Prototype vanishing integrals

$$\int_{U \in \mathrm{O}(n)} s_\lambda(U) = 0$$

unless λ is of the form 2μ.

$$\int_{U \in \mathrm{Sp}(2n)} s_\lambda(U) = 0$$

unless λ is of the form μ^2.

- Nonzero values are "nice" (in fact, 1)
- Come from symmetric spaces $\mathrm{U}(n)/\mathrm{O}(n)$, $\mathrm{U}(2n)/\mathrm{Sp}(2n)$

Orthogonal cases:

$$\int s_\lambda(\ldots x_i, x_i^{-1} \ldots) \prod_{\substack{1\le i<j\le n \\ s_1,s_2=\pm 1}} (1 - x_i^{s_1} x_j^{s_2}) = 0$$

$$\int s_\lambda(\ldots x_i, x_i^{-1} \ldots, 1, -1) \prod_{\substack{1\le i\le n \\ s=\pm 1}} (1 - x_i^{2s}) \prod_{\substack{1\le i<j\le n \\ s_1,s_2=\pm 1}} (1 - x_i^{s_1} x_j^{s_2}) = 0$$

unless λ is of the form 2μ or $2\mu + 1^{2n}$

$$\int s_\lambda(\ldots x_i, x_i^{-1} \ldots, 1) \prod_{\substack{1\le i\le n \\ s=\pm 1}} (1 - x_i^{s}) \prod_{\substack{1\le i<j\le n \\ s_1,s_2=\pm 1}} (1 - x_i^{s_1} x_j^{s_2}) = 0$$

$$\int s_\lambda(\ldots x_i, x_i^{-1} \ldots, -1) \prod_{\substack{1\le i\le n \\ s=\pm 1}} (1 + x_i^{s}) \prod_{\substack{1\le i<j\le n \\ s_1,s_2=\pm 1}} (1 - x_i^{s_1} x_j^{s_2}) = 0$$

unless λ is of the form 2μ or $2\mu + 1^{2n+1}$

Symplectic case:

$$\int s_\lambda(\ldots x_i, x_i^{-1} \ldots) \prod_{\substack{1\le i\le n\\ s=\pm 1}} (1-x_i^{2s}) \prod_{\substack{1\le i<j\le n\\ s_1,s_2=\pm 1}} (1-x_i^{s_1}x_j^{s_2}) = 0$$

unless λ is of the form μ^2

Macdonald polynomials

Define an inner product on symmetric functions by:

$$\langle \prod_{1\le i} p_i^{c_i}, \prod_{1\le i} p_i^{d_i} \rangle_{q,t} = \prod_{1\le i} \delta_{c_i d_i} c_i! \frac{i(1-q^i)}{1-t^i}$$

(I.e., p_i are independent complex Gaussian,

$$E(|p_i|^2) = \frac{i(1-q^i)}{1-t^i}$$

The Macdonald polynomials $P_\lambda(;q,t)$ are defined by:

$$P_\lambda(;q,t) = \sum_{\mu\le\lambda} c_{\lambda\mu} m_\lambda, \quad c_{\lambda\lambda} = 1$$

$$\langle P_\lambda(;q,t), P_\mu(;q,t)\rangle_{q,t} \propto \delta_{\lambda\mu}$$

(When $q=t$, these are the Schur functions)

Are there vanishing integrals for these?

Nonzero values

Evaluating at $x_1^{\pm 1}, \dots x_n^{\pm 1}$ has kernel

$$\langle e_i - e_{2n-i} : 1 \le i < n \rangle$$

In particular, applying the $Sp(2n)$ vanishing integral to

$$(e_1 - e_{2n-1}) P_\lambda(; q, t)$$

gives 0. Expanding in Macdonald polynomials, only two terms survive: this gives a recurrence for the nonzero values. (Values are "nice")

For orthogonal cases, each of the generators of the kernel gives at most two terms with nonzero integrals; the different recurrences all agree. So exists unique symmetric formal Laurent series giving vanishing "integral".

Symplectic: can only prove uniqueness.

