0 and μ

then

<u>thm</u>: [Biane, Capitaine & G][2002]

$$\chi^{**} \leq \chi \leq \chi^* \qquad (*)$$

χ^{**} similar to χ^*

$$\Rightarrow \sup_{\tau \in \mathcal{M}_d} \{\chi^{**}(P) - \tau(P)\} \leq \varliminf_{N \uparrow \infty} \frac{1}{N^2} \log \int e^{-N \operatorname{tr} P(A_1^N, \dots, A_d^N)} dA_1^N \dots dA_d^N$$

$$\leq \varlimsup_{N \uparrow \infty} \frac{1}{N^2} \log \int e^{-N \operatorname{tr} P(A_1^N, \dots, A_d^N)} dA_1^N \dots dA_d^N$$

$$\leq \sup_{\tau \in \mathcal{M}_d} \{\chi^*(P) - \tau(P)\}$$

→ I believe equality holds

→ Computation of matrix integrals & free probability entropies are intimately related

→ (*) JW T. Cabanal Duvillard $\chi(\tau) \leq \chi^*(\tau)$

for τ with good J's (maybe all ?)