Univariate cases 1

Want density $\Delta(x)$ on unit circle T^1 such that

$$\int P_\lambda(x, 1/x; q, t)\Delta(x) = 0$$

unless $\lambda_1 \equiv \lambda_2 \pmod 2$, $\lambda_3 = 0$.

Solve for coefficients of Laurent series...

When $t = q^{2k+1}$, get (apparent) polynomial:

$$\begin{aligned}\Delta(x) &= \frac{(qx^{\pm 2}; q^2)}{(tx^{\pm 2}; q^2)}? \\ &= \frac{(x^{\pm 2}; q)}{(x^{\pm 1}, -x^{\pm 1}, \sqrt{t}x^{\pm 1}, -\sqrt{t}x^{\pm 1}; q)}\end{aligned}$$

Special case of Askey-Wilson integral:

$$\Delta(x; a, b, c, d; q) := \frac{(x^{\pm 2}; q)}{(ax^{\pm 1}, bx^{\pm 1}, cx^{\pm 1}, dx^{\pm 1}; q)}$$

Univariate cases 2

Other orthogonal cases:

$$\int P_\lambda(x,1/x,1;q,t)\Delta(x;t,-1,\sqrt{t},-\sqrt{t};q)$$

$$\int P_\lambda(x,1/x,-1;q,t)\Delta(x;1,-t,\sqrt{t},-\sqrt{t};q)$$

$$\int P_\lambda(x,1/x,1,-1;q,t)\Delta(x;t,-t,\sqrt{t},-\sqrt{t};q)$$

Symplectic case:

$$\int P_\lambda(x,1/x;q,t)\Delta(x;\sqrt{t},-\sqrt{t},\sqrt{qt},-\sqrt{qt};q)$$

There is a natural multivariate analogue of the Askey-Wilson integral!

Koorwinder density:

$$\begin{aligned}
&\Delta^{(n)}(x_1, x_2, \dots x_n; a, b, c, d; q, t) \\
&\quad = \prod_{1\le i\le n} \frac{(x^{\pm 2}; q)}{(ax^{\pm 1}, bx^{\pm 1}, cx^{\pm 1}, dx^{\pm 1}; q)} \\
&\qquad \prod_{1\le i<j\le n} \frac{(x_i^{\pm 1} x_j^{\pm 1}; q)}{(tx_i^{\pm 1} x_j^{\pm 1}; q)}
\end{aligned}$$

Conjecture: Replacing Askey-Wilson density by Koornwinder density gives multivariate vanishing integrals.

$q = t$: reduces to Schur function case

The Schur case of the vanishing integral has another generalization:

$$E_{U\in \mathsf{O}(n)} s_{2\lambda}(AU) = \lim_{q\to 1} P_\lambda(AA^t; q, q^{1/2})$$
$$E_{U\in \mathsf{Sp}(n)} s_{\lambda^2}(AU) = \lim_{q\to 1} P_\lambda(AJA^t; q, q^2)$$

(nonzero otherwise)

Consider

$$E_{\substack{U_1\in \mathsf{O}(2n)\\ U_2\in \mathsf{Sp}(2n)}} s_\lambda(U_1U_2)$$

This is 0 unless λ has the form $2\mu^2$, so

$$E_{U\in \mathsf{Sp}(n)} \lim_{q\to 1} P_\lambda(UU^t; q, q^{1/2})$$
$$E_{U\in \mathsf{O}(n)} \lim_{q\to 1} P_\lambda(UJU^t; q, q^2)$$

are vanishing integrals.

The corresponding densities are the limits of the conjectured Koornwinder densities.

$$n \to \infty$$

Express the *formal* logarithm of the Koornwinder density in terms of $p_k(\dots x_i, 1/x_i \dots)$. Take

$$\log(x;q) \sim -\sum_{1\le k} \frac{x^k}{k(1-q^k)}.$$

Obtain:

$$\begin{aligned}
&\log \Delta^{(n)}(;a,b,c,d;q,t)\\
&\quad\sim \sum_{1\le k} \frac{(a^k+b^k+c^k+d^k)p_k - p_{2k}}{k(1-q^k)}\\
&\qquad\qquad - \frac{1-t^k}{k(1-q^k)} \frac{p_k^2 - n - p_{2k}}{2}
\end{aligned}$$

This is quadratic in p_k, so suggests a (real) independent Gaussian limit.

Diaconis-Shashahani: This works for $\mathrm{O}^{\pm}(n)$, $\mathrm{Sp}(2n)$.

The general Gaussian limit follows.

$n \to \infty$, continued

Macdonald polynomials satisfy:

$$\sum_\lambda P_\lambda(x)P_\lambda(y) = \prod_{1\le i,j} (tx_iy_j;q)/(x_iy_j;q)$$
$$= \exp\left(\sum_{k\ge 1} \frac{p_k(x)p_k(y)(1-t^k)}{k(1-q^k)}\right)$$

Integrate both sides over the $n \to \infty$ limit of the $O^\pm(n)$ vanishing integrals. LHS becomes:

$$\sum_\lambda c_\lambda P_{2\lambda}(y)$$

while RHS can be computed by completing the square. Result: Macdonald's Littlewood identities for Macdonald polynomials.

So conjectures hold in $n \to \infty$ limit.

Note: Littlewood identities are dual to each other!

Rationality in $T = t^n$

Theorem. For any partition λ, there exists a rational function $F_\lambda(a, b, c, d; q, t; T)$ such that

$$\int P_\lambda(x_1^{\pm 1}, \dots x_n^{\pm 1}; q, t) \Delta^{(n)}(x_1, \dots x_n; a, b, c, d; q, t) = F_\lambda(a, b, c, d; q, t; t^n)$$

for generic $a, b, c, d, q, t < 1$ and all $n \geq 1$.

(Genericity conditions are tractable)

Duality:

$$F_\lambda(a, b, c, d; q, t; T) \propto F_{\lambda'}\left(\frac{-\sqrt{qt}}{a}, \frac{-\sqrt{qt}}{b}, \frac{-\sqrt{qt}}{c}, \frac{-\sqrt{qt}}{d}; t, q; 1/T\right)$$

With other symmetries, we find that all five conjectures are equivalent.

More special cases:

Theorem. If $\ell(\lambda) \leq 1$, then the $O^{\pm}(n)$ vanishing conjecture holds for P_λ.

Sketch: Use Cauchy identity with only one variable y; obtain a basic hypergeometric generating function. The conjectures then becomes a (known) quadratic transformation.

No nontrivial multivariate quadratic transformations are known!

Theorem. If $\lambda_1 \leq 4$, then the $O^{\pm}(n)$ vanishing conjecture holds for P_λ.

Sketch: Compare the Cauchy identity for Macdonald polynomials to the Cauchy identity for Koornwinder polynomials. Reduce to D_4-type Macdonald polynomials, and apply triality.

(Dual statements hold for $Sp(2n)$)

Smallest remaining case is $\lambda = 51$

Other symmetric spaces:

$G \times G/G$: known orthogonality results for Macdonald and Koornwinder polynomials.

For other cases, uniqueness falls through; harder to guess densities!

Approach 1: Look at the classical case, and make a wild guess

$U(p+q)/U(p) \times U(q)$: just take product of independent Macdonald densities. (Works for small p, q) Nonzero values appear to be "nice".

Other Grassmannian cases should be analogous, but are difficult to test (we must replace Macdonald polynomials by Koornwinder polynomials).

Approach 2: Use other cosets to fix nonzero values.

For $U(2n)/U(n) \times U(n)$, exchange involution gives another coset; can solve for nonzero values and guess density.

$$\int P_{\mu\overline{\nu}}(\cdots \pm \sqrt{x_i}; q, t) \prod_{i \neq j} \frac{(x_i/x_j; q^2)}{(t^2 x_i/x_j; q^2)} \propto \delta_{\mu\nu}$$

Suitable sign changes give nonzero values for identity coset. Gives different density from Approach 1! Conjectured density is "nice", but even normalization isn't known (Both approaches agree when $q = t$, and give "nice" nonzero values)

Approach 2 also gives conjectures for other classical symmetric spaces; Macdonald polynomials become Koornwinder polynomials.